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ON THE FERMAT PROBLEM FOR ELLIPSE

Zvonko Čerin and Gian Mario Gianella

Kopernikova 7
10010 Zagreb, Croatia
e-mail: cerin@math.hr

Dipartimento di Matematica
Universita di Torino
Torino, Italy
e-mail: gianella@dm.unito.it

Abstract

We extend a curious observation by Fermat about points on a semicircle on a side of a special rectangle to a theorem that holds for all points on an ellipse.

1. Introduction

Among the numerous questions that Pierre Fermat formulated, the following geometric problem has drawn some attention (see Figure 1).

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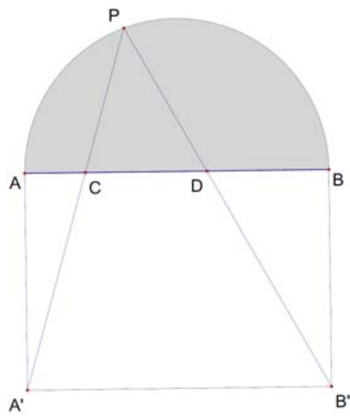

Figure 1. The Fermat configuration for a semicircle.

Fermat Problem. Let P be a point on the semicircle that has the top side AB of the rectangle $ABB'A'$ as a diameter. Let $\frac{|AB|}{|AA'|} = \sqrt{2}$. Let the segments PA' and PB' intersect the side AB in the points C and D. Then $|AD|^2 + |BC|^2 = |AB|^2$.

The great Leonard Euler in [2] has provided the first rather long proof, which is old fashioned (for his time), and avoids the analytic geometry (which offers rather simple proofs as we shall see below). Several more concise synthetic proofs are now known (see [6], [3, pp. 602, 603], [1, pp. 168, 169] and [4, pp. 181, 264]).

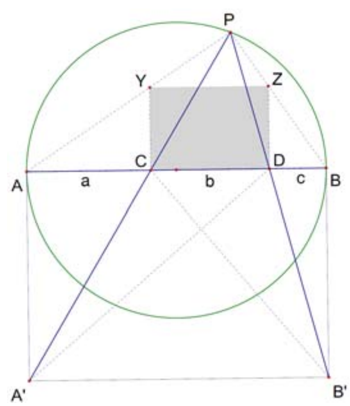

Figure 2. The Fermat configuration for a circle.

Here is an adaptation of Lionnet's proof from [3, p. 602] that applies wherever P might be on the complete circle (see Figure 2).

Let the directed lengths $\overrightarrow{AC}$, $\overrightarrow{CD}$, $\overrightarrow{DB}$ be a, b, c. Then

$$AD^2 + BC^2 - AB^2 = (a+b)^2 + (b+c)^2 - (a+b+c)^2 = b^2 - 2ac.$$

So

$$AD^2 + BC^2 = AB^2 \Leftrightarrow b^2 = 2ac. \tag{1}$$

Now draw CY and DZ perpendicular to AB, with Y on PA and Z on PB. Using pairs of similar triangles, we have

$$\frac{YC}{AA'} = \frac{PC}{PA'} = \frac{CD}{A'B'} = \frac{PD}{PB'} = \frac{ZD}{BB'}.$$

Hence, $YCDZ$ is a rectangle similar to $AA'B'B$. The triangles YCA, BDZ are equiangular, so $\frac{YC}{a} = \frac{c}{DZ}$. But $YC = DZ = \frac{CD}{\sqrt{2}} = \frac{b}{\sqrt{2}}$. Thus $b^2 - 2ac$ vanishes, hence $AD^2 + BC^2 = AB^2$.

The analytic proof was recently recalled in [5] where it was observed that the above relation holds for all points on the circle with the segment AB as a diameter.

