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ON DEFORMATION TECHNIQUE OF THE HYPERBOLIC SECANT DISTRIBUTION

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Abstract

Based on the deformation technique for the hyperbolic secant (HS) function, the *p*-deformed hyperbolic secant (*p*-DHS) distribution is presented. This technique is used to deform the balance between

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exponential growth and decay parts in the hyperbolic secant (HS) distribution. Some fundamental properties of the constructed *p*-DHS distribution are discussed. Our study aims to derive some important corresponding functions in closed forms, to explain some fundamental measures and to summarize some results for the mentioned distribution.

1. Introduction

Many techniques for generalization the probability distributions have been studied in [5-8, 11-12]. In terms of density functions – provided their existence – most of these techniques can be introduced as multiplication of a probability density function (pdf) of the original distribution by an appropriate weighting function of the original cumulative distribution function (cdf) with parameter vector on a fixed interval [9]. Recently, a deformation technique has been suggested generally and it has been proposed in some works in the mathematics, physics and statistics [1-6, 10]. In [6], the authors discussed the deformation technique of the HS distribution with respect to the exponential decay part and introduced the so-called *q*-DHS distribution. This study will be concerned on the exponential growth part of the HS distribution. We construct the *p*-DHS distribution by applying the *p*-deformation technique to the HS distribution. According to [9] and other literatures, we study some fundamental properties of the constructed *p*-deformed hyperbolic secant family of distributions.

This paper is organized as follows: Section 2 reviews the HS distribution and its fundamental properties. In Section 3, we introduce the concept of the deformed hyperbolic functions and the *p*-DHS distribution. A moment-generating function (mgf), a characteristic function (cf) and the moments of the *p*-DHS distribution are proposed in Section 4. Section 5 gives some attention to the maximum likelihood estimates (MLE) for the parameter *p*. We summarize the results and give some features and comments in Section 6.

2. Definition of the Hyperbolic Secant Distribution

According to [5-6, 8], the continuous random variable X has a HS distribution if its pdf is given by

$$f_{HS}(x) = \frac{1}{2}\operatorname{sech}\left(\frac{\pi x}{2}\right) = \frac{1}{\exp\left(\frac{\pi x}{2}\right) + \exp\left(-\frac{\pi x}{2}\right)}; \quad x \in \mathbb{R}. \tag{1}$$

The cdf of X is

$$F_{HS}(x) = \frac{1}{2} + \frac{1}{\pi}\arctan\left[\sinh\left(\frac{\pi x}{2}\right)\right]; \quad x \in \mathbb{R}, \tag{2}$$

with the corresponding inverse cdf (critical value or quantile function)

$$x_{\alpha}^{HS} = F_{HS}^{-1}(\alpha) = \frac{2}{\pi}\operatorname{arcsinh}\left[\tan\left(\pi\left(\frac{1}{2} - \alpha\right)\right)\right], \tag{3}$$

where

$$P[X > x_{\alpha}^{HS}] = \alpha, \ \alpha \in (0, 1).$$

This distribution is symmetric with zero mean and unit variance, and it has zero value for the skewness γ and its excess kurtosis β is equal to 2. Moreover, its mgf and cf are given respectively as follows:

$$M_{HS}(t) = \sec t, \quad \Psi_{HS}(t) = \operatorname{sech} t; \quad |t| < \frac{\pi}{2}.$$

3. The *p*-deformed Hyperbolic Secant Distribution

3.1. Definition of the *p*-DHS distribution

Throughout this paper, we consider the deformation technique for which a positive parameter is introduced as scalar factor of the exponential growth part of the HS function. The p-DHS distribution is defined by means of the p-deformation for the hyperbolic functions [2-4, 6, 10]. Firstly, we explain the concept and some properties of the deformed hyperbolic functions.

