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COINCIDENCE POINTS AND FIXED POINT THEOREMS FOR MAPPINGS IN *G*-METRIC SPACES

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Abstract

In this paper, the existence of coincidence points of single-valued and multi-valued mappings in G-metric spaces is proven. Moreover, we also prove the coincidence points and fixed point theorems for single-valued mappings satisfying the contractive conditions concerning the mapping ϕ in G-metric spaces.

1. Introduction

In 2006, Mustafa and Sims [10] introduced a generalization of metric spaces, namely, G-metric spaces. Since then, the fixed point theorems in metric spaces have been extended to G-metric spaces. In 1989, Mizoguchi and Takahashi [8] proved the generalization of Banach contraction principle as the following:

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Theorem 1.1 ([8, Theorem 5]). *Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$ *be a multi-valued mapping satisfying*

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y),$$

for all $x, y \in K$, *where* $\varphi : [0, \infty) \to [0, 1)$ *is a function such that* $\sup_{r \to t^+} \varphi(r) < 1$ *for every* $t \in [0, \infty)$. *Then T has a fixed point in X.*

In this paper, we prove a generalization of Theorem 1.1 ([8, Theorem 5]) in *G*-metric spaces.

The common fixed point theorems for mappings satisfying certain contractive conditions in metric spaces have been continually studied for decade (see [2, 4-7, 13] and references contained therein). In this paper, we obtain the unique common fixed point theorem for a pair of weakly compatible single-valued mappings in *G*-metric spaces. Furthermore, we prove the existence of coincidence points for single-valued mappings satisfying a certain contractive condition and this result is a generalization of Theorem 2.1 [3].

2. Preliminaries

We now recall some of the basic concepts and results in *G*-metric spaces that have been established in [10].

Definition 2.1. Let *X* be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables), and

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a *generalized metric* or more specifically a *G-metric* on X, and the pair (X, G) is called a *G-metric space*.

Since then, the fixed point theory in G-metric spaces has been studied and developed by many authors (see [1, 3, 10-12, 14]).

Definition 2.2. A G-metric is said to be *symmetric* if $G(x, y, y) = G(y, x, x)$, for all $x, y \in X$.

Proposition 2.3. *Every G-metric space* (X, G), *defines a metric space* (X, d_G) *by*

$$d_G(x, y) = G(x, y, y) + G(x, x, y), \text{ for all } x, y \in X.$$

Definition 2.4. Let (X, G) be a G-metric space. Then we say that a sequence $\{x_n\}$ in X is:

(i) a *G-convergent sequence* if, for any $\varepsilon > 0$, there exist $x \in X$ and $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$;

(ii) a *G-Cauchy sequence* if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$.

Theorem 2.5. *Let* (X, G) *be a G-metric space and* $\{x_n\}$ *be a sequence in X. Then the following are equivalent*:

(i) $\{x_n\}$ *is G-convergent to x*,

(ii) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$,

(iii) $G(x_n, x, x) \to 0$ *as* $n \to \infty$,

(iv) $G(x_m, x_n, x) \to 0$ *as* $m, n \to \infty$.

Theorem 2.6. *Let* (X, G) *be a G-metric space and* $\{x_n\}$ *be a sequence in X. Then the following are equivalent*:

(i) $\{x_n\}$ *is G-Cauchy*.

(ii) *For every* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* $G(x_n, x_m, x_m) < \varepsilon$, *for all* $n, m \geq N$.

(iii) $\{x_n\}$ *is a Cauchy sequence in the metric space* (X, d_G).

A *G*-metric space *X* is said to be *complete* if every *G*-Cauchy sequence in *X* is a *G*-convergent sequence in *X*.

Proposition 2.7. *Let* (X, G) *be a G-metric space. Then the function* $G(x, y, z)$ *is jointly continuous in all three of its variables*.

Definition 2.8. Let *f* and *g* be single-valued self mappings on a set *X*. If $w = fx = gx$ for some $x \in X$, then *x* is called a *coincidence point* of *f* and *g*, and *w* is called a *point of coincidence* of *f* and *g*.

