Far East Journal of Applied Mathematics
Volume 62, Number 2, 2012, Pages 117-128
Published Online: March 2012
Available online at http://pphmj.com/journals/fjam.htm
Published by Pushpa Publishing House, Allahabad, INDIA

THE SINE-COSINE METHOD FOR GENERALIZED KdV EQUATION WITH GENERALIZED EVOLUTION

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Abstract

In this paper, we study a class generalized KdV equation with generalized evolution by using sine-cosine method. As a result, more types of new exact solutions to the generalized KdV equation with generalized evolution are obtained, which include more general soliton solutions, compactons solutions and solitary patterns solutions.

1. Introduction

Studies of various physical structures of nonlinear dispersive equations had attracted much attention in connection with the important problems that

2010 Mathematics Subject Classification: 74J35, 35Q51.

Keywords and phrases: solitary wave, compactons, solitary patterns solutions, generalized KdV equation with generalized evolution.

This research is supported by the National Natural Science Foundation of China (11061010, 11161013) and the Science Foundation of Guangxi province (No. 2011jjA10047).

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Received December 27, 2011

arise in scientific applications. Mathematically, these physical structures have been studied by using various analytical methods such as inverse scattering method [1], Darboux transformation method [2], Hirota bilinear method [3], Fan-expansion method [4] and so on. Practically, there is no unified technique that can be employed to handle all types of nonlinear differential equations.

Recently, Ismail and Biswas [5] studied the following generalized Korteweg-de Vries equation with generalized evolution (the GKdV$(1, n)$ equation in short)

$$(q^l)_t + aq(q^n)_x + b[q(q^n)_{xx}]_x + cq(q^n)_{xxx} = 0, \tag{1.1}$$

where a, b, c are arbitrary constants and obtained the topological 1-soliton solution by using the solitary wave ansätz method. More recently, Sturdevant and Biswas [6] studied equation (1.1) and the topological 1-soliton solution by using another solitary wave ansätz method. These equations with $l = 1$ first appeared in 2004 [7]. It was studied by Wazwaz with $l = 1$ and a number of soliton solutions were obtained.

In this paper, we will study equation (1.1) by using the sine-cosine method and the GKdV$(-1, -n)$ equation will be examined as well. The sine-cosine method is one of most direct and effective algebraic methods for finding exact solutions of nonlinear diffusion equations (see [7, 8] and the references therein). The sine-cosine algorithm, that provides a systematic framework for many nonlinear dispersive equations, will be employed to back up our analysis to determine solitons, compactons and solitary patterns traveling wave solutions.

In what follows, we highlight the main steps of the sine-cosine algorithm.

2. The Sine-cosine Method

(1) We introduce the wave variable $\xi = (x - vt)$ into the nonlinear PDE

$$P(q, q_t, q_x, q_{xx}, q_{xxx}, ...) = 0, \tag{2.1}$$

where $q(x, t)$ is the traveling wave solution. This enables us to use the following changes:

$$\frac{\partial}{\partial t} = -v\frac{d}{dt}, \quad \frac{\partial^2}{\partial t^2} = v^2\frac{d^2}{dt^2}, \quad \frac{\partial}{\partial x} = \frac{d}{d\xi}, \quad \frac{\partial^2}{\partial x^2} = \frac{d^2}{d\xi^2}. \tag{2.2}$$

We can immediately reduce the nonlinear PDE (2.1) into a nonlinear ODE

$$Q(q, q', q'', q''', ...) = 0. \tag{2.3}$$

The ordinary differential equation (2.3) is then integrated as long as all terms contain derivatives, where we neglect integration constants.

(2) The sine-cosine algorithm admits the use of the ansätz

$$q(x, t) = \begin{cases} A\cos^p(B\xi), & |\xi| \le \frac{\pi}{2B}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.4}$$

or the ansätz

$$q(x, t) = \begin{cases} A\sin^p(B\xi), & |\xi| \le \frac{\pi}{B}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.5}$$

where A, B and p are parameters that will be determined.

(3) Substituting (2.4) or (2.5) into the reduced ODE obtained above after integrating (2.3) gives a trigonometric equation of cosine or sine terms.

