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# EXPONENTIAL STABILITY OF STOCHASTIC DELAY DIFFERENTIAL EQUATION DRIVEN BY LÉVY NOISE

**Ranjing**

Department of Mathematics
Southwest Jiaotong University
09401002, Chengdu, P. R. China
e-mail: rainy.0902@qq.com

## Abstract

Our main aim is to study $p$th moment exponential stability of stochastic delay differential equation driven by Lévy noise. The key tools such as Itô's formula for general semimartingales and Lyapunov function are used in this paper.

## 1. Introduction

It is well known that stochastic delay differential equation can describe a number of different scientific and practical problems which appear in many different fields such as mathematics, physics, biology, mechanics and financial market. Stability is one of the most important problems of stochastic delay differential equation and many authors have studied about it such as Arnold, Kolmanovskii, Mao, Mohammed have devoted their interests to this area, but there are few results about stochastic delay differential equation driven by Lévy noise.



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One of the most important tools in study of the stability problem is the Lyapunov function. However, it is generally much more difficult to construct the Lyapunov function in the case of delay than the Lyapunov function in the case of non-delay. Mao and Shah [4] compared the stochastic delay differential equation

$$dx(t) = f(t, x(t), x(t-\tau))dt + g(t, x(t), x(t-\tau))dw(t)$$

with the corresponding non-delay equation

$$dx(t) = f(t, x(t)), x(t)dt + g(t, x(t), x(t))dw(t)$$

and assumed that the non-delay equation is exponentially stable. It is proved that the original stochastic delay differential equation remains exponentially stable provided the time lag $\tau$ is sufficiently small, and a bound for such $\tau$ is obtained.

An extensive study of linear stochastic differential equation driven by Lévy noise has been carried our by Li et al. [3] while Grigoriu [2] has studied some special cases (both linear and non-linear) for stochastic differential equation driven by compound Poisson processes. Precisely, Applebaum and Siakalli [5] considered the stochastic differential equation driven by Lévy noise of the form

$$\begin{aligned} dx(t) = {} & f(x(t-))dt + g(x(t-))dw(t) \\ & + \int_{|y|<c} H(x(t-), y)\tilde{N}(dt, dy), \quad t \geq 0, \end{aligned} \tag{1.1}$$

and found conditions under which the solutions to the stochastic differential equation driven by Lévy noise are stable in probability, almost surely and moment exponentially stable. However, the *p*th moment exponentially stable of stochastic delay differential equation driven by Lévy process has not been considered.

The purpose of this paper is to extend Mao and Shah [4] and Applebaum and Siakalli [5] techniques to the case of non-linear stochastic delay differential equation driven by Lévy noise of the form

$$dx(t) = f(x(t-\tau))dt + g(x(t-\tau))dw(t)$$

$$+ \int_{|y|<c} H(x(t), y)\tilde{N}(dt, dy), \quad t \geq 0.$$

Compared the stochastic delay differential equation driven by Lévy noise with the corresponding (1.1) and assumed that the solution of (1.1) is $ph$ moment exponentially. Let $V(t, x): \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, and $V(t, x) \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$. By Itô's formula for general semimartingales, we defined an operator $\mathcal{L}$ associated to equation (1.1) which is given by

$$\mathcal{L}V(t, x) = V_t(t, x) + V_x(t, x) f(x) + \frac{1}{2} trace[g^T(x)V_{xx}(t, x)g(x)]$$

$$+ \int_{|y|<c} [V(t, x) + H(x, y) - V(t, x) - V_x(t, x)H(x, y)]\nu(dy). \quad (1.2)$$

Then, by using Itô's formula for general semimartingales, applications of Brownian motion and the fundamental results about Lévy process, we find the conditions under which the solution of the stochastic differential equation with delay driven by Lévy are $p$th moment exponential stable.

The content of this paper is arranged as follows: In Section 2, some necessary definitions, notation used in this paper are introduced and by establishing a lemma, some sufficient conditions about the $p$th exponential stability are derived.

## 2. Preliminaries and Main Results

In this section, we set some notations that will be used throughout the paper and study the exponential stability of the trivial solution of (2.1).