Abbas and Rhoades [1] proved the unique common fixed point for a pair of weakly compatible mappings by using the following key proposition.

Proposition 2.9 ([1, Proposition 1.5])**.** *Let f and g be weakly compatible self mappings on a set X. If f and g have a unique point of coincidence* $w = fx = gx$, *then w is the unique common fixed point of f and g*.

Let *X* be a *G*-metric space. We shall denote $CB(X)$ the family of all nonempty *G*-closed bounded subsets of *X*. Let $H_G(\cdot, \cdot, \cdot)$ be the Hausdorff *G*-distance on $CB(X)$, i.e.,

$$H_G(A, B, C) = \max\{\sup_{x \in A} G(x, B, C), \sup_{x \in B} G(x, C, A), \sup_{x \in C} G(x, A, B)\},$$

where

$$G(x, B, C) = d_G(x, B) + d_G(B, C) + d_G(x, C),$$

$$d_G(x, B) = \inf\{d_G(x, y) : y \in B\},$$

$$d_G(A, B) = \inf\{d_G(a, b) : a \in A, b \in B\}.$$

A mapping $T : X \to 2^X$ is called a *multi-valued mapping*. A point $x \in X$ is called a *fixed point* of T if $x \in Tx$.

3. Coincidence Points and Fixed Points

We first prove the following lemma.

Lemma 3.1. *Let (X, G) be a G-metric space and let $A, B \in CB(X)$. Suppose that $\varepsilon > 0$ and $H_G(A, B, B) < \varepsilon$. Then for each $a \in A$, there exists $b \in B$ such that $G(a, b, b) < \varepsilon$.*

Proof. Suppose that there exists $a \in A$ such that

$$G(a, b, b) \geq \varepsilon, \text{ for all } b \in B.$$

Therefore,

$$\varepsilon \leq G(a, b, b) \leq G(a, b, b) + G(a, a, b) = d_G(a, b), \text{ for all } b \in B.$$

It follows that

$$\varepsilon \leq d_G(a, B) \leq G(a, B, B) \leq H_G(A, B, B),$$

which contradicts to the assumption. This completes the proof. □

Theorem 3.2. *Let (X, G) be a G-metric space. Suppose that $T : X \to CB(X)$ is a multi-valued mapping and $f : X \to X$ is a single-valued mapping satisfying*

(i) $T(X) \subseteq f(X)$,

(ii) *$f(X)$ is complete,*

(iii) *there exists a function $\varphi : [0, \infty) \to [0, 1)$ such that*

$$\limsup_{r \to t^+} \varphi(r) < 1, \textit{ for all } t \in [0, \infty), \tag{1}$$

and for all $x, y, z \in X$,

$$H_G(Tx, Ty, Tz) \leq \varphi(G(fx, fy, fz))G(fx, fy, fz). \tag{2}$$

Then T and f have a coincidence point in X. That is, there exists $p \in X$ such that $fp \in Tp$.

Proof. Let x_0 be an arbitrary element in X. Since $T(X) \subseteq f(X)$, there exists $x_1 \in X$ such that $fx_1 \in Tx_0$. Define a function $\omega : [0, \infty) \to [0, 1)$ by $\omega(t) = \frac{\varphi(t) + 1}{2}$, for all $t \in [0, \infty)$. Thus we can see that

$$\limsup_{r \to t^+} \omega(r) < 1, \ \varphi(t) < \omega(t), \text{ and } 0 < \omega(t) < 1, \text{ for all } t \in [0, \infty).$$

By the definition of Hausdorff G-distance and (2), we have

$$\begin{aligned} G(fx_1, Tx_1, Tx_1) &\leq H_G(Tx_0, Tx_1, Tx_1) \\ &\leq \varphi(G(fx_0, fx_1, fx_1))G(fx_0, fx_1, fx_1) \\ &< \omega(G(fx_0, fx_1, fx_1))G(fx_0, fx_1, fx_1). \end{aligned}$$

