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THE EXISTENCE OF WEAK SOLUTIONS OF $p(x)$-BIHARMONIC EQUATION WITH NAVIER BOUNDARY CONDITIONS

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Abstract

Applying monotone mapping theory, this paper studies the existence of weak solutions of $p(x)$-biharmonic equation on variable exponential space.

1. Introduction and Preliminaries

In the theory of elasticity, we often encounter with $\int_{\Omega} |Du(x)|^{p(x)} dx$ and the integrals constructed from more general function with growth conditions, where $p(x)$ is the function defined on Ω. These integrals reflect an important physical phenomenon called "*Anisotropy*".

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Fan and others have systematically investigated $p(x)$-Laplacian on variable exponential space more than ten years. However, the results on $n\ (n > 2)$ order equations with non-standard growth conditions are fewer. Applying pseudomonotone operator, Zhao and Fan [1] discussed the existence of weak solutions of $2m$ order elliptic equations with $p(x)$-growth condition. Moreover, applying calculus of variations, Zang discussed the existence of solutions and multiplicity solutions in [2].

Deeply impressed by [3], we shall discuss the existence of solutions of Navier boundary problem with monotone mapping theory. To be precise, we show that

$$(p)\begin{cases} -\Delta^2_{p(x)}u = f(x) & \text{a.e. on } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p(x) \in C(\overline{\Omega})$, $\Omega \subset R^n$ is a bounded smooth domain and $1 < p_- = \operatorname{essinf}_{x\in\overline{\Omega}} p(x) \leq p_+ = \operatorname{esssup}_{x\in\overline{\Omega}} p(x) < \frac{N}{K}$.

Let $p(x)$ is log-Hölder continuous, i.e., for the bounded exponent $p(x) > 1$ and $\forall x \in \Omega$, we have

$$|\, p(x) - p(y)\,| \leq \frac{C}{\log|\, x - y\,|} \quad x,\ y \in \Omega,\ |\, x - y\,| \leq \frac{1}{2}, \tag{1}$$

$$|\, p(x) - p(y)\,| \leq \frac{C}{\log(e + |\, x\,|)} \quad x,\ y \in \Omega,\ |\, y\,| \geq |\, x\,|. \tag{2}$$

When $p(x) = p$ is constant, the existence and non-existence of the solutions can be referred to [4, 5]. For $x \in \overline{\Omega}$, $-\Delta^2_{p(x)}u = \Delta(|\,\Delta u\,|^{p(x)-2}\Delta u)$ is called $p(x)$-*biharmonic operator*.

In order to study boundary value problem (p), we first introduce some notations and results in variable exponent Sobolev space $W^{k,\, p(x)}$ [6].

We use $S(\Omega)$ to denote the set of all real measurable functions defined on Ω. For $x, y \in S(\Omega)$, x, y are considered as the same element if they equal almost everywhere. Set

$$L_{+}^{\infty}(\Omega) = \{p(x) \in L_{\Omega}^{\infty}, \operatorname{essinf}_{\Omega} p(x) = p^{-} \geq 1\}.$$

For $p \in L_{+}^{\infty}(\Omega)$, we further denote

$$L^{p(x)}(\Omega) = \left\{u \in S(\omega) : \int_{\omega} |u|^{p(x)} dx < \infty\right\}.$$

The corresponding norms are

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{\lambda > 0 : \int_{\Omega} \left|\frac{u}{\lambda}\right|^{p(x)} dx \leq 1\right\},$$

$$W^{K, p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^{\alpha}u \in L^{p(x)}(\Omega), 1 \leq |\alpha| \leq K\},$$

$$\|u\|_{K, p(x)} = \|u\|_{W^{K, p(x)}(\Omega)} = \sum_{|\alpha| \leq K} |D^{\alpha}u|_{L^{p(x)}(\Omega)}.$$

Let $\|u\|_{J, p(x), \Omega} = \sum_{\alpha = J} |D^{\alpha}u|_{L^{p(x)}(\Omega)}$, where J is multiple index and $|\alpha|$ is differential order.

Let $W_{0}^{K, p(x)}(\Omega)$ is the closure of $C_{0}^{\infty}(\Omega)$ in $W^{K, p(x)}(\Omega)$.

Property 1.1 [6]. $L^{p(x)}(\Omega)$, $W^{K, p(x)}(\Omega)$ and $W_{0}^{K, p(x)}(\Omega)$ are all reflexive Banach space.

The norm of $X = W_{0}^{1, p(x)}(\Omega) \cap W_{0}^{2, p(x)}(\Omega)$ is defined as

$$\begin{aligned} \|u\| &= \|u\|_{W_{0}^{1, p(x)}(\Omega) \cap W_{0}^{2, p(x)}(\Omega)} \\ &= \|u\|_{1, p(x)} + \|u\|_{2, p(x)} \\ &= 2|u|_{p(x)} + 2|Du|_{p(x)} + \sum_{|\alpha| = 2} |D^{\alpha}u|_{p(x)}. \end{aligned}$$

Set

$$p^*(x) = \begin{cases} Np(x)/(N - Kp(x)) & p(x) < \dfrac{N}{K}, \\ \infty & p(x) \geq \dfrac{N}{K}. \end{cases}$$

Property 1.2 [7]**.** Let Ω be a bounded region, $p(x)$ be log-Hölder continuous and $p^+ < \frac{N}{2}$ restricted in space $W_0^{1, p(x)}(\Omega) \cap W_0^{2, p(x)}(\Omega)$. Then $\| \cdot \|$ and $| \Delta |_{p(x)}$ are equivalent norms.