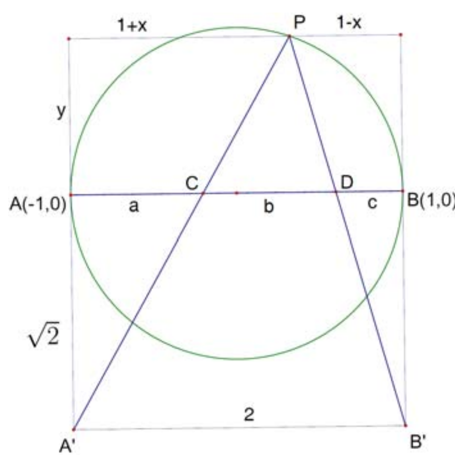

Figure 3. Simple analytic proof.

We can also use the equivalence (1) to obtain a simple analytic proof (see Figure 3). There is no loss of generality in taking the radius of the circle as the unit of length. By similar triangles $\frac{a}{\sqrt{2}} = \frac{1+x}{\sqrt{2}+y}$, $\frac{b}{y} = \frac{2}{\sqrt{2}+y}$ and $\frac{c}{\sqrt{2}} = \frac{1-x}{\sqrt{2}+y}$. Simplifying $b^2 - 2ac$ and putting $x^2 = 1 - y^2$, we find that this difference vanishes.

Our goal is to extend the above problem to any ellipse. Instead of a point P on a (semi)circle that has the side AB as a diameter, we consider a point P on an ellipse that has the segment AB as a principal diameter. It turns out that when $\frac{|AB|}{|AA'|} = \sqrt{\frac{2}{1-e^2}}$, where e denotes the eccentricity of the ellipse, then the relation $|AD|^2 + |BC|^2 = |AB|^2$ again holds for all points P (see Figure 4).

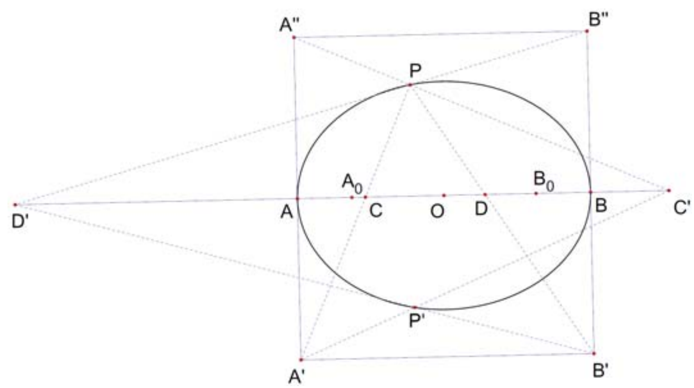

Figure 4. The Fermat configuration for an ellipse.

For an ellipse, we shall consider a slightly more general situation when the quotient $\frac{|AB|}{|AA'|}$ is a positive real number m.

For four points P_1, P_2, P_3, P_4, let $\varphi(P_1P_2, P_3P_4) = \frac{|P_1P_2|^2 + |P_3P_4|^2}{|AB|^2}$.

In this notation, the above Fermat Problem for ellipse is the implication (a) $\Rightarrow$ (b) in the following theorem. Let $f = \sqrt{1 - e^2}$.

Theorem 1. *The following statements are equivalent*:

(a) *The ratio m of side lengths of the rectangle $ABB'A'$ is* $\dfrac{\sqrt{2}}{f}$.

(b) $\varphi(AD, BC) = 1$.

Proof. We shall use analytic geometry that offers a simple proof. Let the origin of the rectangular coordinate system be the midpoint O of the side AB so that the points A and B have coordinates $(-a, 0)$ and $(a, 0)$ for some positive real number a. The foci are the points $A_0(-ae, 0)$ and $B_0(ae, 0)$ and the equation of the ellipse is a standard $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $e = \sqrt{1 - \dfrac{b^2}{a^2}} < 1$ and $b = a\sqrt{1 - e^2} = af$.