If $fx_0 = fx_1$, then f and T have a coincidence point. Assume that $fx_0 \neq fx_1$. Therefore, by Lemma 3.1, there exists $x_2 \in X$ such that

$$fx_2 \in Tx_1 \text{ and } G(fx_1, fx_2, fx_2) < \omega(G(fx_0, fx_1, fx_1))G(fx_0, fx_1, fx_1).$$

Again, by the definition of Hausdorff G-distance and (2), we obtain that

$$\begin{aligned} G(fx_2, Tx_2, Tx_2) &\leq H_G(Tx_1, Tx_2, Tx_2) \\ &\leq \varphi(G(fx_1, fx_2, fx_2))G(fx_1, fx_2, fx_2) \\ &< \omega(G(fx_1, fx_2, fx_2))G(fx_1, fx_2, fx_2). \end{aligned}$$

If $fx_1 = fx_2$, then f and T have a coincidence point. Assume that $fx_1 \neq fx_2$. Therefore, by Lemma 3.1, there exists $x_3 \in X$ such that

$$fx_3 \in Tx_2 \text{ and } G(fx_2, fx_3, fx_3) < \omega(G(fx_1, fx_2, fx_2))G(fx_1, fx_2, fx_2).$$

Continuing this process, we can construct a sequence $\{fx_n\}$ such that $fx_{n+1} \in Tx_n$ and

$$\begin{aligned} G(fx_{n+1}, fx_{n+2}, fx_{n+2}) &< \omega(G(fx_n, fx_{n+1}, fx_{n+1}))G(fx_n, fx_{n+1}, fx_{n+1}) \\ &< G(fx_n, fx_{n+1}, fx_{n+1}). \end{aligned}$$

From the above argument, we can conclude that the sequence

$\{G(fx_n, fx_{n+1}, fx_{n+1})\}$ is a nonincreasing sequence in $[0, \infty)$. This implies that $\{G(fx_n, fx_{n+1}, fx_{n+1})\}$ is convergent. Since $\limsup_{r \to t^+} \omega(r) < 1$, we obtain that

$$\limsup_{n\to\infty} \omega(G(fx_n, fx_{n+1}, fx_{n+1})) = s \text{ for some } s \in [0, 1).$$

Therefore, for each $k \in (s, 1)$, there exists $N \in \mathbb{N}$ such that

$$\omega(G(fx_{n-1}, fx_n, fx_n)) < k, \text{ for all } n \geq N.$$

For each $n \geq N$, we have

$$\begin{aligned} G(fx_n, fx_{n+1}, fx_{n+1}) &< \omega(G(fx_{n-1}, fx_n, fx_n))G(fx_{n-1}, fx_n, fx_n) \\ &< kG(fx_{n-1}, fx_n, fx_n). \end{aligned}$$

Thus, for each $m > n \geq N$, we obtain that

$$\begin{aligned} G(fx_n, fx_m, fx_m) &\leq G(fx_n, fx_{n+1}, fx_{n+1}) + \cdots + G(fx_{m-1}, fx_m, fx_m) \\ &\leq (k^{n-N} + \cdots + k^{m-N-1})G(fx_N, fx_{N+1}, fx_{N+1}) \\ &\leq \frac{k^{n-N}}{1-k} G(fx_N, fx_{N+1}, fx_{N+1}). \end{aligned}$$

Taking the limit of both sides, we get that $G(fx_n, fx_m, fx_m) \to 0$ as $m, n \to \infty$. It follows that $\{fx_n\}$ is a G-Cauchy sequence. By the completeness of $f(X)$, we have $\{fx_n\}$ is G-convergent to some $q \in X$. Therefore, there exists $p \in X$ such that $fp = q$. By using (2), we obtain that

$$\begin{aligned} G(fx_{n+1}, Tp, Tp) &\leq H_G(Tx_n, Tp, Tp) \\ &\leq \varphi(G(fx_n, fp, fp))G(fx_n, fp, fp) \\ &< G(fx_n, fp, fp). \end{aligned}$$

Taking the limit of both sides as $n \to \infty$, we have $G(fp, Tp, Tp) = 0$ and hence $fp \in Tp$. □

Corollary 3.3. *Let* (X, G) *be a G-metric space. Suppose that* $T : X \to CB(X)$ *is a multi-valued mapping and* $f : X \to X$ *is a single-valued mapping satisfying*

(i) $T(X) \subseteq f(X)$,

(ii) $f(X)$ *is complete,*

(iii) $H_G(Tx, Ty, Tz) \leq kG(fx, fy, gz)$, *for all* $x, y, z \in X$, *where* $0 \leq k < 1$.