Property 1.3 [8]**.** Let Ω be a bounded region, $p(x) \in C(\overline{\Omega})$, $q(x) \geq p(x)$ a.e. $x \in \overline{\Omega}$ and $\operatorname{essinf} | p^*(x) - q(x) | > 0$. Then $W^{K, p(x)}(\Omega)$ compact embedding in $L^{q(x)}$.

Definition 1.1 [9]**.** Let *E* be a real Banach space and let E^* be the dual space of *E*. Then $M \subset E \times E^*$ is called *monotone set* if $(y_1 - y_2, x_1 - x_2) \geq 0$, for $[x_1, y_1], [x_2, y_2] \in M$. And the monotone set $M \subset E \times E^*$ is called *maximal monotone* if $E \times E^*$ is not the real subset of any monotone set.

Definition 1.2 [9]**.** The multi-value map $T : E \to 2^{E^*}$ is monotonous if its image $G(T) = \{(x, y) : x \in D, y = Tx\}$ is a monotone set of $E \times E^*$. *T* is *maximal monotonous* if its image $G(T)$ is a maximal monotone set of $E \times E^*$.

Definition 1.3 [9]**.** Let $D \subset E$, map $T : D \to E^*$ and $x_0 \in D$. If $h \in E$, $t_n > 0$, $x_0 + t_n h \in D$, $t_n \to 0 \Rightarrow T(x_0 + t_n h) \to T(x_0)$, then *T* is called *semi-continuous* in x_0. Moreover, *T* is called *semi-continuous* in *D*, if *T* is semi-continuous in every point of *D*.

2. Main Results

Lemma 2.1 [9]**.** *If* $T : E \to E^*$ *is semi-continuous and monotonous, then T must be maximal monotonous.*

Lemma 2.2 [2]**.** *The map* $T : X \to X^*$ *is defined as follows*:

$$(v, Tu) = -\int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx, \quad \forall u, v \in X.$$

Then T is defined everywhere and bounded and monotonous. Thus, T is maximal monotonous.

Lemma 2.3 [9] (Minty-Browder)**.** *Let E be reflexive, map* $T : E \to E^*$ *is semi-continuous and monotonous. And let T be mandatory, i.e.,* $\lim_{\|x\| \to +\infty} \frac{(Tx, x)}{\|x\|} = +\infty$. *Then T must be surjective, i.e.,* $T(E) = E^*$.

Theorem 2.1. *For* $f \in L^{p(x)}(\Omega)$, *the boundary value problem* (p) *have solutions in the space* $W_0^{2, p(x)}(\Omega)$.

Proof. For $u \in W_0^{2, p(x)}(\Omega)$ and $u \neq 0$, then

$$\begin{aligned}
\|Tu\|_{(W_0^{2, p(x)}(\Omega))^*} &= \sup_{v \neq 0, v \in W_0^{2, p(x)}(\Omega)} \left| \frac{(v, Tu)}{\|v\|_{W_0^{2, p(x)}(\Omega)}} \right| \\
&= \sup \left| \frac{-\int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx}{\|v\|_{2, p(x)}} \right| \\
&\geq \frac{\int_\Omega |\Delta u|^{p(x)} dx}{\|u\|_{2, p(x)}} \\
&\geq M \|u\|_{2, p(x)}^{p-1},
\end{aligned}$$

where *M* is a constant.

By Property 1.2, $\forall u \in W_0^{2, p(x)}(\Omega)$, when $\|u\|_{2, p(x)} \to \infty$, $\|Tu\|_{(W_0^{2, p(x)}(\Omega))^*} \to \infty$, i.e.,

$$\lim_{u \in W_0^{2, p(x)}(\Omega), \|u\|_{2, p(x)} \to \infty} \|Tu\|_{(W_0^{2, p(x)}(\Omega))^*} \to \infty.$$

Due to Minty-Browder theorem, we obtain

$$R(B) = (W_0^{2,\, p(x)}(\Omega))^*.$$

Meanwhile, $\forall f \in L^{p(x)}(\Omega)$, we could derive from Property 1.1 and Property 1.3 that

$$f \in (W_0^{2,\, p(x)}(\Omega))^*.$$

Therefore

$$\exists u \in W_0^{2,\, p(x)}(\Omega) \quad \text{s.t.} \quad f = Tu.$$

Consequently, $\forall \varphi \in C_0^\infty(\Omega)$, it follows that

$$\begin{aligned}(\varphi,\ f) &= -\int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta \varphi dx \\ &= \sum_{i=1,\ j=1} \left[\varphi \cdot \frac{\partial}{\partial x_i \partial x_j} \left(|\Delta u|^{p(x)-2} \frac{\partial u}{\partial x_i \partial x_j} \right) \right] \\ &= (\varphi,\ -\Delta^2_{p(x)} u).\end{aligned}$$

Obviously,

$$f = -\Delta^2_{p(x)} u \quad \text{a.e.} \quad x \in u.$$

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