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ON THE DECOMPOSITION OF PRIME IDEALS OF ORDERED Γ-SEMIGROUPS INTO THEIR $\mathcal{N}$-CLASSES

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Abstract

In this paper, we show that every ideal of an $\mathcal{N}$-class of an ordered Γ-semigroup does not contain proper prime ideals. Similar results on ordered semigroups were presented by Kehayopulu and Tsingelis in [2] and on semigroups can be founded in [3, II.2.11].

1. Preliminaries

In 1986, Sen and Saha [4] defined Γ-semigroup as a generalization of semigroup as follows:

Definition 1.1. Let S and Γ be two nonempty sets. Then S is called a Γ-*semigroup* if there is a mapping $S \times \Gamma \times S \to S$, written as $(x, \gamma, y) \mapsto x\gamma y$, such that $(x\gamma y)\beta z = x\gamma(y\beta z)$ for all $x, y, z \in S$ and all $\gamma, \beta \in \Gamma$.

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Let $(S, \cdot)$ be a semigroup and Γ be a nonempty set. For $x, y \in S$ and $\gamma \in \Gamma,$ let $x\gamma y$ be defined by $x\gamma y = x \cdot y.$ Then S is a Γ-semigroup.

Let S be a Γ-semigroup. For $A, B \subseteq S,$ let

$$A\Gamma B = \{a\gamma b \mid a \in A, b \in B, \gamma \in \Gamma\}.$$

For $x \in S,$ let $A\Gamma x = A\Gamma\{x\}$ and $x\Gamma A = \{x\}\Gamma A.$

In [5], Sen and Seth introduced an ordered Γ-semigroup as a generalization of a Γ-semigroup as follows:

Definition 1.2. A Γ-semigroup S is called an *ordered* Γ-*semigroup* (*po*-Γ-*semigroup*) if there is a relation $\leq$ on S such that $x \leq y$ implies $x\gamma z \leq y\gamma z$ and $z\gamma x \leq z\gamma y$ for any $x, y, z \in S$ and all $\gamma \in \Gamma.$

Let S be a Γ-semigroup. For $x, y \in S,$ let $x \leq y$ if $x = y.$ Then S is an ordered Γ-semigroup.

Definition 1.3. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A nonempty subset T of S is called a Γ-*subsemigroup* of S if $T\Gamma T \subseteq T.$

Definition 1.4. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A nonempty subset I of S is called an *ideal* of S if the following hold:

(i) $S\Gamma I \subseteq I$ and $I\Gamma S \subseteq I.$

(ii) If $x \in I$ and $y \in S$ such that $y \leq x,$ then $y \in I.$

Definition 1.5. An ideal I of an ordered Γ-semigroup $(S, \Gamma, \leq)$ is said to be *prime* if for $x, y \in S$ and $\gamma \in \Gamma,$ $x\gamma y \in I$ implies $x \in I$ or $y \in I.$

In [1], the author introduced filters in ordered Γ-semigroups as follows:

Definition 1.6. A Γ-subsemigroup F of an ordered Γ-semigroup $(S, \Gamma, \leq)$ is called a *filter* of S if the following hold:

(i) For $x, y \in S$ and $\gamma \in \Gamma,$ $x\gamma y \in F$ implies $x \in F$ and $y \in F.$

(ii) For $x \in F$ and $y \in S,$ $x \leq y$ implies $y \in F.$

An ideal (resp. filter) F of an ordered Γ-semigroup S is said to be *proper* if $F \neq S$.

Let $F = \{F_i \mid i \in I\}$ be a nonempty family of filters of an ordered Γ-semigroup $(S, \Gamma, \leq)$. If $\bigcap F \neq \varnothing$, then $\bigcap F \neq \varnothing$ is a filter of S. In fact: Assume that $\bigcap F \neq \varnothing$, then $\bigcap F \neq \varnothing$ is a Γ-subsemigroup of S. Let $x, y \in S$ and $\gamma \in \Gamma$ be such that $x\gamma y \in \bigcap F$. Since $x\gamma y \in F_i$ for all $i \in I$, we have $x \in \bigcap F$ and $y \in \bigcap F$. Let $x \in \bigcap F$ and $y \in S$ be such that $x \leq y$. For $i \in I$, since $x \in F_i$, we obtain $y \in F_i$. Thus $y \in \bigcap F$.