The coordinates of the points A', B', A'' and B'' are $\left(-a, -\dfrac{2a}{m}\right)$, $\left(a, -\dfrac{2a}{m}\right)$, $\left(-a, \dfrac{2a}{m}\right)$ and $\left(a, \dfrac{2a}{m}\right)$. For any real number t, let $u = 1 - t^2$, $v = 1 + t^2$, $w = fm$, $z = fmt$, $\eta = v - z$ and $\vartheta = v + z$. An arbitrary point P on the ellipse has the coordinates $\left(\dfrac{au}{v}, \dfrac{2az}{mv}\right)$. Writing down the linear equations of lines joining two points and solving systems of two equations to determine their intersections, we easily find that $C\left(\dfrac{a(u - z)}{\vartheta}, 0\right)$ and $D\left(\dfrac{a(u + z)}{\vartheta}, 0\right)$. The equivalence of the statements (a) and (b) follows from the identity $\varphi(AD, BC) - 1 = \dfrac{t^2(w^2 - 2)}{\vartheta^2}$. □

In Figure 5, we see how the implication (a) $\Rightarrow$ (b) (for an ellipse) could be proved using the Fermat Problem (for a circle). Since an ellipse is the affine image of a circle, all we have to do is stretch the y-coordinate of every point P on the ellipse for a factor $\frac{a}{b}$ in order to get the associated point Q on the circle. The points A' and B' must be pushed down for the same ratio to the points X and Y. The points A, B, C and D remain fixed so that the implication (a) $\Rightarrow$ (b) follows from the claim of the Fermat Problem.

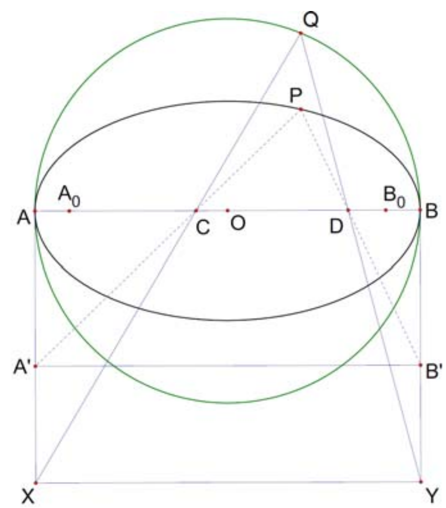

Figure 5. Ellipse with its associated circle.

Let A'', B'', P' be the reflections of the points A', B', P in the line AB. We close this introduction with a remark that most of our results come in related pairs. The second version, which requires no extra proof, comes (for example in Theorem 1) by replacing the points C and D with the points C' and D', which are the intersections of the line AB with the lines PA'' (or $P'A'$) and PB'' (or $P'B'$) (see Figure 4).

2. Invariants of the Fermat Configuration

Our first goal is to introduce several statements similar to (b) that could be added to Theorem 1. In other words, we explore what other relationships in the Fermat configuration remain invariant as the point P changes position on the ellipse.

We begin with the diagonals of the trapezium $A'B'DC$ and the sides of the trapezium $A'B'C'D'$.

(c) $\varphi(A'D, B'C) = \varphi(A'D', B'C') = 1 + f^2$.

Proof of (c). With straightforward computations, one can easily check that $1 + f^2 - \varphi(A'D, B'C) = \dfrac{(w^2 - 2)(\vartheta^2 - m^2t^2)}{m^2\vartheta^2}$.

Pythagoras' theorem might be useful in computing the function φ. For instance, for $\xi = \varphi(A'D, B'C)$, we have

$$A'D^2 + B'C^2 = (AA')^2 + (BB')^2 + AD^2 + BC^2.$$

But $(AA')^2 + (BB')^2 = f^2AB^2$ and $AD^2 + BC^2 = AB^2$, so $\xi = 1 + f^2$. □

For points X and Y, let $X \oplus Y$ be the centre of the square built on the segment XY such that the triangle $X(X \oplus Y)Y$ has the positive orientation (counterclockwise). When the point $X \oplus Y$ is shortened to M, then M^* denotes $Y \oplus X$.

The midpoints G, H, G', H' of the segments AC, BD, AC', BD' and the top N of the semicircle over AB are used in the next two statements. In other words, $N = B \oplus A$. The centre O of the circle (i.e. the midpoint of the segment AB; the origin of the rectangular coordinate system) appears in the statement (e).

(d) $\varphi(NG, NH) = \varphi(NG', NH') = \dfrac{3}{4}$.

(e) $\varphi(OG, OH) = \varphi(OG', OH') = \dfrac{1}{4}$.

