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CONSTRUCTION OF A FOUR DIMENSIONAL THIRD ORDER ROTATABLE DESIGN THROUGH BALANCED INCOMPLETE BLOCK DESIGNS

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Abstract

The aim of this paper is to construct a new third order rotatable design in four dimensions through balanced incomplete block design. By choosing balanced incomplete block designs in four factors where each block contains three treatments, factorial combinations are obtained. Other point sets each of the same factors are suitably chosen so that their associate combinations are obtained.

A new third order rotatable design in four dimensions is obtained with 120 points through balanced incomplete block designs.

Introduction

In Agriculture, Science and Technology, experiments performed at

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predetermined levels of the controllable factors mean that an experimental design must be selected prior to experimentation. A cyclical group of a certain order in a particular degree of a polynomial forms a rotatable design if it satisfies both moment and non-singularity conditions of the rotatability method or criterion. In agriculture and science, we observe what happens, based on these observations form a theory as to what may be true; test the theory by further observations and by experiments; and watch to see if the predictions based on the theory are fulfilled, while technology is the application of agricultural and scientific knowledge to practical tasks in all of which statistical theory is crucial in the formulation of theories or hypotheses and evolution of predictions. Suppose that an experimenter is interested in determining the relationship between a response and several independent variables. The independent variables may be controlled by the experimenter or observed without control. Suppose, further, that these independent variables represent all the factors that contribute to the response, and that the exact relationship between the response and the independent variables is the response function and, geometrically, it defines a surface called the *response surface*.

Response Surface Methodology (RSM) is a collection of mathematical and statistical techniques useful for analyzing problems where several independent variables influence a dependent variable. The fitting of the response surface can be complex and costly if done haphazardly. To cut on costs, an experimenter has to make a choice of the experimental design prior to experimentation.

The idea of making a design rotatable is a useful one in practice. It enables the experimental information to be obtained equally in all directions, at the same distance from the origin, and in the space of real design variables $x_1, x_2, ..., x_k$.

Third order rotatability assumes that it is desired to fit a cubic polynomial in $x_1, x_2, ..., x_k$ to the available experimental data.

Rotatability is a natural and desirable property, which requires that the variance of a predicted response at a point remains constant at all points that are equidistant from the design center. To achieve stability in prediction variance, this important property of rotatability was evolved and was formerly developed by Box and Hunter [1], assuming that the errors in the observations are uncorrelated. Herzberg [10] presented two other 72 points third order rotatable designs in four dimensions among the designs which contains 72 points listed by Das and Narasimham [7]. Huda [12] constructed three dimensional designs from two dimensional designs from what is known from Box and Hunter [1], and Gardiner et al. [9], that a set of $N' (\geq 7)$ points equally spaced on a circle centered at the origin satisfies the moment requirements of a third order rotatable. Koske et al. [16] extended the method and constructed a third order rotatable design in five dimensions with 320 points while Koske et al. [17] generalized the construction for units of size $s = k - 1 = 2.$ Here units of size $s = k - 1 = 3$ is utilized.

Draper [4] stated that this set of points forms a rotatable arrangement of third order in *k* factors if the following relations hold:

$$\sum_{u=1}^{N} x_{iu}^2 = A \quad (i = 1, 2, ..., k),$$

$$\sum_{u=1}^{N} x_{iu}^4 = 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 3B,$$

$$\sum_{u=1}^{N} x_{iu}^6 = 5\sum_{u=1}^{N} x_{iu}^2 x_{ju}^4 = 15\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 x_{lu}^2 = 15C,$$

$$i \neq j \neq l = 1, 2, ..., k, \quad u = 0, 1, ..., N \tag{1}$$

and all other sums of powers and products up to order six are zero, where $A = N\lambda_2,$ $B = N\lambda_4$ and $C = N\lambda_6.$

The arrangement of points is said to *form a rotatable design of third order* only if it forms a non-singular third order design (if the points give rise to a non-singular matrix).

Gardiner et al. [9] derived the non-singularity conditions as:

$$D: \frac{NB}{A^2} > \frac{k}{(k+2)},$$

$$E: \frac{AC}{B^2} > \frac{(k+2)}{(k+4)}. \tag{2}$$

These are the conditions required for a set of points to form a third order rotatable design. Draper [5] proved that it is impossible for the inequalities of (2) to be reversed. He further showed that in order to get usable third order rotatable designs, at least two spherical sets of points with different positive radii but centered at the origin of the design must be combined (Draper [6]). Readers are advised to read more on construction of rotatable designs in the following: [2, 3, 8, 11, 13-15].