(4) The main task is to balance the exponents of the trigonometric functions cosine or sine. Collect all terms with same power in $\cos^k(B\xi)$ or $\sin^k(B\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknowns A, B and p. The problem is now completely reduced to an algebraic one. Having determined A, B and p by algebraic calculations or by using computerized symbolic calculations, the solutions proposed in (2.4) and in (2.5) follow immediately. The algorithm described above certainly works well for a large class of very interesting nonlinear equations.

3. Using the Sine-cosine Method

3.1. For positive exponents

We first consider the GKdV$(1, n)$ equation

$$(q^l)_t + aq(q^n)_x + b[q(q^n)_{xx}]_x + cq(q^n)_{xxx} = 0, \quad l, n \in Z^+. \tag{3.1}$$

Using the wave variable $\xi = x - vt$ carries (3.1) into the ODE

$$-v(q^l)' + aq(q^n)' + b[q(q^n)'']' + cq(q^n)''' = 0. \tag{3.2}$$

Substituting (2.4) into (3.2) gives

$$\begin{aligned} &A^l Blpv \cos^{lp-1}(B\xi) \\ &+ A^{n+1}Bnp\{-a + B^2n^2p^2[b(n+1) + cn]\}\cos^{(n+1)p-1}(B\xi) \\ &-A^{n+1}B^3np(np-1)\{b[(n+1)p - 2] + c(np-2)\}\cos^{(n+1)p-3}(B\xi) = 0. \end{aligned} \tag{3.3}$$

Equation (3.3) is satisfied only if the following system of algebraic equations holds:

$$\begin{aligned} &lp - 1 = (n+1)p - 3, \quad np - 1 \neq 0, \\ &A^{n+1}Bnp\{-a + B^2n^2p^2[b(n+1) + cn]\} = 0, \\ &A^l Blpv - A^{n+1}B^3np(np-1)\{b[(n+1)p - 2] + c(np-2)\} = 0, \end{aligned} \tag{3.4}$$

or

$$\begin{aligned} &lp - 1 = (n+1)p - 1, \\ &-A^{n+1}B^3np(np-1)\{b[(n+1)p - 2] + c(np-2)\} = 0, \\ &A^l Blpv + A^{n+1}Bnp\{-a + B^2n^2p^2[b(n+1) + cn]\} = 0. \end{aligned} \tag{3.5}$$

Solving the system (3.4) and (3.5) gives

Case I.

$$p = \frac{2}{n+1-l},\ B^2 = \frac{a(n+1-l)^2}{4n(bn+b+cn)},\ v = \frac{aA^{n+1-l}(n-1+l)(bl+cl-c)}{2l(bn+b+cn)},$$

$$n+1 \neq l,\quad (bl+cl-c)(bn+cn+b) \neq 0,\quad A = A \neq 0. \tag{3.6}$$

Case II.

$$l = n+1,\ p = \frac{2(b+c)}{bn+b+cn},\ B^2 = \frac{[an-v(n+1)](bn+cn+b)}{4n^2(b+c)^2},$$

$$np-1 \neq 0,\quad [v(n+1)-an](bn+cn+b) \neq 0,\quad b+c \neq 0. \tag{3.7}$$

Case III.

$$l = n+1,\quad p = \frac{1}{n},\quad B^2 = \frac{an-v(n+1)}{bn+cn+b},$$

$$v(n+1)-an \neq 0,\quad bn+cn+b \neq 0,\quad A = A \neq 0. \tag{3.8}$$

The results (3.6)-(3.8) can be easily obtained if we also use the sine method (2.5). Combining (3.6)-(3.8) with (2.4) and (2.5), we obtain the following compactons solutions:

$$q_1 = \begin{cases} A\cos^{\frac{2}{n+1-l}}\left[\dfrac{n+1-l}{2}\sqrt{\dfrac{a}{n(bn+cn+b)}}\left(x - \dfrac{aA^{n+1-l}(n-1+l)(bl+cl-c)}{2l(bn+cn+b)}t\right)\right], \\ 0 < |\,x-vt\,| < \dfrac{\pi}{|\,n+1-l\,|}\sqrt{\dfrac{a}{n(bn+cn+b)}},\ a(bn+cn+b) > 0,\ v = v,\ A = A, \\ 0, \qquad \text{otherwise}, \end{cases}$$