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space that is equipped with a filtration $(\mathcal{F}_t, t > 0)$ which satisfies the usual hypotheses of right continuity and completeness. Assume that we give an $m$-dimensional $\mathcal{F}_t$-adapted Brownian motion $W = (W(t), t > 0)$ with which

$$W(t) = (w_1(t), w_2(t), \ldots, w_m(t))$$

and an independent $\mathcal{F}_t$-adapted Poisson random measure $N$ defined on $\mathbb{R}^+ \times (\mathbb{R}^m - \{0\})$ with compensator $\tilde{N}$ of the form $\tilde{N}(dt, dy) = N(dt, dy) - \nu(dy)dt$ and $\int_{\mathbb{R}^m - \{0\}} (|y|^2 \wedge 1)\nu(dy) < \infty$, where $\nu$ is a Lévy measure. We consider stochastic delay differential equation driven by Lévy noise of the form

$$dx(t) = f(x(t-\tau))dt + g(x(t-\tau))dw(t) + \int_{|y|<c} H(x(t), y)\tilde{N}(dt, dy), \quad t \geq 0 \tag{2.1}$$

with initial value $x(t) = \xi(t)\ (-\tau \leq t \leq 0)$. Here $\{\xi(t) : -\tau \leq t \leq 0\} \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$. Denote by $L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ the family of $\mathbb{R}^n$-valued stochastic processes $\xi(s, \omega), -\tau \leq t \leq 0$ such that $\xi(s, \omega)$ is $\mathcal{B}([-\tau, 0]) \times \mathcal{F}_0$-measurable and $\int_{-\tau}^0 E|\xi(s)|^p ds < \infty$. Assume that the mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the usual global Lipschitz and growth conditions. $c \in (0, \infty]$ is the maximum allowable jump size. We assume that $f(0) = 0$, $g(0) = 0$ and $H(0, y) = 0$ for all $|y| < c$, then (2.1) has a unique solution $x(t; 0) = 0$ for all $t \geq 0$, which is called the *trivial solution* or simply *solution* of (2.1). The trivial solution of (2.1) is said to be $p$th *moment exponential stable* if there is a constant $C > 0$ and $\lambda > 0$ such that

$$E[|x(t; \xi)|^p] \leq C|x_0|^p \exp(-\lambda(t - t_0))$$

for all $t \geq 0$, $\xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$. We call the quantity

$$\limsup_{t\to\infty} \frac{1}{t} \log(E|x(t; \xi)|^p)$$

the *pth moment Lyapunov exponent*.

Now, we will study the $p$th moment exponential stability of the trivial solution of (2.1). For this propose, we will provide some assumptions on $f(x)$, $g(x)$ and $V(t, x)$ as following:

(1) Let $V(t, x) \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$. Then there exist three constants $k_1, k_2, k_3 > 0$ such that

(i) $k_1 |x|^p \leq V(t, x) \leq k_2 |x|^p$,

(ii) $\mathcal{L}V(t, x) \leq -k_3 |x|^p$.

(2) For $f: \mathbb{R}^n \to \mathbb{R}^n$, $g: \mathbb{R}^n \to \mathbb{R}^{n\times m}$, there exist two constants $\theta_1, \alpha_1 > 0$ such that

(iii) $|f(x) - f(y)| \leq \theta_1 |x - y|$,

(iv) $trace\{[g(x) - g(y)]^T [g(x) - g(y)]\} \leq \alpha_1^2 |x - y|^2$.

**Lemma 2.1.** *Let* $u, v, \varepsilon_1, \varepsilon_2$ *be positive numbers. Then*

$$u^{p-1}v \leq \frac{\varepsilon_1(p-1)u^p}{p} + \frac{v^p}{p\varepsilon_1^{p-1}}, \tag{2.2}$$

$$u^{p-2}v^2 \leq \frac{\varepsilon_2(p-2)u^p}{p} + \frac{2v^p}{p\varepsilon_2^{\frac{p-2}{2}}}. \tag{2.3}$$

This lemma can easily be proved. Here we omit the details.

**Lemma 2.2.** *Assume* $f(x)$, $g(x)$ *satisfy the assumptions* (1) *and* (2). *Write* $x(t; \xi) = x(t)$ *simply. Then*

$$\int_0^t e^{\varepsilon s} E|x(s) - x(s-\tau)|^p ds \leq c_1 + \tau e^{\varepsilon t}\eta_1 \int_0^t e^{\varepsilon s} E|x(s)|^p ds \tag{2.4}$$

*for all* $t \geq 0$, *where*

$$\eta_1 = \theta_1^p (2\tau)^{p-1} + \alpha_1^p (2\tau)^{\frac{p-2}{2}} [p(p-1)]^{\frac{p}{2}}$$

*and* $c_1$ *is a constant larger than* $\int_0^\tau e^{\varepsilon s} E|x(s) - x(s-\tau)|^p ds$.