Proof of (d) and (e). This time the differences $\varphi(NG, NH) - \dfrac{3}{4}$ and

$\varphi(OG, OH) - \frac{1}{4}$ both simplify to the following quotient $\frac{t^2(w^2-2)}{4\vartheta^2}$ that has the factor $w^2 - 2$ again.

The following synthetic proof shows that the statements (a) and (b) together imply (d) and (e).

A dilatation with centre A and scale factor 2 maps GO on to CB, thus $GO = \frac{CB}{2}$; similarly $OH = \frac{AD}{2}$. Consequently for $\varphi(OG, OH)$, we have $OG^2 + OH^2 = \frac{BC^2}{4} + \frac{AD^2}{4} = \frac{1}{4}AB^2$. Similarly, for $\varphi(NG, NH)$, we have $NG^2 = NO^2 + OG^2$ and $NH^2 = NO^2 + OH^2$ so that

$$NG^2 + NH^2 = 2NO^2 + \frac{BC^2 + AD^2}{4} = \frac{AB^2}{2} + \frac{AB^2}{4} = \frac{3}{4}AB^2. \qquad \square$$

Let G_s, H_s, G_s', H_s' be the points that divide the segments NG, NH, NG', NH' in the same ratio $s \neq -1$ (i.e. $NG_s : G_sG = s : 1$, etc.).

(f) $\varphi(OG_s, OH_s) = \varphi(OG_s', OH_s') = \frac{s^2 + 2f^2}{4(s+1)^2}$.

(g) $\varphi(NG_s, NH_s) = \varphi(NG_s', NH_s') = \frac{(1 + 2f^2)s^2}{4(s+1)^2}$.

Proof of (f). Since $G_s = \left(\frac{-ast(w+t)}{(s+1)\vartheta}, \frac{a}{s+1}\right)$ and $H_s = \left(\frac{as(1+z)}{(s+1)\vartheta}, \frac{a}{s+1}\right)$, the difference $\varphi(OG_s, OH_s) - \frac{s^2 + 2f^2}{4(s+1)^2}$ is $\frac{s^2t^2(w^2-2)}{4(s+1)^2\vartheta^2}$. $\square$

Let N_1, N_2, N_3, N_4 denote the highest points on the semicircles built on the segments AC, BD, AC', BD' above the line AB. In other words, $N_1 = C \oplus A$, $N_2 = B \oplus D$, $N_3 = C' \oplus A$, $N_4 = B \oplus D'$.

(h) $\varphi(BN_1, AN_2) = \varphi(BN_3, AN_4) = \frac{3}{2}$.

(i) $\varphi(NN_1, NN_2) = \varphi(NN_3, NN_4) = \frac{1}{2}$.

Proof of (h). Since $N_1 = \left(\frac{-at(w+t)}{\vartheta}, \frac{a}{\vartheta}\right)$ and $N_2 = \left(\frac{a(1+z)}{\vartheta}, \frac{at^2}{\vartheta}\right)$, we get that $\varphi(BN_1, AN_2) - \frac{3}{2}$ is $\frac{t^2(w^2-2)}{2\vartheta^2}$.

The synthetic proof of the implication (a) $\Rightarrow$ (h) uses the right-angled triangles AHN_2 and BGN_1 to get $AN_2^2 = \left(a + b + \frac{c}{2}\right)^2 + \frac{c^2}{4}$ and $BN_1^2 = \left(\frac{a}{2} + b + c\right)^2 + \frac{a^2}{4}$. But $b^2 = 2ac$ implies $AN_2^2 + BN_1^2 = \frac{3}{2} AB^2$.

Let $a' = \frac{a}{\sqrt{2}}$, etc. From the isosceles right-angled triangles ABN, ACN_1, BDN_2, we get $NN_1^2 = (b' + c')^2$ and $NN_2^2 = (a' + b')^2$. But $b^2 = 2ac$ implies $NN_1^2 + NN_2^2 = \frac{1}{2} AB^2$. □

The following statements also use the summits N_1, N_2, N_3 and N_4. However, they do not use the function φ.