Definition 1. Let p be a positive real parameter. We define the *deformed hyperbolic functions* to be a family of the functions $\sinh_p x$, $\cosh_p x$, $\tanh_p x$, $\operatorname{sech}_p x$, $\coth_p x$ and $\operatorname{csch}_p x$ as

$$\sinh_p x = \frac{pe^x - e^{-x}}{2}, \; \cosh_p x = \frac{pe^x + e^{-x}}{2}, \; \tanh_p x = \frac{\sinh_p x}{\cosh_p x},$$

$$\coth_p x = \frac{\cosh_p x}{\sinh_p x}, \; \operatorname{sech}_p x = \frac{1}{\cosh_p x}, \; \operatorname{csch}_p x = \frac{1}{\cosh_p x}; \; x \in \mathbb{R}. \quad (4)$$

The parameter p is called *deformation parameter*. □

Lemma 1 [1-4]. *A family of the deformed hyperbolic functions satisfies the following relations*:

$$(\sinh_p x)' = \cosh_p x, \quad (\cosh_p x)' = \sinh_p x,$$

$$(\tanh_p x)' = p \operatorname{sech}_p^2 x, \quad (\operatorname{sech}_p x)' = -\operatorname{sech}_p x \tanh_p x,$$

$$\cosh_p^2 x - \sinh_p^2 x = p, \quad \tanh_p^2 x = 1 - p \operatorname{sech}_p^2 x. \quad (5)$$

Furthermore, if $p \neq 1$, *then* $\sinh_p x$ *is not odd function and* $\cosh_p x$ *is not even function, i.e.*, $\sinh_p(-x) = -p \sinh_{\frac{1}{p}} x$, $\cosh_p(-x) = p \cosh_{\frac{1}{p}} x$; $x \in \mathbb{R}$. □

As an immediate consequence of previous definition and lemma, one can write the following function:

$$g(x;\, p) = \frac{1}{2} \operatorname{sech}_p\left(\frac{\pi x}{2}\right); \; p > 0, \; x \in \mathbb{R}, \quad (6)$$

which satisfies that $g(x;\, p) > 0, \; \forall x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} g(x;\, p) dx = \frac{1}{\sqrt{p}}$.

Definition 2. Let $X_{p\text{-}DHS}$ be a continuous random variable. The variable $X_{p\text{-}DHS}$ has a p-DHS distribution with a positive real parameter p, if its pdf given by,

$$f_{p\text{-}DHS}(x;\, p) = \sqrt{p}\, g(x;\, p) = \frac{\sqrt{p}}{2} \operatorname{sech}_p\left(\frac{\pi x}{2}\right); \quad x \in \mathbb{R}. \quad (7)$$

In this case, $X_{p\text{-}DHS}$ is said to be a *p-DHS random variable with one parameter p*, defined over $\mathbb{R}$. Furthermore, the cdf $F_{p\text{-}DHS}(x;\ p)$ is

$$F_{p\text{-}DHS}(x;\ p) = \frac{1}{2} + \frac{1}{\pi}\arctan\left[\frac{1}{\sqrt{p}}\sinh_p\left(\frac{\pi x}{2}\right)\right]; \quad x \in \mathbb{R} \tag{8}$$

with the inverse cdf (critical value)

$$x_{\alpha}^{p\text{-}DHS} = F_{p\text{-}DHS}^{-1}(\alpha;\ p) = \frac{2}{\pi}\left[\operatorname{arcsinh}\left[\tan\left(\pi\left(\frac{1}{2} - \alpha\right)\right)\right] - \ln\sqrt{p}\right], \tag{9}$$

where

$$P[X > x_{\alpha}^{p\text{-}DHS}] = 1 - F_{p\text{-}HS}(x_{\alpha}^{p\text{-}DHS}) = \alpha;\ \alpha \in (0, 1). \qquad \square$$

The values $x_{\alpha}^{p\text{-}DHS}$ for some different values of p using equation (9) can be computed.

3.2. Properties of the *p*-DHS distribution

By the definition of the expectation of $X_{p\text{-}DHS}$ and $X_{p\text{-}DHS}^2$, respectively, take the forms

$$\mu = E[X_{p\text{-}DHS}] = \frac{2}{\pi}\ln\frac{1}{\sqrt{p}},\ E[X_{p\text{-}DHS}^2] = 1 + \frac{4}{\pi^2}\left(\ln\frac{1}{\sqrt{p}}\right)^2, \tag{10}$$

which implies that $\sigma^2 = 1$.

Proposition 1. *The p-DHS distribution with a positive real valued parameter p is symmetric about* 0 *for* $p = 1$. *Moreover, it skewed more to the right for* $p \in (1, \infty)$ *and skewed more to the left for* $p \in (0, 1)$. *For all positive real values of the parameter p, the kurtosis is always constant.* $\square$

Different densities for the *p*-DHS distribution with $p \in (0, 1)$ and their corresponding densities with $p \in (1, \infty)$ for some positive real values of *p* are plotted in Figure 1.