The proof of this lemma is as Lemma 2.4 in Mao and Shah [4].

**Theorem 2.3.** *Suppose that the assumptions* (1) *and* (2) *are satisfied. Then the trivial solution of* (2.1) *is pth moment exponentially stable if*

$$\tau < \frac{1}{4\theta_1^2} \left(\sqrt{\beta_3^2 + 2^{p+1}\theta_1^p \beta_2} - \beta_3\right)^{\frac{2}{p}},$$

*where*

$$\beta_1 = k_2 p\theta_1 + k_2 p(p-1)\alpha_1^2,$$

$$\beta_2 = \begin{cases} \left( \dfrac{\sqrt{\beta_1^2 + 2k_2 k_3 p(p-1)\alpha_1^2} - \beta_1}{k_2 p(p-1)\alpha_1^2} \right)^p, & \alpha_1 \neq 0, \\ \left( \dfrac{k_3}{k_2 p\theta_1} \right)^p, & \alpha_1 = 0, \end{cases}$$

$$\beta_3 = 2^{\frac{p-2}{2}} \alpha_1^p [p(p-1)]^{\frac{p}{2}}$$

*for all* $x, y \in \mathbb{R}^n$.

**Proof.** Note $x(t; \xi) = x(t)$, by Itô's formula for general semimartingales

$$dV(t, x(t)) = \Big\{ V_t(t, x(t)) + V_x(t, x(t)) f(x(t-\tau))$$

$$+ \frac{1}{2} trace[g^T(x(t-\tau)) V_{xx}(t, x(t)) g(x(t-\tau))] \Big\} dt$$

$$+ V_x(t, x(t)) g(x(t-\tau)) dw(t) + \int_{|y|<c} [V(t, x(t) + H(x(t), y))$$

$$- V(t,\ x(t)) - V_x(t,\ x(t))H(x(t),\ y)]\nu(dy)ds$$

$$+ \int_{|y|<c} [V(t,\ x(t) + H(x(t),\ y)) - V(t,\ x(t))]\tilde{N}(ds,\ dy). \quad (2.5)$$

From the assumptions (i) and (iii), we have

$$\begin{aligned} &V_x(t,\ x(t))\, f(x(t-\tau)) \\ &\le V_x(t,\ x(t))\theta_1 |\, x(t) - x(t-\tau)\,| + V_x(t,\ x(t))\, f(x(t)) \\ &\le k_2 p\theta_1 |\, x(t)\,|^p |\, x(t) - x(t-\tau)\,| + V_x(t,\ x(t))\, f(x(t)). \end{aligned} \quad (2.6)$$

Also, note that $gx(t) = g_1,\ gx(t-\tau) = g_2,$ from the assumptions (i) and (iv), we obtain

$$\begin{aligned} trace[g_2^T V_{xx} g_2] &= trace\{[g_1^T - (g_1 - g_2)^T]V_{xx}[g_1 - (g_1 - g_2)]\} \\ &= trace(g_1^T V_{xx} g_1) - 2trace[g_1^T V_{xx}(g_1 - g_2)] \\ &\quad + trace[(g_1 - g_2)^T V_{xx}(g_1 - g_2)] \\ &\le trace(g_1^T V_{xx} g_1) + \| V_{xx} \| trace[(g_1 - g_2)^T (g_1 - g_2)] \\ &\quad + 2\| V_{xx} \| \{trace(g_1^T g_1) trace[(g_1 - g_2)^T (g_1 - g_2)]\}^{\frac{1}{2}} \\ &\le trace(g_1^T V_{xx} g_1) + 2p(p-1)k_2\alpha_1^2 |\, x(t)\,|^{p-1} |\, x(t) - x(t-\tau)\,| \\ &\quad + p(p-1)k_2\alpha_1^2 |\, x(t)\,|^{p-2} |\, x(t) - x(t-\tau)\,|^2. \end{aligned} \quad (2.7)$$