(j) $|N_1N_2| = |N_3N_4| = |AN|$.

(k) $|N_1N_2|^2 + |N_2N_3|^2 + |N_3N_4|^2 + |N_4N_1|^2 = 2|AB|^2$.

Proof of (j). The differences $|N_1N_2|^2 - |AN|^2$ and $|N_3N_4|^2 - |N_1N_2|^2$ are equal to $\frac{2a^2t^2(w^2-2)}{\vartheta^2}$ and $\frac{8a^2t^2zv(w^2-2)}{\eta^2\vartheta^2}$, respectively. □

Let us note that $|N_1N_2^*|^2 + |N_2^*N_3^*|^2 + |N_3^*N_4|^2 + |N_4N_1|^2 = 2|AB|^2$ if and only if $m = \frac{1}{f}$.

Let $N_5 = A \oplus D$, $N_6 = C \oplus B$, $N_7 = A \oplus D'$ and $N_8 = C' \oplus B$.

(l) $\varphi(AN_5,\, BN_6) = \varphi(AN_7,\, BN_8) = \frac{1}{2}$.

(m) $\varphi(GN_6,\, HN_5) = \varphi(G'N_8,\, H'N_7) = \frac{3}{4}$.

(n) $\varphi(NN_5,\, NN_6) = \varphi(NN_7,\, NN_8) = \frac{3}{2}$.

Proof of (l). Since $N_5 = \left(\frac{-at^2}{\vartheta}, \frac{-a(1+z)}{\vartheta}\right)$ and $N_6 = \left(\frac{a}{\vartheta}, \frac{-at(w+t)}{\vartheta}\right)$, the difference $\varphi(AN_5,\, BN_6) - \frac{1}{2}$ is again $\frac{t^2(w^2-2)}{2\vartheta^2}$.

From the isosceles right-angled triangles ADN_5 and BCN_6, we get $AN_5^2 + BN_6^2 = \frac{AD^2}{2} + \frac{BC^2}{2} = \frac{1}{2}AB^2$. □

The next six statements use the centres of squares on the segments CD and $C'D'$. Let $M_1 = C \oplus D$ and $M_2 = C' \oplus D'$.

(o) $|NM_1| = |NM_2| = |AN|$.

Proof of (o). Since $M_1 = \left(\frac{au}{\vartheta}, -\frac{az}{\vartheta}\right)$, the difference $|M_1N|^2 - |AN|^2$ is equal to $\frac{2a^2t^2(w^2-2)}{\vartheta^2}$. Similarly, $|M_2N|^2 - |M_1N|^2 = \frac{8a^2t^2vz(w^2-2)}{\eta^2\vartheta^2}$.

From the right-angled triangle NN_1M_1, since $\frac{b^2}{2} = ac$, we get $NM_1^2 = (a'+b')^2 + (b'+c')^2 = a'^2 + b^2 + c'^2 + ab + bc = (a'+b'+c')^2 = NA^2$.

□

The same argument proves also the following:

(p) $\varphi(M_1N_1, M_1N_2) = \varphi(M_2N_3, M_2N_4) = \frac{1}{2}$.

For any point X in the plane, let G_1, G_2, G_3, G_4, G_5 and G_6 denote the centroids of the triangles ACX, CDX, DBX, $AC'X$, $C'D'X$ and $BD'X$, respectively.

(q) $\varphi(G_2G_1, G_2G_3) = \varphi(G_5G_4, G_5G_6) = \frac{1}{9}$.

Proof of (q). If $X = (x, y)$, then the points G_1, G_2 and G_3 have the same ordinate $\frac{y}{3}$ while their abscissae are $\frac{x}{3} - \frac{2at(w+t)}{3\vartheta}$, $\frac{x}{3} + \frac{2au}{3\vartheta}$ and $\frac{x}{3} + \frac{2a(z+1)}{3\vartheta}$. It follows that the difference $\varphi(G_2G_1, G_2G_3) - \frac{1}{9}$ is $\frac{t^2(w^2-2)}{9\vartheta^2}$.