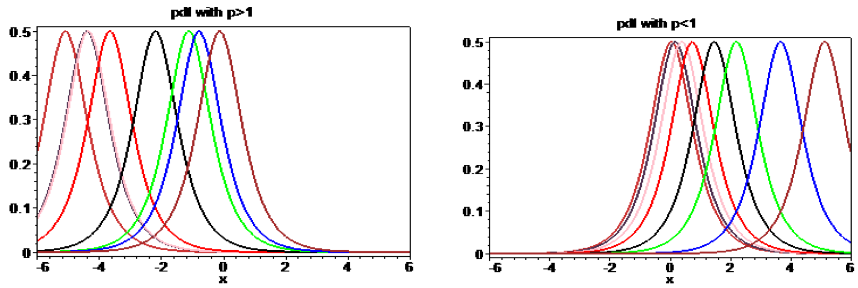

Figure 1. Probability density function for the p-DHS distribution for different values of the parameter p.

Moreover, Figure 2 illustrates the pdf for the p-DHS distribution with $p = 1$.

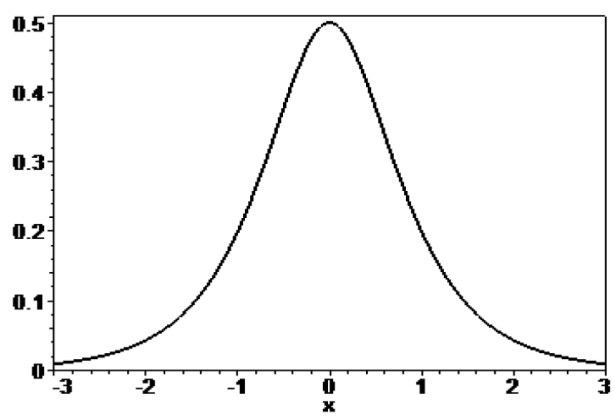

Figure 2. Probability density function for the p-DHS distribution for the case $p = 1$.

By analysis the previous figures, Proposition 1 is valid. Graphically and in compare with the results in [6], it is clear that the DHS distribution with a unit value parameter is symmetric and independent on the used deformation parameters. Moreover, the direction of the skewness of shape of the DHS distribution depends on both the deformed exponential part in the corresponding used HS function and the value of the deformation parameter. Computationally, we can find that, the density corresponding to (8) has smaller mean when p is increasing and it has larger mean when p is decreasing. In compare with [6] we find that, the mean value of the density corresponding to (8) depends on the deformed exponential part in the HS function of the studied original distribution.

Proposition 2. *The score function* $S_{p\text{-}DHS}(x;\ p)$ *of* $X_{p\text{-}DHS}$ *is given by*

$$S_{p\text{-}DHS}(x;\ p) = \frac{\pi}{2}\tanh_p \frac{\pi x}{2}; \quad p > 0. \tag{11}$$

Setting $p = 1$, *the last equation reduces to* $S_{HS}(x) = \frac{\pi}{2}\tanh\frac{\pi x}{2}$, *where* $S_{HS}(x)$ *is the score function of the HS distribution.*

Proof. Based on [6] and other literatures, the score function of a probability distribution is defined by $S(x) = -\frac{(\mathrm{pdf})'}{\mathrm{pdf}}$. By using (7), we can obtain on the form (11) of $S_{p\text{-}DHS}(x;\ p)$ with the reduced case $S_{HS}(x)$ for $p = 1$. □

Proposition 3. *The p-DHS distribution with* $p > 0$ *is unimodal.*

Proof. The probability density function for the p-DHS distribution is given by equation (7). We want to show this density is unimodal for all choices of p. Since $f_{p\text{-}DHS}(x;\ p)$ is a continuously differentiable function, the only critical points for this function satisfy the equation $f'_{p\text{-}DHS}(x;\ p) = 0$. Thus, we want to prove that the last equation has exactly one root, and that this yields a relative maximum. Since $\lim_{x\to\pm\infty} f_{p\text{-}DHS}(x;\ p) = 0$, if there is one critical point, it must yield the absolute maximum, so we need to prove there is exactly one root to the derivative equation.