For an element x of an ordered Γ-semigroup $(S, \Gamma, \leq)$, let $N(x)$ be the filter of S generated by $x(N(x)$ is the intersection of all filters of S containing $x)$. The equivalent relation $\mathcal{N}$ is defined on S by

$$\mathcal{N} = \{(x, y) \in S \times S \mid N(x) = N(y)\}.$$

For $x \in S$, the $\mathcal{N}$-class of S containing x will be denoted by $(x)_{\mathcal{N}}$. $\mathcal{N}$ is a congruence on S (that is, for $x, y, z \in S$ and $\gamma \in \Gamma$, $(x, y) \in \mathcal{N}$ implies $(x\gamma z, y\gamma z) \in \mathcal{N}$ and $(z\gamma x, z\gamma y) \in \mathcal{N}$). Using this fact, the set $S/\mathcal{N} = \{(x)_{\mathcal{N}} \mid x \in S\}$ forms a Γ-semigroup defined by

$$(x)_{\mathcal{N}} \gamma (y)_{\mathcal{N}} = (x\gamma y)_{\mathcal{N}}$$

for all $x, y \in S$ and $\gamma \in \Gamma$. For $x, y \in S$, the following hold:

(1) $(x, x\gamma x) \in \mathcal{N}$ for all $\gamma \in \Gamma$. Indeed: Since $x\gamma x \in N(x)$, we have $N(x) \subseteq N(x\gamma x)$. Since $x\gamma x \in N(x\gamma x)$, we obtain $x \in N(x\gamma x)$. Then $N(x\gamma x) \subseteq N(x)$.

(2) $(x\gamma y, y\beta x) \in \mathcal{N}$ for all $\gamma, \beta \in \Gamma$. In fact: Since $x\gamma y \in N(x\gamma y)$, we have $x \in N(x\gamma y)$ and $y \in N(x\gamma y)$. Since $y\beta x \in N(x\gamma y)$, we have $N(y\beta x) \subseteq N(x\gamma y)$. Similarly, $N(x\gamma y) \subseteq N(y\beta x)$.

(3) $(x)_{\mathcal{N}}$ is a Γ-subsemigroup of S. Indeed: Clearly, $x \in (x)_{\mathcal{N}} \neq \varnothing$. Let $y, z \in (x)_{\mathcal{N}}$ and $\gamma \in \Gamma$. Since $(y)_{\mathcal{N}} = (x)_{\mathcal{N}}$ and $(z)_{\mathcal{N}} = (x)_{\mathcal{N}}$, we have $(y\gamma z)_{\mathcal{N}} = (y)_{\mathcal{N}} \gamma (z)_{\mathcal{N}} = (x)_{\mathcal{N}} \gamma (x)_{\mathcal{N}} = (x\gamma x)_{\mathcal{N}} = (x)_{\mathcal{N}}$. Then $y\gamma z \in (x)_{\mathcal{N}}$.

The purpose of this paper is to show that every ideal of an $\mathcal{N}$-class of an ordered Γ-semigroup does not contain proper prime ideals. Similar results on ordered semigroups were presented by Kehayopulu and Tsingelis in [2].

2. Main Results

Lemma 2.1. *Let* $(S, \Gamma, \leq)$ *be an ordered* Γ*-semigroup and* $x, y \in S$. *If* $x \leq y$, *then* $(x, x\gamma y) \in \mathcal{N}$ *for all* $\gamma \in \Gamma$.

Proof. Assume that $x \leq y$ and $\gamma \in \Gamma$. Since $x \in N(x)$ and $x \leq y$, we have $y \in N(x)$. Since $x\gamma y \in N(x)$, we obtain $N(x\gamma y) \subseteq N(x)$. Since $x\gamma y \in N(x\gamma y)$, we have $x \in N(x\gamma y)$. Then $N(x) \subseteq N(x\gamma y)$. Therefore, $N(x) = N(x\gamma y)$.