Construction of the Design

Huda [12] considered the doubly balanced incomplete block designs with $t = k$ and $s = k - 1$ and assumed that for the first block $h_i = x$ for $i = 1, \ldots, k-1$ and $h_i = 0$ for $i = k - 1 + 1, \ldots, k$. Then for each block of $\mathrm{BIB}(k, b, r, k-1, \lambda, \mu)$ and each $u\,(u = 1, \ldots, N)$, there is an associated point generated by replacing the $(k-1)$ nonzero entries of $(h_1, \ldots, h_k)$ by $x_{1u}, \ldots, x_{k-1\,u}$ in any order without repeating the x_{iu}'s $(i = 1, \ldots, k-1)$. Let $\mathrm{ABIB}(k, b, r, k-1, \lambda, \mu)$ denote the set of all such points generated from the corresponding block design and the given $(k-1)$ dimensional arrangement. Then by combining $\mathrm{ABIB}(k, b, r, k-1, \lambda, \mu)$ with symmetric point sets or their suitably balanced subsets, it may often be possible to obtained a design satisfying (1) and (2) and containing $(x_{1u}, \ldots, x_{k-1,u}, 0, \ldots, 0)$ $(u = 1, \ldots, N')$.

Following the same method lay down by Huda [12] and extended by Koske et al. [16], here a new design is constructed utilizing units of size $s = k - 1 = 3$.

Let $\mathrm{BIB}(v, b, k-1, s, \lambda, \mu)$ denote the set of all such points generated from the corresponding block design and the given $(k-1)$ dimensional arrangement. Other three point sets with unknown levels are proposed to be taken and denoted by x, y and z excepting that some of these may be zero also and to get a factorial design in four factors, say, out of these unknown levels. Then by combining $\mathrm{BIB}(v, b, r, s, \lambda, \mu)(v = 4)$ with symmetric point sets, it is possible to obtain a design of four dimensions satisfying conditions derived containing $(x_{1u}, x_{2u}, x_{3u}, 0)(u = 1, 2, ..., N)$.

First block $h_i = a\,(i = 1, 2, 3)$ and $h_i = 0\,(i = 4, ..., k)$ is assumed. Then for each block of $\mathrm{BIB}(v, b, k-1, s, \lambda, \mu)$, there is an associated point generated by replacing the $(k-1)$ nonzero entries of $(h_1, ..., h_k)$ by $x_{1u}, ..., x_{k-1,u}$ in any order without replacing the x_{iu}'s being placed in the ascending order of the i's in the first block and to get a factorial design in four factors, say, out of these unknown levels.

The Design

Considering a third order rotatable design in two dimensions consisting of N points equally spaced on a circle of radius ρ, then from Gardiner et al. [9], it is known that for this arrangement

$$A = \frac{N'}{k}\rho^2, \quad B = \frac{N'}{k(k+2)}\rho^4 \quad \text{and} \quad C = \frac{N'}{k(k+2)(k+4)}\rho^6,$$

where A, B and C are as defined in equation (1).

Consider four dimensional point set $\mathrm{BIB}(4, 4, 3, 3, 2)$ generated from this arrangement and combining the points of this set with points of $S(x, x, x, 0)$, $S(y, y, y, y)$ and $S(z, z, z, z)$ with $z^2 = ty^2$, $t \geq 0$. This arrangement forms a third order rotatable design in four dimensions if

$$x^2 = \left(\frac{9}{88}C\right)^{\frac{1}{3}}, \quad y^2 = \left(\frac{21C}{24}\frac{1}{(1+t^3)}\right)^{\frac{1}{3}}.$$

We now consider a design with $k = 3$ and $N' = 15$, then $B = \rho^4$, $C = \frac{1}{7}\rho^6$ and

$$\frac{(1+2t^2)^3}{(1+2t^3)^2} = 3$$

such that $t = 1$.

Combining 64 design points from BIB$(4, 4, 3, 3, 2)$ generated from this arrangement with 56 points from the point sets $S(x, x, x, 0)$, $S(y, y, y, y)$ and $S(z, z, z, z)$, a third order rotatable design in four dimensions with 120 points exists. Using the relations given in (1), the set of points

$$S(x, x, x, 0) + S(y, y, y, y) + S\left(\frac{\rho}{\sqrt{2}}, \frac{\rho}{\sqrt{2}}, \frac{\rho}{\sqrt{2}}, 0\right)$$

$$+ S(z, z, z, z) + S\left(\frac{\rho}{\sqrt{2}}, \frac{\rho}{\sqrt{2}}, 0, 0\right)$$

forms a third order rotatable design with

$$x^2 = 0.2445\rho^2, \quad y^2 = 0.0625\rho^2 \quad \text{and} \quad z^2 = 0.0625\rho^2.$$

Results and Conclusion

During last few decades, rotatable designs using BIBDs have been introduced by a number of workers. This introduces a new method of constructing a third order rotatable design with 120 points.

In situations where for example, the experimenter might be interested in some of the $(k-1)$ subsets of factors, these designs may be desirable since these subsets may be identified with the blocks generating a BIB$(k, b, r, k-1, \lambda)$ so that the four dimensional third order rotatable design contains third order rotatable designs in $(k-1)$ dimensions involving the subsets of factors the experimenter is interested in.

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