After simplification, this can be seen to be equivalent to proving

$$\left(\operatorname{sech}_p \frac{\pi x}{2}\right)\left(\tanh_p \frac{\pi x}{2}\right) = 0$$

has exactly one root. Set $x = \frac{2}{\pi}[y - \ln\sqrt{p}]$, then the last statement is equivalent to showing $\operatorname{sech}(y)\tanh(y) = 0$ has exactly one root $y = 0$ in the interval $\mathbb{R}$. This means that the equation $f'_{p\text{-}DHS}(x;\ p) = 0$ has only the root $x^* = \frac{2}{x}\ln\frac{1}{\sqrt{p}}$ in $\mathbb{R}$. Since $f''_{p\text{-}DHS}(x^*;\ p) = -\frac{\pi^2}{8} < 0,$ the point x^* is the maximum value of the p-DHS distribution. It then also follows this yields a relative maximum (and hence absolute maximum) since $f'_{p\text{-}DHS}(x;\ p)$ is positive to the left of the root x^*, and negative to the right, see Figure 3. □

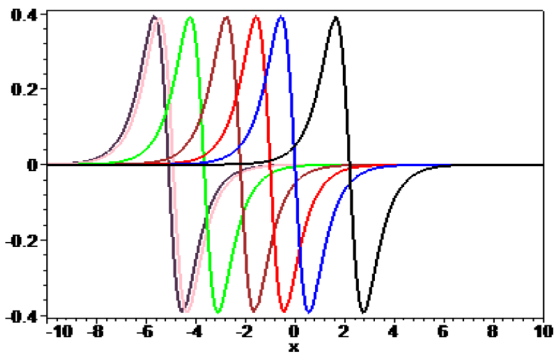

Figure 3. Derivative of the unimodal pdf of p-DHS distributions with $p \in (0, \infty)$.

Note that, the mode for the p-DHS distribution has the root x^* in the last proposition and this value is equal to the obtained value of the mean in (10) for the p-DHS distribution.

Proposition 4. *The median and the mode for the p-DHS distribution with* $p \in (0, \infty)$ *have the same value of the mean*.

Proof. Due to the unimodality of the given distribution and the previous results and the fact that the median of the unimodal distribution lies between the mean and the mode, we can find that the mode and the median for the p-DHS distribution have the same value of the mean as in (10). □

Note that, the case for the unit value of the deformation parameter, the HS distribution, is recovered. In compare with [6] we find that the unimodality of the DHS distribution is valid and it is independent on the used deformation parameters. Moreover, the mean, the median and the mode for the DHS distribution have the same value in each of the two directions, the exponential growth part and the decay part of the HS function.

4. Moment-generating Function of the *p*-DHS Distribution

In this section, the formula for the mgf, the moments and also the cf of the *p*-DHS distribution will be derived. Consequently, we can determine the skewness and kurtosis coefficients of the studied distribution.

Proposition 5. *The mgf of the p-DHS variable* $X_{p\text{-}DHS}$ *with* $p > 0$ *is given by*

$$M_{p\text{-}DHS}(t;\ p) = e^{\frac{2t}{\pi}\ln\frac{1}{\sqrt{p}}}\sec t; \quad |\, t\,| < \frac{\pi}{2}. \tag{12}$$

In particular, all moments of the p-DHS distribution exist.

Proof. By the definition of the mgf for the given variable and the substitutions $x = \frac{2}{\pi}[y - \ln\sqrt{p}]$ and $B = \frac{2t}{\pi}$, we find that

$$M_{p\text{-}DHS}(t;\ p) = \frac{1}{2}\int_{-\infty}^{\infty} e^{tx}\mathrm{sech}_p\left(\frac{\pi x}{2}\right)dx = \frac{1}{\pi}e^{B\ln\frac{1}{\sqrt{p}}}\int_{-\infty}^{\infty} e^{By}\mathrm{sech}\, y dy. \tag{13}$$

According to [6], we can find the integration in (13) as

$$\int_{-\infty}^{\infty} e^{By}\mathrm{sech}\, y dy = \pi\sec t; \quad |\, B\,| < 1, \tag{14}$$

which can be worked out with the help of Maple or Mathematica. Substitute (14) in (13), then the required formula (12) of the mgf of $X_{p\text{-}DHS}$ can be obtained. □