$$\tag{3.9}$$

$$q_2 = \begin{cases} A\cos^{\frac{2(b+c)}{b(n+1)+cn}}\left[\dfrac{1}{2n(b+c)}\sqrt{[an-v(n+1)](bn+cn+b)}(x-vt)\right],\ A = A,\ v = v, \\ [an-v(n+1)](bn+cn+b) > 0,\ 0 < |\,x-vt\,| < \dfrac{|\,b+c\,|\pi}{\sqrt{[an-v(n+1)](bn+cn+b)}}, \\ 0, \qquad \text{otherwise}, \end{cases}$$

$$\tag{3.10}$$

$$q_3 = \begin{cases} A\cos^{\frac{1}{n}}\left[\sqrt{\dfrac{an-v(n+1)}{bn+cn+b}}(x-vt)\right], 0<|x-vt|<\dfrac{\pi}{2}\sqrt{\dfrac{bn+cn+b}{an-v(n+1)}}, \\ A=A, v=v, (bn+cn+b)(an-v(n+1))>0, \\ 0, \qquad \text{otherwise} \end{cases} \tag{3.11}$$

and

$$q_4 = \begin{cases} A\sin^{\frac{2}{n+1-l}}\left[\dfrac{n+1-l}{2}\sqrt{\dfrac{a}{n(bn+cn+b)}}\left(x-\dfrac{aA^{n+1-l}(n-1+l)(bl+cl-c)}{2l(bn+cn+b)}t\right)\right], \\ 0<|x-vt|<\dfrac{2\pi}{|n+1-l|}\sqrt{\dfrac{a}{n(bn+cn+b)}}, a(bn+cn+b)>0, v=v, A=A, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{3.12}$$

$$q_5 = \begin{cases} A\sin^{\frac{2(b+c)}{b(n+1)+cn}}\left[\dfrac{1}{2n(b+c)}\sqrt{[an-v(n+1)](bn+cn+b)}(x-vt)\right], v=v, A=A, \\ 0<|x-vt|<\dfrac{2|b+c|\pi}{\sqrt{[an-v(n+1)](bn+cn+b)}}, [an-v(n+1)](bn+cn+b)>0, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{3.13}$$

$$q_6 = \begin{cases} A\sin^{\frac{1}{n}}\left[\sqrt{\dfrac{an-v(n+1)}{bn+cn^2+b}}(x-vt)\right], 0<|x-vt|<\pi\sqrt{\dfrac{bn+cn^2+b}{an-v(n+1)}}, \\ A=A, v=v, (bn+cn^2+b)(an-v(n+1))>0, \\ 0, \qquad \text{otherwise}. \end{cases} \tag{3.14}$$

However, for $B^2<0$, we obtain the following solitary patterns solutions:

$$q_7 = A\cosh^{\frac{2}{n+1-l}} \cdot \left[\frac{n+1-l}{2}\sqrt{\frac{-a}{n(bn+cn+b)}}\left(x - \frac{aA^{n+1-l}(n-1+l)(bl+cl-c)}{2l(bn+cn+b)}t\right)\right],$$

$$a(bn+cn+b) < 0, \tag{3.15}$$

$$q_8 = A\cosh^{\frac{2(b+c)}{b(n+1)+cn}}\left[\frac{1}{2n(b+c)}\sqrt{[an - v(n+1)](bn+cn+b)}(x - vt)\right],$$

$$[an - v(n+1)](bn+cn+b) > 0, \tag{3.16}$$

$$q_9 = A\cosh^{\frac{1}{n}}\left[\sqrt{\frac{v(n+1)-an}{bn+cn+b}}(x - vt)\right],$$

$$(v(n+1) - an)(bn+cn+b) > 0, \tag{3.17}$$

and

$$q_{10} = -A\sinh^{\frac{2}{n+1-l}} \cdot \left[\frac{n+1-l}{2}\sqrt{\frac{-a}{n(bn+cn+b)}}\left(x - \frac{aA^{n+1-l}(n-1+l)(bl+cl-c)}{2l(bn+cn+b)}t\right)\right],$$

$$a(bn+cn+b) < 0, \tag{3.18}$$

$$q_{11} = -A\sinh^{\frac{2(b+c)}{b(n+1)+cn}}\left[\frac{1}{2n(b+c)}\sqrt{[an - v(n+1)](bn+cn+b)}(x - vt)\right],$$

$$[an - v(n+1)](bn+cn+b) > 0. \tag{3.19}$$

Remark 1. To the best of our knowledge, solutions (3.9)-(3.14) and (3.16)-(3.19) obtained for equation (1.1) have not been reported in literature.

