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OSCILLATORY PROPERTIES OF CERTAIN CATEGORY OF SECOND ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Zeng Qingwu

Guangdong Songshan Polytechnic College
Shaoguan, Guangdong 512126
P. R. China

Abstract

This paper discusses the oscillatory properties of a certain category of second order nonlinear functional differential equations and concludes with conditions in which oscillations occur.

1. Introduction

The oscillation of functional differential equations has important applications across the ecosystem and the control theory. The studies on the oscillation of solutions to functional differential equations are closely followed by people that share the interest, and some compelling findings have been made so far.

This paper examines the oscillation of the solutions to the second order nonlinear functional differential equation

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$$\frac{\partial}{\partial t}u(x, t) = a(t)\sum_{i=1}^{n}\frac{\partial^2 u(x, t)}{\partial x_i^2}$$
$$+\sum_{k=1}^{s}a_k(t)\sum_{i=1}^{n}\frac{\partial^2 u(x, t-p_k)}{\partial x_i^2}-\sum_{j=1}^{m}q_j(t)u(x, t-\sigma_j) \quad (1)$$

and obtains the necessary and sufficient condition, where $(x, t)\in\Omega\times[0, +\infty)$ $= G,$ Ω being the bounded domain of $\partial\Omega$ with piecewise smooth boundaries in R^n.

In this paper, we agree that the following conditions are always satisfied:

(R_1) $m, n, s, i, j, k \in N,$ and $1 \le i \le n, 1 \le j \le m, 1 \le k \le s,$

(R_2) $a, a_k, q_j \in C([0, +\infty), [0, +\infty)),$

(R_3) p_k, σ_j is a nonnegative constant, and the boundary condition is

$$\frac{\partial u(x, t)}{\partial N} = 0, \quad (x, t)\in\partial\Omega\times[0, +\infty), \quad (2)$$

N is the outward unit normal vector of the boundary $\partial\Omega$.

The solution to equation (1), $u(x, t)\in C^2(\overline{G}, R),$ oscillates within G, if for any positive number μ exists a point $(x_0, t_0)\in\Omega\times[\mu, +\infty)$ that satisfies $u(x_0, t_0) = 0$.

2. Main Findings

The following theorems are the main findings of this paper.

Theorem 1. *The necessary and sufficient condition for each solution of equation* (1) *to oscillate within G is the differential inequality*

$$v'(t)+\sum_{j=1}^{m}q_j(t)v(t-\sigma_j)\le 0. \quad (3)$$

No final positive solution is attainable.

To verify the theorem, the result in [1] is cited.

Lemma [1]**.** *Equation*

$$v'(t) + \sum_{j=1}^{m} q_j(t) v(t - \sigma_j) = 0. \tag{4}$$

The necessary and sufficient condition to solve the oscillation is that (3) returns no final positive solution.

To verify Theorem 1:

(i) Sufficiency. Suppose $u(x, t)$ is one of the non-oscillatory solutions of (1) on $\Omega \times [t_0, +\infty)$, $t_0 > 0$. Let $t_1 \geq t_0$, when $(x, t) \in \partial\Omega \times [0, +\infty)$, $u(x, t) > 0$, $u(x, t - p_k) > 0$ and $u(x, t - \sigma_j) > 0$ are satisfied at the same time.

With respect to the integral *x* on Ω on either side of (1), we get

$$\begin{aligned} \frac{d}{dt}\left(\int_\Omega u(x, t)dx\right) = a(t)\int_\Omega \sum_{i=1}^{n} \frac{\partial^2 u(x, t)}{\partial x_i^2} dx \\ + \sum_{k=1}^{s} a_k(t) \int_\Omega \sum_{i=1}^{n} \frac{\partial^2 u(x, t - p_k)}{\partial x_i^2} dx \\ - \sum_{j=1}^{m} q_j(t) \int_\Omega u(x, t - \sigma_j) dx, \quad t \geq t_1. \end{aligned} \tag{5}$$