The implication (b) $\Rightarrow$ (q) could be proved as follows. Let I denote the midpoint of the segment CD. A dilatation with centre C and scale factor 2 maps GI on to AD, thus $GI = \frac{AD}{2}$; similarly $HI = \frac{BC}{2}$. Hence $\varphi(GI, HI) = \frac{1}{4}$. On the other hand, a dilatation with centre X and scale factor $\frac{3}{2}$ maps G_1G_2 on to GI, thus $G_1G_2 = \frac{2}{3}GI$; similarly $G_3G_2 = \frac{2}{3}HI$. Hence $\varphi(G_1G_2, G_3G_2) = \frac{1}{9}$. □

Let U and V be the midpoints of the segments CC' and DD'.

(r) $\varphi(NU, NV) = \frac{1+f^2}{2}$.

(s) $\varphi(OU, OV) = \varphi(N_6U, N_5V) = \varphi(N_8U, N_7V) = \frac{1}{2}$.

Proof of (r) and (s). Since $U = \left(\frac{a(uv + z^2)}{\eta\vartheta}, 0\right), V = \left(\frac{a(uv - z^2)}{\eta\vartheta}, 0\right),$ we get $\varphi(NU, NV) - \frac{1+f^2}{2} = \varphi(OU, OV) - \frac{1}{2} = \frac{t^2v^2(w^2-2)}{\eta^2\vartheta^2}.$ □

Let $W = U \oplus V.$

(t) *The centre W lies on the circle that has the segment AB as a diameter.*

Proof of (t). Since the coordinates of the point W is the pair $\left(\frac{auv}{\eta\vartheta}, \frac{az^2}{\eta\vartheta}\right)$, we get that $|WO|^2 - a^2$ equals $\frac{2a^2t^2v^2(w^2-2)}{\eta^2\vartheta^2}.$ □

(u) $\varphi(W^*O, WO) = \varphi(WN_i, WN_j) = \frac{1}{2}$, *for* $i \in \{1, 3\}$ *and* $j \in \{2, 4\}.$

(v) $\varphi(W^*N, WN) = \frac{1+f^2}{2}.$

(w) *The lines* WN_1 *and* WN_2 *are perpendicular.*

(x) *The lines* WN_3 *and* WN_4 *are perpendicular.*

Proof of (w). The lines WN_1 and WN_2 have the equations

$$(M + z)x + (M - z)y = \lambda \text{ and } (M' + z)x - (M' - z)y = \mu,$$

where $M = z^2 - v$, $M' = w^2 - v$ and λ and μ are real numbers. These lines are perpendicular if and only if the sum $M + M'$ is zero. However, $M + M' = v(w^2 - 2).$ □

Let $K_1 = B \oplus N_1$, $K_2 = N_2 \oplus A$, $K_3 = B \oplus N_3$, $K_4 = N_4 \oplus A$. These points can be defined more simply. They all are at the same height as N and vertically above the points N_6, N_5, N_8, N_7, respectively. The next four statements use rather exotic numbers.

(y) $\varphi(A'K_2,\, B'K_1) = \varphi(A'K_4,\, B'K_3) = \frac{3}{4} + f\sqrt{2} + f^2.$

(z) $\varphi(A'K_2^*,\, B'K_1^*) = \varphi(A'K_4^*,\, B'K_3^*) = \frac{3}{4} - f\sqrt{2} + f^2.$

Proof of (y). Since $K_2 = \left(-\frac{at^2}{\vartheta},\, a\right)$ and $K_1 = \left(\frac{a}{\vartheta},\, a\right)$, it follows that $\frac{3}{4} + f\sqrt{2} + f^2 - \varphi(A'K_2,\, B'K_1)$ is equal to $\frac{M(w - \sqrt{2})}{4m^2\vartheta^2}$, where M is the sum $(4(m+1)\vartheta^2 - m^2t^2)\sqrt{2} + w(2\vartheta + mt)(2\vartheta - mt)$. □

Replacing A' and B' with A and B in (y) and (z), we get the number $\frac{3}{4}$ as the common value of the function φ in all four cases.