Lemma 2.2. *An ordered* Γ*-semigroup* $(S, \Gamma, \leq)$ *does not contain proper filters if and only if S does not contain proper prime ideals.*

Proof. $(\Rightarrow)$ Assume that an ordered Γ-semigroup $(S, \Gamma, \leq)$ does not contain proper filters. Suppose that I is a proper prime ideal of S. Then $S \backslash I \neq \varnothing$. Note that $S \backslash (S \backslash I)$ is a prime ideal of S. Moreover, $S \backslash I$ is a filter. Indeed: Let $x, y \in S \backslash I$ and $\gamma \in \Gamma$. Since $x, y \notin I$ and I is prime, we have $x\gamma y \notin I$. Thus $x\gamma y \in S \backslash I$ for all $\gamma \in \Gamma$. Since I is an ideal of S, it follows that for $x, y \in S$ and $\gamma \in \Gamma$, $x \in I$ or $y \in I$ implies $x\gamma y \in I$. Let $x \in S \backslash I$ and $y \in S$ be such that $x \leq y$. If $y \in I$, then $x \in I$, a contradiction. Therefore, $S \backslash I$ is a filter of S. By assumption, $S \backslash I = S$. Then $I = \varnothing$. A contradiction.

$(\Leftarrow)$ Assume that S does not contain proper prime ideals. Let T be a proper filter of S. Then $S \backslash T \neq \varnothing$. Let $z \in (S \backslash T)\Gamma S$ and $z \notin S \backslash T$. Then $z = x\gamma y$ for some $x \in S \backslash T$, $\gamma \in \Gamma$ and $y \in S$. Since $z \in T$, $x\gamma y \in T$. Since T is filter, we have $x \in T$ and $y \in T$. Thus $x \in T$. A contradiction $(x \in S \backslash T)$. This proves that $(S \backslash T)\Gamma S \subseteq S \backslash T$. Similarly, $S\Gamma(S \backslash T) \subseteq S \backslash T$. For $x, y \in S$ and $\gamma \in \Gamma$, if $x, y \notin S \backslash T$, then $x\gamma y \notin S \backslash T$. Therefore, $S \backslash T$ is a prime ideal of S. Since $S \backslash T = S$, we obtain $T = \varnothing$. A contradiction.

Lemma 2.3. *Let T be a filter of an ordered* Γ*-semigroup* $(S, \Gamma, \leq)$ *and* $z, x \in S$. *If* $x \in T \cap (z)_{\mathcal{N}}$, *then* $(z)_{\mathcal{N}} \subseteq T$.

Proof. Assume that $x \in T \cap (z)_{\mathcal{N}}$. Since $x \in (z)_{\mathcal{N}}$, we have $(x)_{\mathcal{N}} = (z)_{\mathcal{N}}$, that is, $N(x) = N(z)$. Let $y \in (z)_{\mathcal{N}}$. Then $N(y) = N(z) = N(x)$. So $y \in N(x)$. Since $x \in T$, we have $N(x) \subseteq T$. Thus $y \in T$.

Now, we prove the main result.

Theorem 2.4. *Let* $(S, \Gamma, \leq)$ *be an ordered* Γ*-semigroup and* $z \in S$. *If I is an ideal of* $(z)_{\mathcal{N}}$, *then I does not contain proper prime ideals of I.*

Proof. Assume that *I* is an ideal of $(z)_{\mathcal{N}}$. By Lemma 2.2, we shall show that *I* does not contain proper filters. Let *F* be a filter of *I* and $a \in F$. Let

$$T = \{x \in S \mid a\Gamma a\Gamma x \subseteq F\}.$$

(1) $F = T \cap I$. Indeed: Let $y \in F$. Clearly, $y \in I$. Since $a\Gamma a \subseteq F$, we have $a\Gamma a\Gamma y \subseteq F$. Then $y \in T$. Thus $F \subseteq T \cap I$. Let $y \in T \cap I$. Since $y \in T$, we have $a\Gamma a\Gamma y \subseteq F$. Since *F* is filter, $y \in F$. Then $T \cap I \subseteq F$.

(2) *T* is a filter of *S*. In fact: since $a\Gamma a\Gamma a \subseteq F$, $a \in T \neq \emptyset$.