Then T and f have a coincidence point in X. That is, there exists $p \in X$ *such that* $fp \in Tp$.

Proof. Define $\varphi : [0, \infty) \to [0, 1)$ by $\varphi(s) = k$, for all $s \in [0, \infty)$. Therefore, (1) and (2) in Theorem 3.2 are now satisfied. This completes the proof. □

By setting f in Theorem 3.2 to be the identity function on X, we immediately have the following corollary:

Corollary 3.4. *Let* (X, G) *be a complete G-metric space and* $T : X \to CB(X)$ *be a multi-valued mapping satisfying*

$$H_G(Tx, Ty, Tz) \leq \varphi(G, (x, y, z))G(x, y, z),$$

for all $x, y, z \in X$, *where* $\varphi : [0, \infty) \to [0, 1)$ *is a function such that* $\limsup_{r \to t^+} \varphi(r) < 1$, *for all* $t \in [0, \infty)$. *Then T has a fixed point in X.*

Theorem 3.5. *Let* (X, G) *be a G-metric space. Suppose that* $f, g : X \to X$ *are single-valued mappings satisfying*

(i) $f(X) \subseteq g(X)$,

(ii) $g(X)$ *is complete,*

(iii) *there exists a function* $\varphi : [0, \infty) \to [0, 1)$ *such that*

$$\limsup_{r \to t^+} \varphi(r) < 1, \text{ for all } t \in [0, \infty), \tag{3}$$

and for all $x, y, z \in X$,

$$G(fx, fy, fz) \leq \varphi(G(gx, gy, gz))G(gx, gy, gz). \tag{4}$$

Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. By applying Theorem 3.2, we obtain that *f* and *g* have a point of coincidence in *X*, say *p*. We now prove that *f* and *g* have a unique point of coincidence. Suppose that $gq = fq$ for some $q \in X$. By applying (4), we get that

$$\begin{aligned} G(gp, gq, gq) &= G(fp, fq, fq) \\ &\leq \varphi(G(gp, gq, gq))G(gp, gq, gq). \end{aligned}$$

This implies that $G(gp, gq, gq) = 0$ and hence $gp = gq$. Therefore, *f* and *g* have a unique point of coincidence. By Proposition 2.9, we obtain that *f* and *g* have a unique common fixed point. □

Corollary 3.6. *Let* (X, G) *be a G-metric space. Suppose that* $f, g : X \to X$ *are single-valued mappings satisfying*

$$G(fx, fy, fz) \leq kG(gx, gy, gz), \tag{5}$$

for all $x, y, z \in X$, *where* $0 \leq k < 1$. *Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.*

From now on, let $\phi : [0, \infty) \to [0, \infty)$ be a nondecreasing function satisfying

(ϕ_1) $\phi(0) = 0$,

(ϕ_2) $\phi(t) < t$, for all $t \in (0, \infty)$,

(ϕ_3) $\sum_{n=1}^{\infty} \phi^n(t) < \infty$, for all $t \in (0, \infty)$.

We next prove the existence of coincidence points of two single-valued mappings concerning the mappings ϕ which is mentioned as above.