Using the Green Formula and (R_3), we get

$$\int_\Omega \sum_{i=1}^{n} \frac{\partial^2 u(x, t)}{\partial x_i^2} dx = \int_{\partial\Omega} \frac{\partial u(x, t)}{\partial N} ds = 0, \quad t \geq t_1, \tag{6}$$

$$\int_\Omega \sum_{i=1}^{n} \frac{\partial^2 u(x, t - p_k)}{\partial x_i^2} dx = \int_{\partial\Omega} \frac{\partial u(x, t - p_k)}{\partial N} ds = 0, \quad t \geq t_1. \tag{7}$$

Substituting (6), (7) into (5) and let $v(t) = \int_\Omega u(x, t)dx$, $t \geq t_1$, we get

$$v'(t) + \sum_{j=1}^{m} q_j(t) v(t - \sigma_j) = 0, \quad t \geq t_1. \tag{8}$$

The foregoing expression shows that $v(t) = \int_\Omega u(x, t)dx > 0$ is the solution to (3), which contradicts the lemma.

(ii) Necessity. Suppose (3) has a final positive solution. Then (4) has a non-oscillatory solution as suggested by the lemma.

Let $\overline{v}(t) > 0$ and $t_1 \geq t_0 \geq 0$ is a solution to (4). Based on (4), we get

$$\overline{v}'(t) = -\sum_{j=1}^{m} q_j(t)\overline{v}(t - \sigma_j), \quad t \geq t_0. \tag{9}$$

Apparently, $\sum_{i=1}^{n} \frac{\partial^2 \overline{v}(t)}{\partial x_i^2} = 0$, and substituting $\sum_{i=1}^{n} \frac{\partial^2 \overline{v}(t - p_k)}{\partial x_i^2} = 0$ into (9), we get

$$\frac{\partial \overline{v}(t)}{\partial t} = a(t)\sum_{i=1}^{n} \frac{\partial^2 \overline{v}(t)}{\partial x_i^2} + \sum_{k=1}^{s} a_k(t)\sum_{i=1}^{n} \frac{\partial^2 \overline{v}(t - p_k)}{\partial x_i^2} - \sum_{j=1}^{m} q_j(t)\overline{v}(t - \sigma_j),$$

$$t \geq t_0.$$

This shows $\overline{v}(x, t) = \overline{v}(t) > 0$ is a solution to (1), which contradicts the condition. Theorem 1 is verified.

Based on Theorem 1 and the lemma, we get

Conclusion. The necessary and sufficient condition for each solution to equation (1) to oscillate within G is that every solution to equation (4) oscillates.

Using the deduction and [1], we get

Theorem 2. *Suppose* $q_j(t) \geq q_j \geq 0, \; j = 1, 2, ..., m,$ *if*

$$\sum_{j=1}^{m} q_j \sigma_j > \frac{1}{e} \;\; or \;\; \sum_{j=1}^{m} \sigma_j \left(\sum_{j=1}^{m} q_j \right)^{\frac{1}{m}} > \frac{1}{e},$$

then each solution to equation (1) *oscillates within G.*

For example,

$$\frac{\partial u(x,\, t)}{\partial t} = (1 + e^{t})\frac{\partial^2 u(x,\, t)}{\partial x^2} + (2 + e^{-t})\frac{\partial^2 u\left(x,\, t - \frac{3\pi}{2}\right)}{\partial x^2}$$

$$- (3 + e^{-t})u\left(x,\, t - \frac{\pi}{2}\right) - (1 + e^{t})u(x,\, t - \pi),$$

$$(x,\, t) \in (0,\, \pi) \times [0,\, +\infty)$$

and

$$\frac{\partial u(0,\, t)}{\partial x} = \frac{\partial u(\pi,\, t)}{\partial x} = 0, \quad t \geq 0.$$

This verifies easily that it meets conditions for Theorem 2. Therefore, the solution to this equation oscillates within $(0,\, \pi) \times [0,\, +\infty)$. In fact, $u(x,\, t) = \cos x \sin t$ is the oscillatory solution.

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