Let S_1 and T_1 denote the midpoints of the segments $A'C$ and $B'D$. Similarly, let S_2 and T_2 be the midpoints of the segments $A'C'$ and $B'D'$. Note that $\varphi(G_sS_1,\, H_sT_1) = \varphi(G_s'S_2,\, H_s'T_2) = \frac{f^2(s+1+\sqrt{2})^2 + 1}{4(s+1)^2}$.

(a1) $\varphi(NS_1,\, NT_1) = \varphi(NS_2,\, NT_2) = \frac{3}{4} + \frac{f\sqrt{2}}{2} + \frac{f^2}{4}.$

(b1) $\varphi(OS_1,\, OT_1) = \varphi(OS_2,\, OT_2) = \frac{1}{4} + \frac{f^2}{4}.$

Proof of (b1). From the right-angled triangles OGS_1 and OHT_1 and (e), we get that the sum $OS_1^2 + OT_1^2 = (OG^2 + GS_1^2) + (OH^2 + HT_1^2)$ is $(OG^2 + OH^2) + \frac{f^2AB^2}{4} = \frac{1+f^2}{4}AB^2.$ □

By replacing the point $N = (0,\, a)$ with its reflection $N^* = (0,\, -a)$ in (a1) and (b1) on the right hand side, the first + changes into –.

For points X and Y, let ϱ_X^Y be the reflection of the point X in the point Y. Let $Q = \varrho_A^D$, $R = \varrho_B^C$, $Q' = \varrho_A^{D'}$, $R' = \varrho_B^{C'}$.

(c1) $\varphi(A'Q,\ B'R) = \varphi(A'Q',\ B'R') = 4 + f^2$.

Proof of (c1). Since $Q = \left(\frac{a(u+3z+2)}{\vartheta}, 0\right)$ and $R = \left(\frac{a(3u-3z-2)}{\vartheta}, 0\right)$, we get that $4 + f^2 - \varphi(A'Q,\ B'R)$ is equal to $\frac{\eta(w^2-2)(\vartheta+2z)}{m^2\vartheta^2}$.

From the right-angled triangles $AA'Q$ and $BB'R$, we get that $A'Q^2$ and $B'R^2$ are $4(a+b)^2 + A'A^2$ and $4(b+c)^2 + B'B^2$. By adding we conclude from (1) that $\varphi(A'Q,\ B'R) = 4 + f^2$. Also, we have $\varphi(AQ,\ BR) = 4$ and $\varphi(N_5Q,\ N_6R) = \frac{5}{2}$. □

In the last two statements, we use the foci A_0 and B_0 of the ellipse. Let $n_1 = C \oplus A_0$, $n_2 = B_0 \oplus D$, $n_3 = C' \oplus A_0$, $n_4 = B_0 \oplus D'$. Let I and J denote the midpoints of the segments NM_1^* and NM_2^*.

(d1) $\varphi(An_2,\ Bn_1) = \varphi(An_4,\ Bn_3) = \frac{3 + 2e + e^2}{4}$.

(e1) $\varphi(A_0I,\ B_0I) = \varphi(A_0J,\ B_0J) = \frac{3 - 2f^2}{4}$.

3. Common Properties for All Ratios

Of course, there are many properties that hold for all ratios m. The following is an example of such properties.

Theorem 2. *The triangles ADP and BCP have the same orthocenter that lies on a circle with the segment CD as a diameter if and only if* $f = 1$ (*i.e. the ellipse is a circle*).

The triangles $AD'P$ *and* $BC'P$ *have a common orthocenter. It lies on a circle with the segment* $C'D'$ *as a diameter if and only if* $f = 1$.

Proof. The orthocenters of the triangles *ADP* and *BCP* both have the coordinates $\left(\frac{au}{v}, \frac{2amt^2}{\vartheta v}\right)$.

If *K* is the midpoint of the segment *CD* and *L* is the orthocenter of the triangle *ADP*, then $|CK|^2 - |KL|^2 = \frac{4a^2m^2t^4(f-1)(f+1)}{\vartheta^2 v^2}$. Hence, this difference is equal to zero if and only if $f = 1$. □

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