Theorem 3.7. *Let (X, G) be a G-metric space. Suppose that the mappings $f, g : X \to X$ satisfy*

$$\begin{aligned} & G(fx, fy, fz) \\ & \leq \phi\Big(\max\Big\{G(gx, gy, gz), G(gx, fx, fx), G(gy, fy, fy), G(gz, fz, fz), \\ & \quad \frac{[G(gx, fy, fy) + G(gz, fx, fx)]}{2}, \frac{[G(gx, fy, fy) + G(gy, fx, fx)]}{2} \\ & \quad \frac{[G(gy, fz, fz) + G(gz, fy, fy)]}{2}, \frac{[G(gx, fz, fz) + G(gz, fx, fx)]}{2}\Big\}\Big), \qquad (6) \end{aligned}$$

for all $x, y, z \in X$. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a coincidence point in X. That is, there exists $p \in X$ such that $fp = gp$.

Proof. Let x_0 be an arbitrary element in X. Since $f(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $gx_1 = fx_0$. Let $a \in \mathbb{R}$ be such that $\phi(G(gx_0, gx_1, gx_1)) \leq \phi(a)$. Again, since $f(X) \subseteq g(X)$, there exists $x_2 \in X$ such that $gx_2 = fx_1$. By (6), we have

$$\begin{aligned} & G(gx_1, gx_2, gx_2) \\ & = G(fx_0, fx_1, fx_1) \\ & \leq \phi\Big(\max\Big\{G(gx_0, gx_1, gx_1), G(gx_0, fx_0, fx_0), G(gx_1, fx_1, fx_1), \\ & \quad G(gx_1, fx_1, fx_1), \frac{[G(gx_0, fx_1, fx_1) + G(gx_1, fx_0, fx_0)]}{2}, \\ & \quad \frac{[G(gx_0, fx_1, fx_1) + G(gx_1, fx_0, fx_0)]}{2}, \\ & \quad \frac{[G(gx_1, fx_1, fx_1) + G(gx_1, fx_1, fx_1)]}{2}, \end{aligned}$$

$$\left.\left.\frac{[G(gx_0, fx_1, fx_1) + G(gx_1, fx_0, fx_0)]}{2}\right\}\right)$$

$$\leq \phi\Bigg(\max\Bigg\{G(gx_0, gx_1, gx_1), G(gx_0, gx_1, gx_1), G(gx_1, gx_2, gx_2),$$

$$G(gx_1, gx_2, gx_2), \frac{[G(gx_0, fx_2, fx_2) + G(gx_1, gx_1, gx_1)]}{2},$$

$$\frac{[G(gx_0, gx_2, gx_2) + G(gx_1, gx_1, gx_1)]}{2},$$

$$\frac{[G(gx_1, gx_2, gx_2) + G(gx_1, gx_2, gx_2)]}{2},$$

$$\left.\left.\frac{[G(gx_0, gx_2, gx_2) + G(gx_1, gx_1, gx_1)]}{2}\right\}\right)$$

$$\leq \phi\left(\max\left\{G(gx_0, gx_1, gx_1), G(gx_1, gx_2, gx_2), \frac{G(gx_0, gx_2, gx_2)}{2}\right\}\right)$$

$$\leq \phi\Bigg(\max\Bigg\{G(gx_0, gx_1, gx_1), G(gx_1, gx_2, gx_2),$$

$$\left.\left.\frac{G(gx_0, gx_1, gx_1) + G(gx_1, gx_2, gx_2)}{2}\right\}\right)$$

$$\leq \phi(\max\{G(gx_0, gx_1, gx_1), G(gx_1, gx_2, gx_2)\}).$$

If $G(gx_0, gx_1, gx_1) \leq G(gx_1, gx_2, gx_2)$, then

$$G(gx_1, gx_2, gx_2) \leq \phi(G(gx_1, gx_2, gx_2)).$$

This implies that $G(gx_1, gx_2, gx_2) = 0$ and thus $gx_1 = fx_1$. Therefore, f and g have a coincidence point.

