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ON THE CONDITIONS OF OSCILLATION OF FIRST ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract

We discuss the oscillations of first order nonlinear functional differential equations and obtain the conditions that ensure all solutions oscillate. These conditions are necessary and sufficient when the coefficient function reduces to a constant.

1. Introduction

Oscillatory properties of first order linear differential equations of neutral type are studied in [1-6]. In this paper, we discuss a nonlinear equation of neutral type,

$$\frac{d}{dt}[x(t) - p(t)x(t-r)] + q(t)\prod_{i=1}^{n} x^{\alpha_i}(t-\sigma_i) = 0, \tag{1}$$

where $p, q \in C([t_0, +\infty), R)$, $r, \sigma_i \in (0, +\infty)$, $\alpha_i \geq 0$, and $\sum_{i=1}^{n} \alpha_i = 1$, $i = 1, 2, ..., n$.

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Denote $m = \max\{r, \sigma_i, i = 1, 2, ..., n\}$, and $p(t)$ is not identically zero on any closed subinterval of the interval $[t_0, +\infty)$. If $n = 1$, then (1) reduces to

$$\frac{d}{dt}[x(t) - p(t)x(t-r)] + q(t)x(t-\sigma) = 0. \tag{2}$$

As customary, a solution of (1) is called *oscillatory* if it has arbitrarily large zeros. Equation (1) is said to be *oscillatory* if all its solutions are oscillatory.

2. Main Result

In order to obtain main result, we need the following lemmas:

Lemma 1. *Assume q is positive and q is bounded and nonnegative, and there exists a* $t^* \geq t_0$ *such that*

$$p(t^* + nr) \leq 1, \quad n = 0, 1, 2, \ldots. \tag{3}$$

Let $x(t)$ *be an eventually positive solution of equation* (1) *and set*

$$z(t) = x(t) - p(t)x(t-r). \tag{4}$$

Then eventually $z(t) > 0$ *and* $z'(t) < 0.$

Proof. It follows from (1) that $z'(t) < 0$ eventually. It remains to show that $z(t) > 0$ eventually. Otherwise, $z(t)$ is eventually negative. Thus, there exists a sufficiently large T such that $z(t) < -d < 0$ for $t \geq T$, where d is a positive constant. Hence

$$x(t) \leq -d + p(t)x(t-r) \text{ for } t \geq T.$$

In particular, $x[t^* + (n+N)r] \leq -nd + x[t^* + (N-1)r]$, $n = 1, 2,$ if $t^* + Nr \geq T$, hence $x(t)$ cannot be eventually positive. This contradiction proves the lemma.

Lemma 2. *Assume*

$$\sigma > 0, \quad q \in C([t_0, +\infty), (0, +\infty)), \quad \lambda \in C([t_0 - \sigma, +\infty), (0, +\infty))$$

satisfying

$$\liminf_{t \to +\infty} \int_{t-\sigma}^{t} q(s)ds > 0 \tag{5}$$

and

$$\lambda(t) \geq q(t)\exp\left(\int_{t-\sigma}^{t} \lambda(s)ds\right), \quad t \geq t_0. \tag{6}$$

Then

$$\liminf_{t \to +\infty} \int_{t-\sigma}^{t} \lambda(s)ds < +\infty. \tag{7}$$

Proof. Define $Q(t) = \int_{t_0}^{t} q(s)ds$, $t \geq t_0$, (5) implies that $\lim_{t \to +\infty} Q(t) = +\infty$, and $Q(t)$ is strictly increasing. Then $Q^{-1}(t)$ is well defined, strictly increasing, and $\lim_{t \to +\infty} Q^{-1}(t) = +\infty$, (5) implies that there exist $c > 0$ and $T_1 \geq t_0$ such that $Q(t) - Q(t - \sigma) \geq \frac{c}{2}$ for $t \geq T_1$ and thus

$$Q^{-1}\left(Q(t) - \frac{c}{2}\right) \geq t - \sigma, \quad t \geq T_1.$$

Set $\Phi(t) = \exp\left(-\int_{T}^{t} \lambda(s)ds\right)$, then

(6) implies that $\Phi'(t) \leq -q(t)\Phi(t - \sigma)$, $t \geq t_0$,

[2] (5) implies the (7) is true.