Then, combining (2.5), (2.6), (2.7) and using the linear operator $\mathcal{L}$ defined in (1.2), we arrive at

$$\begin{aligned} &dV(t,\ x(t)) \\ &\le \Big\{ \mathcal{L}V(t,\ x(t)) + (k_2 p\theta_1 + 2p(p-1)k_2\alpha_1^2) |\, x(t)\,|^{p-1} |\, x(t) - x(t-\tau)\,| \\ &\quad + \frac{1}{2} p(p-1)k_2\alpha_1^2 |\, x(t)\,|^{p-2} |\, x(t) - x(t-\tau)\,|^2 \Big\} dt \end{aligned}$$

$$+ V_x(t, x(t)) g(x(t-\tau)) dw(t)$$

$$+ \int_{|y|<c} [V(t, x(t) + H(x(t), y)) - V(t, x(t))] \tilde{N}(ds, dy). \quad (2.8)$$

Applying Lemma 2.1 and assumption (ii), we obtain

$$dV(t, x(t)) \le [(\eta_2 - k_3)|x(t)|^p + \eta_3 |x(t) - x(t-\tau)|^p] dt$$

$$+ V_x(t, x(t)) g(x(t-\tau)) dw(t)$$

$$+ \int_{|y|<c} [V(t, x(t) + H(x(t), y)) - V(t, x(t))] \tilde{N}(ds, dy), \quad (2.9)$$

where

$$\eta_2 = \frac{\varepsilon_1 (p-1)}{p} [k_2 p \theta_1 + p(p-1) k_2 \alpha_1^2] + \frac{k_2 \varepsilon_2 \alpha_1^2 (p-1)(p-2)}{2p}$$

$$\eta_3 = \frac{1}{p \varepsilon_1^{p-1}} [k_2 p \theta_1 + p(p-1) k_2 \alpha_1^2] + \frac{k_2 \alpha_1^2 p(p-1)}{p \varepsilon_2^{\frac{p-2}{2}}}.$$

From Theorem 41 in Protter [1, p. 30] and applications of Brownian motion. Fix the initial data $\xi$ arbitrarily and write $x(t, \xi) = x(t)$ simply. For any $\varepsilon > 0$ by Lemma 2.2, we obtain

$$E[e^{\varepsilon t} V(t, x(t))] \le EV(0, \xi(0)) + (k_2 \varepsilon + \eta_2 - k_3) \int_0^t e^{\varepsilon s} E|x(s)|^p ds$$

$$+ \eta_3 \int_0^t e^{\varepsilon s} E|x(s) - x(s-\tau)|^p ds$$

$$\le EV(0, \xi(0)) + c_1 \eta_3$$

$$+ (k_2 \varepsilon + \eta_2 - k_3 + \tau \eta_1 \eta_3 e^{\varepsilon \tau}) \int_0^t e^{\varepsilon s} E|x(s)|^p ds. \quad (2.10)$$

Assume $\tilde{\tau}$ is the root to the equation of $\tau$,

$$\tau \eta_1 \eta_3 + \eta_2 - k_3 = 0. \quad (2.11)$$

Actually, equation (2.11) is a parabolic equation of $\tau^{\frac{p}{2}}$ that is

$$2^{p-1}\eta_3\theta_1^p(\tau^{\frac{p}{2}})^2 + 2^{\frac{p-2}{2}}\alpha_1^p[p(p-1)]^{\frac{p}{2}}\tau^{\frac{p}{2}} + \eta_2 - k_3 = 0. \tag{2.12}$$

One can verify that

$$\tilde{\tau} = \frac{1}{4\theta_1^2}(\sqrt{\beta_3^2 + 2^{p+1}\theta_1^p\beta_2} - \beta_3)^{\frac{2}{p}}$$

is the unique root on $\mathbb{R}_+$ to equation (2.12) and $\tau\eta_1\eta_3 + \eta_2$ is an increasing function on $0 < \tau < \tilde{\tau}$. Obviously, we can find an $\varepsilon > 0$ such that

$$k_2\varepsilon + \eta_2 - k_3 + \tau\eta_1\eta_3 e^{\varepsilon\tau} = 0. \tag{2.13}$$

Consequently,

$$E|x(t)|^p \leq \frac{1}{k_1}[EV(0, \xi(0)) + c_1\eta_3]e^{-\varepsilon t},$$

that is

$$\limsup_{t\to\infty}\frac{1}{t}\log(E|x(t;\xi)|^p) \leq -\varepsilon. \tag{2.14}$$

This completes the proof.

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