3.2. For negative exponents

We consider the variant GKdV$(-1, -n)$ equation

$$(q^{-l})_t + aq(q^{-n})_x + b[q(q^{-n})_{xx}]_x + cq(q^{-n})_{xxx} = 0, \quad l, n, \in Z^+. \tag{3.20}$$

Using the wave variable $\xi = x - vt$ carries (3.20) into the ODE

$$-v(q^{-l})' + aq(q^{-n})' + b[q(q^{-n})'']' + cq(q^{-n})''' = 0. \tag{3.21}$$

Substituting (2.4) into (3.21) gives

$$-A^{-l}Blpv\cos^{-lp-1}(B\xi)$$

$$+ A^{1-n}Bnp\{a + B^2n^2p^2[b(1-n) - cn]\}\cos^{(1-n)p-1}(B\xi)$$

$$+ A^{1-n}B^3np(np+1)\{b[2-(1-n)p] + c(np+2)\}\cos^{(1-n)p-3}(B\xi) = 0. \tag{3.22}$$

Equation (3.22) is satisfied only if the following system of algebraic equations holds:

$$-lp - 1 = (1-n)p - 3, \quad -np - 1 \neq 0,$$

$$-A^{-l}Blpv + A^{1-n}B^3np(np+1)\{b[(2-1-n)p] + c(np+2)\} = 0,$$

$$A^{1-n}Bnp\{a + B^2n^2p^2[b - bn - cn]\} = 0, \tag{3.23}$$

or

$$lp - 1 = (1-n)p - 1,$$

$$-A^{-l}Blpv + A^{1-n}Bnp\{a + B^2n^2p^2[b(1-n) - cn]\} = 0,$$

$$A^{1-n}B^3np(np+1)\{b[2-(1-n)p] + c(np+2)\} = 0. \tag{3.24}$$

Solving the system (3.23) and (3.24) gives

Case I.

$$p = \frac{2}{l+1-n}, \; B^2 = \frac{a(l+1-n)^2}{4n(bn-b+cn)}, \; v = \frac{aA^{l+1-n}(n+1+l)(bl+cl+c)}{2l(bn-b+cn)},$$

$$l+1 \neq n, \quad (bl+cl+c)(bn-b+cn) \neq 0, \quad A = A \neq 0. \tag{3.25}$$

Case II.

$$l+1=n,\quad p=-\frac{2(b+c)}{bn-b+cn},\quad B^2=\frac{[an+v(1-n)](bn-b+cn)}{4n^2(b+c)^2},$$

$$A=A\neq 0,\quad (np-1)[an+v(1-n)](bn-b+cn)(b+c)\neq 0. \tag{3.26}$$

Case III.

$$l+1=n,\quad p=-\frac{1}{n},\quad B^2=\frac{an+v(1-n)}{bn-b+cn},$$

$$A=A\neq 0,\quad (an+v(1-n))(bn-b+cn)\neq 0. \tag{3.27}$$

The results (3.25)-(3.27) can be easily obtained if we also use the sine method (2.5). Combining (3.25)-(3.27) with (2.4) and (2.5), we obtain the following compactons solutions:

$$q_1=\begin{cases} A\cos^{\frac{2}{l+1-n}}\left[\frac{l+1-n}{2}\sqrt{\frac{a}{n(bn-b+cn)}}\left(x-\frac{aA^{l+1-n}(n+1+l)(bl+cl+c)}{2l(bn-b+cn)}t\right)\right], \\ 0<|x-vt|<\frac{\pi}{|l+1-n|}\sqrt{\frac{n(bn-b+cn)}{a}},\ a(bn-b+cn)>0, \\ 0, \qquad\qquad \text{otherwise,} \end{cases}$$

(3.28)

$$q_2=\begin{cases} A\cos^{-\frac{2(b+c)}{bn-b+cn}}\left[\frac{1}{2n(b+c)}\sqrt{[an+v(1-n)](bn-b+cn)}(x-vt)\right],\ A=A,\ v=v, \\ [an+v(1-n)](bn-b+cn)>0,\ 0<|x-vt|<\frac{n|b+c|\pi}{\sqrt{[an+v(1-n)](bn-b+cn)}}, \\ 0, \qquad\qquad \text{otherwise,} \end{cases}$$