We are now ready to prove the following result:

Theorem. *In addition to the assumptions of Lemma* 1, *assume* (5) *holds,* *and either*

$$\liminf_{t\to+\infty}\left\{\inf_{\lambda>0}\left[\prod_{i=1}^{n} p^{\alpha_i}(t-\sigma_i)\frac{q(t)}{q(t-r)}e^{\lambda r} + \frac{q(t)}{\lambda}\exp\left(\lambda\sum_{i=1}^{n}\alpha_i\sigma_i\right)\right]\right\} > 0 \quad (8)$$

or

$$\liminf_{t\to+\infty}\left\{\inf_{\lambda>0}\left[\prod_{i=1}^{n} p^{\alpha_i}(t-\sigma_i)\exp\left(\lambda\int_{t-r}^{t} q(s)ds\right)\right.\right.$$

$$\left.\left. + \frac{1}{\lambda}\exp\left(\lambda\sum_{i=1}^{n}\alpha_i\int_{t-\sigma_i}^{t} q(s)ds\right)\right]\right\} > 1. \quad (9)$$

Then every solution of (1) *is oscillatory*.

Proof. First, we assume (8) holds. Without loss of generality, assume that equation (1) has an eventually positive solution $x(t)$. Let $x(t) > 0$, $x(t-m) > 0$, for $t \geq T_1 \geq t_0$. Then by Lemma 1, $z(t) > 0$, $z'(t) < 0$ for $t \geq T_1$, where $z(t)$ is defined by (4). From (1), we have

$$z'(t) = -q(t)\prod_{i=1}^{n} x^{\alpha_i}(t-\sigma_i)$$

$$= -q(t)\prod_{i=1}^{n}[z(t-\sigma_i) + p(t-\sigma_i)x(t-\sigma_i-r)]^{\alpha_i}$$

$$\leq -q(t)\left[\prod_{i=1}^{n} z^{\alpha_i}(t-\sigma_i) + \prod_{i=1}^{n} p^{\alpha_i}(t-\sigma_i)\prod_{i=1}^{n} x^{\alpha_i}(t-\sigma_i-r)\right]$$

$$= -q(t)\prod_{i=1}^{n} z^{\alpha_i}(t-\sigma_i) + \frac{q(t)}{q(t-r)}\prod_{i=1}^{n} p^{\alpha_i}(t-\sigma_i)z'(t-r). \quad (10)$$

Set $\lambda(t) = -\frac{z'(t)}{z(t)}$, then (10) reduces to

$$\lambda(t) \geq \lambda(t-r)\frac{q(t)}{q(t-r)}\prod_{i=1}^{n} p^{\alpha_i}(t-\sigma_i)\exp\left(\int_{t-\tau}^{t}\lambda(s)ds\right)$$

$$+ q(t)\exp\left(\sum_{i=1}^{n}\alpha_i\int_{t-\sigma_i}^{t}\lambda(s)ds\right). \tag{11}$$

It is obvious that $\lambda(t) > 0$ for $t \geq T_1$. From (11), we have

$$\lambda(t) \geq q(t)\exp\left(\overline{\alpha}\int_{t-\sigma_*}^{t}\lambda(s)ds\right),$$

where $\sigma_* = \min_{1\leq i\leq n}\{\sigma_i\}$, $\overline{\alpha} = \max_{1\leq i\leq n}\{\alpha_i\}$. In view of Lemma 2, we have

$$\liminf_{t\to+\infty}\int_{t-\sigma_*}^{t}\lambda(s)ds < +\infty$$

which implies that $\lim_{t\to+\infty}\inf\lambda(t) < +\infty$. Now we show that $\lim_{t\to+\infty}\inf\lambda(t) > 0$.

In fact, if $\lim_{t\to+\infty}\inf\lambda(t) = 0$, then there exists a sequence $\{t_n\}$ such that $t_n \geq T_1$, $\lim_{n\to\infty} t_n = +\infty$ and $\lambda(t_n) \leq \lambda(t)$, for $t \in [T_1, t_n]$. From (11), we have

$$\lambda(t_n) \geq \lambda(t_n)\frac{q(t_n)}{q(t_n-r)}\prod_{i=1}^{n} p^{\alpha_i}(t_n-\sigma_i)\exp(\lambda(t_n)r)$$

$$+ q(t_n)\exp\left(\lambda(t_n)\sum_{i=1}^{n}\alpha_i\sigma_i\right).$$

Hence

$$\frac{q(t_n)}{q(t_n-r)}\prod_{i=1}^{n} p^{\alpha_i}(t_n-\sigma_i)\exp(\lambda(t_n)r) + \frac{1}{\lambda(t_n)}q(t_n)\exp\left(\lambda(t_n)\sum_{i=1}^{n}\alpha_i\sigma_i\right) \leq 1$$