Suppose that $G(gx_1, gx_2, gx_2) \leq G(gx_0, gx_1, gx_1)$. Thus

$$G(gx_1, gx_2, gx_2) \leq \phi(G(gx_0, gx_1, gx_1)) \leq \phi(a).$$

Since the range of *g* contains the range of *f*, we can choose $x_3 \in X$ such that $gx_3 = fx_2$. By (6), we obtain that

$$\begin{aligned}
&G(gx_2, gx_3, gx_3) \\
&= G(fx_1, fx_2, fx_2) \\
&\leq \phi\Big(\max\Big\{G(gx_1, gx_2, gx_2), G(gx_1, fx_1, fx_1), G(gx_2, fx_2, fx_2), \\
&\quad G(gx_2, fx_2, fx_2), \frac{[G(gx_1, fx_2, fx_2) + G(gx_2, fx_1, fx_1)]}{2}, \\
&\quad \frac{[G(gx_1, fx_2, fx_2) + G(gx_2, fx_1, fx_1)]}{2}, \\
&\quad \frac{[G(gx_2, fx_2, fx_2) + G(gx_2, fx_2, fx_2)]}{2}, \\
&\quad \frac{[G(gx_1, fx_2, fx_2) + G(gx_2, fx_1, fx_1)]}{2}\Big\}\Big) \\
&\leq \phi\Big(\max\Big\{G(gx_1, gx_2, gx_2), G(gx_2, gx_3, gx_3), \frac{G(gx_1, gx_3, gx_3)}{2}\Big\}\Big) \\
&\leq \phi(\max\{G(gx_1, gx_2, gx_2), G(gx_2, gx_3, gx_3)\}) \\
&\leq \phi(G(gx_1, gx_2, gx_2)) \\
&\leq \phi^2(a).
\end{aligned}$$

By continuing this process, we can construct a sequence $\{gx_n\}$ such that

$$gx_{n+1} = fx_n \text{ and } G(gx_n, gx_{n+1}, gx_{n+1}) \leq \phi^n(a) \text{ for each } n.$$

We will prove that $\{gx_n\}$ is a *G*-Cauchy sequence. Since

$$\begin{aligned}
&d_G(gx_n, gx_{n+1}) \\
&= G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_n, gx_n, gx_{n+1})
\end{aligned}$$

$$\leq G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_n, gx_{n+1})$$

$$= 3G(gx_n, gx_{n+1}, gx_{n+1})$$

$$\leq 3\phi^n(a),$$

we obtain that

$$\sum_{n=0}^{\infty} d_G(x_n, x_{n+1}) \leq \sum_{n=0}^{\infty} 3\phi^n(a) < \infty.$$

This implies that $\{gx_n\}$ is a Cauchy sequence in (X, d_G). Using Theorem 2.6, we have $\{gx_n\}$ is a G-Cauchy sequence. By the completeness of $g(X)$, we have $\{gx_n\}$ is G-convergent to some $q \in X$. Therefore, there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. By using (6), we obtain that

$$\begin{aligned}
&G(gx_{n+1}, fp, fp) \\
&= G(fx_n, fp, fp) \\
&\leq \phi\Bigg(\max\Bigg\{G(gx_n, gp, gp), G(gx_n, gx_{n+1}, gx_{n+1}), G(gp, fp, fp), \\
&\quad G(gp, fp, fp), \frac{[G(gx_n, fp, fp) + G(gp, gx_{n+1}, gx_{n+1})]}{2}, \\
&\quad \frac{[G(gx_n, fp, fp) + G(gp, gx_{n+1}, gx_{n+1})]}{2}, \\
&\quad \frac{[G(gp, fp, fp) + G(gp, fp, fp)]}{2}, \\
&\quad \frac{[G(gx_n, fp, fp) + G(gp, gx_{n+1}, gx_{n+1})]}{2}\Bigg\}\Bigg).
\end{aligned}$$

Letting $n \to \infty$, we have

$$G(gp, fp, fp) \leq \phi(G(gp, fp, fp)).$$

This implies that $G(gp, fp, fp) = 0$ and so $gp = fp$. □

If we take g in Theorem 3.7 to be the identity on X, then we have the following results:

Corollary 3.8. *Let* (X, G) *be a complete G-metric space. Suppose that the mapping* $f: X \to X$ *satisfies*

$$\begin{aligned} &G(fx, fy, fz) \\ &\leq \phi\Bigg(\max\Bigg\{G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz) \\ &\qquad \frac{[G(x, fy, fy) + G(z, fx, fx)]}{2}, \frac{[G(x, fy, fy) + G(y, fx, fx)]}{2}, \\ &\qquad \frac{[G(y, fz, fz) + G(z, fy, fy)]}{2}, \frac{[G(x, fz, fz) + G(z, fx, fx)]}{2}\Bigg\}\Bigg), \end{aligned} \tag{7}$$

for all $x, y, z \in X$*. Then f has a fixed point in X.*

Corollary 3.9 ([3, Theorem 2.1])**.** *Let* (X, G) *be a complete G-metric space. Suppose that the mapping* $f: X \to X$ *satisfies*

$$\begin{aligned} &G(fx, fy, fz) \\ &\leq k \max\Bigg\{G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz) \\ &\qquad \frac{[G(x, fy, fy) + G(z, fx, fx)]}{2}, \frac{[G(x, fy, fy) + G(y, fx, fx)]}{2}, \\ &\qquad \frac{[G(y, fz, fz) + G(z, fy, fy)]}{2}, \frac{[G(x, fz, fz) + G(z, fx, fx)]}{2}\Bigg\}, \end{aligned} \tag{8}$$

for all $x, y, z \in X$*. Then f has a unique fixed point in X.*

Proof. Define $\phi: [0, \infty) \to [0, \infty)$ by $\phi(t) = kt$, for all $t \in [0, \infty)$. Therefore, ϕ is a nondecreasing function, $\phi(0) = 0$, $\phi(t) < t$ and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$, for all $t \in (0, \infty)$. It follows that the contractive condition (7) in Corollary 3.8 is satisfied. Therefore, f has a fixed point in X. For proving the uniqueness of fixed point of f, see [3, Theorem 2.1]. □

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References

[1] M. Abbas and B. E. Rhoades, Common fixed point results for noncommuting mappings without continuity in generalized metric spaces, Appl. Math. Comput. 215 (2009), 262-269.

[2] I. Beg and M. Abbas, Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition, Fixed Point Theory Appl. 2006 (2006) (Article ID 74503, 7 pages).

[3] R. Chugh, T. Kadian, A. Rani and B. E. Rhoades, Property *P* in *G*-metric spaces, Fixed Point Theory Appl. 2010 (2010) (Ariticle ID 401684, 12 pages).

[4] G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Sci. 9(4) (1986), 771-779.

[5] G. Jungck, Common fixed points for commuting and compatible maps on compacta, Proc. Amer. Math. Soc. 103 (1988), 977-983.

[6] G. Jungck, Common fixed points for noncontinuous nonself maps on non-metric spaces, Far East J. Math. Sci. 4 (1996), 199-215.

[7] G. Jungck and N. Hussain, Compatible maps and invariant approximations, J. Math. Anal. Appl. 325(2) (2007), 1003-1012.

[8] N. Mizoguchi and W. Takahashi, Fixed point theorems for multi-valued mappings on complete metric spaces, J. Math. Anal. Appl. 141 (1989), 177-188.

[9] Z. Mustafa and B. Sims, Some remarks concerning *D*-metric spaces, Proceedings of the International Conference on Fixed Point Theory and Applications, Valencia (Spain), 2003, pp. 189-198.

[10] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7(2) (2006), 289-297.

[11] Z. Mustafa, H. Obiedat and F. Awawdeh, Some common fixed point theorems for mappings on complete *G*-metric spaces, Fixed Point Theory Appl. 2008 (2008) (Article ID 189870, 12 pages).

[12] Z. Mustafa and B. Sims, Fixed point theorems for contractive mappings in complete *G*-metric spaces, Fixed Point Theory Appl. 2009 (2009) (Article ID 917175, 10 pages).

[13] R. P. Pant, Common fixed points of noncommuting mappings, J. Math. Anal. Appl. 188 (1994), 436-440.

[14] W. Shatanawi, Fixed point theory for contractive mapping satisfying Φ-maps in *G*-metric spaces, Fixed Point Theory Appl. 2010 (2010) (Article ID 181650, 9 pages).