(3.29)

$$q_3 = \begin{cases} A\cos^{-\frac{1}{n}}\left[\sqrt{\dfrac{an+v(1-n)}{bn-b+cn}}(x-vt)\right], & 0<|x-vt|<\dfrac{\pi}{2}\sqrt{\dfrac{bn-b+cn}{an+v(1-n)}}, \\ A=A,\ v=v,\ (bn-b+cn)(an+v(1-n))>0, & \\ 0, & \text{otherwise} \end{cases} \tag{3.30}$$

and

$$q_4 = \begin{cases} A\sin^{\frac{2}{l+1-n}}\left[\dfrac{l+1-n}{2}\sqrt{\dfrac{a}{n(bn-b+cn)}}\left(x-\dfrac{aA^{l+1-n}(n+1+l)(bl+cl+c)}{2l(bn-b+cn)}t\right)\right], \\ 0<|x-vt|<\dfrac{2\pi}{|l+1-n|}\sqrt{\dfrac{n(bn-b+cn)}{a}},\ a(bn-b+cn)>0, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{3.31}$$

$$q_5 = \begin{cases} A\sin^{-\frac{2(b+c)}{bn-b+cn}}\left[\dfrac{1}{2n(b+c)}\sqrt{[an+v(1-n)](bn-b+cn)}(x-vt)\right],\ A=A,\ v=v, \\ [an+v(1-n)](bn-b+cn)>0,\ 0<|x-vt|<\dfrac{2n|b+c|\pi}{\sqrt{[an+v(1-n)](bn-b+cn)}}, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{3.32}$$

$$q_6 = \begin{cases} A\sin^{-\frac{1}{n}}\left[\sqrt{\dfrac{an+v(1-n)}{bn-b+cn}}(x-vt)\right], & 0<|x-vt|<\pi\sqrt{\dfrac{bn-b+cn}{an+v(1-n)}}, \\ A=A,\ v=v,\ (bn-b+cn)(an+v(1-n))>0, & \\ 0, & \text{otherwise}. \end{cases} \tag{3.33}$$

However, for $B^2<0$, we obtain the following solitary patterns solutions:

$$q_7 = A\cosh^{\frac{2}{l+1-n}}$$

$$\cdot\left[\frac{l+1-n}{2}\sqrt{\frac{-a}{n(bn-b+cn)}}\left(x - \frac{aA^{l+1-n}(n+1+l)(bl+cl+c)}{2l(bn-b+cn)}t\right)\right],$$

$$a(bn-b+cn) < 0, \tag{3.34}$$

$$q_8 = A\cosh^{-\frac{2(b+c)}{bn-b+cn}}\left[\frac{1}{2n(b+c)}\sqrt{-[an+v(1-n)](bn-b+cn)}(x-vt)\right],$$

$$[an+v(1-n)](bn-b+cn) < 0, \tag{3.35}$$

$$q_9 = A\cosh^{-\frac{1}{n}}\left[\sqrt{-\frac{an+v(1-n)}{bn-b+cn}}(x-vt)\right],$$

$$(an+v(1-n))(bn-b+cn) > 0 \tag{3.36}$$

and

$$q_{10} = -A\sinh^{\frac{2}{l+1-n}}$$

$$\cdot\left[\frac{l+1-n}{2}\sqrt{\frac{-a}{n(bn-b+cn)}}\left(x - \frac{aA^{l+1-n}(n+1+l)(bl+cl+c)}{2l(bn-b+cn)}t\right)\right],$$

$$a(bn-b+cn) < 0, \tag{3.37}$$

$$q_{11} = -A\sinh^{-\frac{2(b+c)}{bn-b+cn}}\left[\frac{1}{2n(b+c)}\sqrt{-[an+v(1-n)](bn-b+cn)}(x-vt)\right],$$

$$[an+v(1-n)](bn-b+cn) < 0. \tag{3.38}$$

4. Discussion

The solitary wave solutions and compactons for the GKdV$(1, n)$ equation and the GKdV$(-1, -n)$ equation are obtained analytically by using the sine-cosine method. The obtained results in this work clearly demonstrate the effect of the purely nonlinear dispersion and the qualitative change made in the genuinely nonlinear phenomenon. This approach may be applied to seek traveling wave solutions for other types of nonlinear dispersion partial differential equations which satisfy certain restrictions.

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