which contradicts (8), and therefore

$$0 < \liminf_{t\to+\infty}\lambda(t) = h < +\infty. \tag{12}$$

From (8), there exists an $\alpha \in (0, 1)$ such that

$$\alpha \liminf_{t\to+\infty}\left\{\inf_{\lambda>0}\left[\prod_{i=1}^{n} p^{\alpha_i}(t-\sigma_i)\frac{q(t)}{q(t-r)}e^{\lambda r} + \frac{q(t)}{\lambda}\exp\left(\lambda\sum_{i=1}^{n}\alpha_i\sigma_i\right)\right]\right\} > 1. \tag{13}$$

In view of (12), we assume that

$$\lambda(t) > \alpha h, \quad t \ge T_2. \tag{14}$$

Substituting (14) into (11), we obtain

$$\lambda(t) > \alpha h\frac{q(t)}{q(t-r)}\prod_{i=1}^{n} p^{\alpha_i}(t-\sigma_i)\exp(\alpha hr) + q(t)\exp\left(\alpha h\sum_{i=1}^{n}\alpha_i\sigma_i\right)$$

for $t \ge T_2 + m.$ Hence

$$h \ge \liminf_{t\to+\infty}\left\{\alpha h\frac{q(t)}{q(t-r)}\prod_{i=1}^{n} p^{\alpha}(t-\sigma_i)\exp(\alpha hr) + q(t)\exp\left(\alpha h\sum_{i=1}^{n}\alpha_i\sigma_i\right)\right\}.$$

Set $\lambda^* = \alpha h,$ then

$$\lambda^* \ge \alpha\liminf_{t\to+\infty}\left\{\lambda^*\frac{q(t)}{q(t-r)}\prod_{i=1}^{n} p^{\alpha_i}(t-\sigma_i)\exp(\lambda^* r) + q(t)\exp\left(\lambda^*\sum_{i=1}^{n}\alpha_i\sigma_i\right)\right\}$$

which contradicts (13) and completes the proof of this theorem under condition (8).

If (9) holds, we let $\lambda(t)q(t) = -\frac{z'(t)}{z(t)}.$ Then (10) becomes

$$\lambda(t) \ge \lambda(t-r)\prod_{i=1}^{n} p^{\alpha_i}(t-\sigma_i)\exp\left(\int_{t-r}^{t}\lambda(s)q(s)ds\right)$$

$$+ \exp\left(\sum_{i=1}^{n}\alpha_i\int_{t-\sigma_i}^{t}\lambda(s)q(s)ds\right). \tag{15}$$

By Lemma 2, we know that

$$\liminf_{t\to+\infty} \int_{t-\sigma}^{t} \lambda(s) q(s) ds < +\infty. \tag{16}$$

From (5) and (16), we conclude that $\liminf_{t\to+\infty} \lambda(t) < +\infty$. From (15),

$$\lambda(t) \geq 1, \text{ so } 0 < \liminf_{t\to+\infty} \lambda(t) = h < +\infty.$$

From (9), there exists an $\alpha \in (0, 1)$ such that

$$\alpha \liminf_{t\to+\infty} \left\{ \inf_{\lambda>0} \left[\prod_{i=1}^{n} p^{\alpha_i}(t-\sigma_i) \exp\left(\lambda \int_{t-r}^{t} q(s) ds \right) \right.\right.$$
$$\left.\left. + \frac{1}{\lambda} \exp\left(\lambda \sum_{i=1}^{n} \alpha_i \int_{t-\sigma_i}^{t} q(s) ds \right) \right] \right\} > 1.$$

By a similar argument to the first part of the proof, we reach a contradiction.

Corollary. *In addition to the assumptions of Lemma* 1 *assume* (5) *holds and*

$$\liminf_{t\to+\infty} \left\{ \inf_{\lambda>0} \left[\prod_{i=1}^{n} p(t-\sigma) \frac{q(t)}{q(t-r)} e^{r} + \frac{1}{\lambda} q(t) e^{\lambda\sigma} \right] \right\} > 1. \tag{17}$$

Then every solution of (2) *is oscillatory*.

Example. Consider

$$\frac{d}{dt}[x(t) - (2 + \sin t) x(t-\pi)] + (3 + 2\sin t) x(t-\pi) = 0, \quad t \geq \pi. \tag{18}$$

It is easy to see that (17) holds. Therefore, every solution of (18) is oscillatory. In fact, $x(t) = \sin t$ is such a solution.

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