Let $x, y \in T$. Since $a\Gamma a\Gamma y \subseteq F$ and $F \subseteq I \subseteq (z)_{\mathcal{N}}$, we have $a\Gamma a\Gamma y \subseteq (z)_{\mathcal{N}}$. Thus $y\Gamma a \subseteq (z)_{\mathcal{N}}$. Since $a \in I$, by assumption, we obtain $y\Gamma a\Gamma a \subseteq I$. Since $(a\Gamma a)\Gamma(y\Gamma a\Gamma a) = (a\Gamma a\Gamma y)\Gamma(a\Gamma a) \subseteq F$, we have $y\Gamma a\Gamma a \subseteq F$. Similarly, $a\Gamma a\Gamma x \subseteq F$ implies $a\Gamma x \subseteq (z)_{\mathcal{N}}$. For $\gamma, \beta \in \Gamma$, we have

$$(a\gamma x\beta y)_{\mathcal{N}} = (a\gamma x)_{\mathcal{N}}\beta(a\gamma y)_{\mathcal{N}} \subseteq (z)_{\mathcal{N}}.$$

This implies $a\Gamma x\Gamma y \subseteq (z)_{\mathcal{N}}$. Therefore, $a\Gamma a\Gamma x\Gamma y \subseteq I$. This proves that $a\Gamma a\Gamma x, y\Gamma a\Gamma a \subseteq F$ implies $(a\Gamma a\Gamma x)\Gamma(y\Gamma a\Gamma a) \subseteq F$. Since $(a\Gamma a\Gamma x\Gamma y)\Gamma(a\Gamma a) \subseteq F$, $a\Gamma a\Gamma x\Gamma y \subseteq F$. Then $x\Gamma y \subseteq T$. Let $x \in T$ and $y \in S$ be such that $x \leq y$. Since $a\Gamma a\Gamma x \subseteq F$, $a\Gamma a\Gamma y \subseteq F$. Then $y \in T$.

(3) Since $a \in T \cap (z)_{\mathcal{N}}$, by Lemma 2.3, $(z)_{\mathcal{N}} \subseteq T$. Then $F = T \cap I = I$.

Corollary 2.5. *Let* $(S, \Gamma, \leq)$ *be an ordered* Γ*-semigroup and I be prime ideal of S. Then*

$$I = \bigcup\{(x)_{\mathcal{N}} \mid x \in I\}.$$

Proof. Let $t \in (x)_{\mathcal{N}}$ for some $x \in I$. Since $(x)_{\mathcal{N}}$ is an ideal of $(x)_{\mathcal{N}}$, by Theorem 2.4, $(x)_{\mathcal{N}}$ does not contain proper prime ideals. We claim that $(x)_{\mathcal{N}} \cap I$ is a prime ideal of $(x)_{\mathcal{N}}$. Using the claim, $(x)_{\mathcal{N}} \cap I = (x)_{\mathcal{N}}$. Thus $t \in I$.

Clearly, $\varnothing \neq (x)_{\mathcal{N}} \cap I \subseteq (x)_{\mathcal{N}}$. We have

$$\begin{aligned}(x)_{\mathcal{N}}\Gamma((x)_{\mathcal{N}} \cap I) &\subseteq (x)_{\mathcal{N}}\Gamma(x)_{\mathcal{N}} \cap (x)_{\mathcal{N}}\Gamma I\\ &\subseteq (x\Gamma x)_{\mathcal{N}} \cap (x)_{\mathcal{N}}\Gamma I\\ &= (x)_{\mathcal{N}} \cap (x)_{\mathcal{N}}\Gamma I\\ &\subseteq (x)_{\mathcal{N}} \cap S\Gamma I\\ &\subseteq (x)_{\mathcal{N}} \cap I\end{aligned}$$

and

$$\begin{aligned}((x)_{\mathcal{N}} \cap I)\Gamma(x)_{\mathcal{N}} &\subseteq (x)_{\mathcal{N}}\Gamma(x)_{\mathcal{N}} \cap I\Gamma(x)_{\mathcal{N}}\\ &\subseteq (x)_{\mathcal{N}} \cap I\Gamma S\\ &\subseteq (x)_{\mathcal{N}} \cap I.\end{aligned}$$

Let $y \in (x)_{\mathcal{N}} \cap I$ and $z \in (x)_{\mathcal{N}}$ be such that $y \leq z$. Since $y \in I$, $z \in I$. Then $z \in (x)_{\mathcal{N}} \cap I$.

Let $y, z \in (x)_{\mathcal{N}} \cap I$ and $\gamma \in \Gamma$ be such that $y\gamma z \subseteq (x)_{\mathcal{N}} \cap I$. Since $x\gamma y \in I$, $x \in I$ or $y \in I$. Thus $y \in (x)_{\mathcal{N}} \cap I$ or $z \in (x)_{\mathcal{N}} \cap I$.

Hence, we have the claim.

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