Proposition 6. *The* 1*st four non-central moments of* $X_{p\text{-}DHS}$ *with* $p > 0$ *are given by*

$$\mu_1' = \frac{2}{\pi}\ln\frac{1}{\sqrt{p}}, \quad \mu_2' = 1 + \frac{4}{\pi^2}\left(\ln\frac{1}{\sqrt{p}}\right)^2,$$

$$\mu_3' = \frac{6}{\pi}\ln\frac{1}{\sqrt{p}} + \frac{8}{\pi^3}\left(\ln\frac{1}{\sqrt{p}}\right)^3,$$

$$\mu_4' = 5 + \frac{24}{\pi^2}\left(\ln\frac{1}{\sqrt{p}}\right)^2 + \frac{16}{\pi^4}\left(\ln\frac{1}{\sqrt{p}}\right)^4. \tag{15}$$

□

Note that the formulas (15) can be worked out with the help of Maple or Mathematica. Using (15) and the relation between the central and non-central moments [6], we can obtain the first four central moments $\mu_1, \mu_2, \mu_3, \mu_4$ of $X_{p\text{-}DHS}$ as $\mu_1 = 0, \mu_2 = 1, \mu_3 = 0, \mu_4 = 5.$ This implies that the skewness γ and the excess kurtosis β of $X_{p\text{-}DHS}$ are 0 and 2, respectively. Using the relation between the cf and the mgf, we obtain the cf of the *p*-DHS distribution as

$$\Psi_{p\text{-}DHS}(t;\ p) = e^{\frac{2it}{\pi}\ln\frac{1}{\sqrt{p}}}\operatorname{sech} t; \quad |t| < \frac{\pi}{2}. \tag{16}$$

5. Maximum Likelihood Parameter Estimation

Here, we review the maximum likelihood (ML) method to obtain the MLE for the deformation parameter *p*. Suppose that $X_1, X_2, \ldots, X_n$ are an iid random sample from a *p*-DHS distribution. Then the likelihood function is given by

$$L(x_1, x_2, \ldots, x_n \mid p) = p^{n/2}\prod_{i=1}^{n}\left[p\exp\left(\frac{\pi x_i}{2}\right) + \exp\left(-\frac{\pi x_i}{2}\right)\right]^{-1}. \tag{17}$$

According to [6] and other literatures, computing the log-likelihood function and taking the partial derivative of this function with respect to the deformation parameter p and finally setting the result equal to 0 yields

$$p\sum_{i=1}^{n} \exp\left(\frac{\pi x_i}{2}\right) \operatorname{sech}_p\left(\frac{\pi x_i}{2}\right) = n. \tag{18}$$

The MLE $\hat{p}$ for the parameter p can be deduced by solving (18) iteratively.

6. Summary

We applied the p-deformation technique of the hyperbolic functions to probability distribution by introducing a positive real valued parameter. We considered the so-called p-DHS distribution which arises as deformation for the HS distribution. This is a class of continuous probability distributions in which the shape parameter can be used to introduce skew. We found that the constructed p-DHS distribution is unimodal with unit variance and mean as a function of p. Moreover, the ML method to determine the MLE for the parameter p has been illustrated. It was shown that all moments exist. Moreover, each of the mgf, cf and score function are given in closed forms. Furthermore, some properties of the p-DHS distribution have been illustrated and discussed.

In compare with other previous studies, the DHS distribution, either in the exponential growth or decay part of the corresponding used HS function, with a unit value parameter is symmetric. Moreover, the direction of the skewness of shape of the DHS distribution depends on the deformed exponential part and the value of the deformation parameter. We found that the mean value of the p-DHS density function depends on the deformed exponential part in the studied original HS distribution. In addition, we found that the unimodality of the DHS distribution is valid and independent on the used deformation parameters. Moreover, the mean, the median and the mode for the DHS distribution have the same value in each of the two exponential parts of the HS function. This paper is merely an initial work; more studies to other interesting hyperbolic or triangular probability distributions with applications should be conducted. In the future, we hope to study the deformation technique to the HS distribution with respect to the both exponential growth and the decay parts simultaneously.

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