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ON THE MULTIVARIATE γ-ORDERED NORMAL DISTRIBUTION

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Abstract

The aim of this paper is to introduce and discuss a multivariate (and elliptically contoured) generalization of the γ-ordered normal distribution. This new family of generalized Normals includes a number of well known distributions such as the multivariate uniform, Normal, Laplace and the degenerated Dirac distributions. The moments and characteristic function of this are also discussed.

1. Introduction

This paper analyzes the properties of the family of γ-ordered Normal distributions $\mathcal{N}_\gamma^p(\mu, \Sigma)$ which in the univariate form is discussed in [7].

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This distribution emerged from the study of the Logarithmic Sobolev Inequalities (LSI) as an extremal for the generalized entropy type measure of information, see [5, 6], and generalized Normal distribution as discussed in [4, 9, 8].

This generalization is obtained as an extremal of the LSI corresponding to a power generalization of the entropy type Fisher's information measure. We comment that the introduced univariate γ-ordered Normal $\mathcal{N}_\gamma^1(\mu, \sigma^2)$ coincides with the existent generalized normal distribution introduced in [12] with density function

$$f(x \mid \mu, \alpha, \beta) = \frac{\beta}{2\alpha\Gamma(1/\beta)} \exp\left\{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}\right\},$$

where $\alpha = \left(\frac{\gamma}{\gamma-1}\right)^{(\gamma-1)/\gamma} \sigma$ and $\beta = \frac{\gamma}{\gamma-1}$, while the multivariate case of the γ-ordered Normal $\mathcal{N}_\gamma^p(\mu, \Sigma)$ coincides with the existent multivariate power exponential distribution $\mathcal{PE}^p(\mu, \Sigma', \beta)$, as introduced in [3], where $\Sigma' = 2^{2(\gamma-1)/\gamma}\Sigma$ and $\beta = \frac{\gamma}{2(\gamma-1)}$. These existent generalizations are technically obtained (involving an extra power parameter β) and not as a theoretical result of a strong mathematical background as the Logarithmic Sobolev Inequalities offer.

The $\mathcal{N}_\gamma^p(\mu, \Sigma)$ family of distributions includes, as special cases, the multivariate and elliptically contoured Uniform, Normal, Laplace and the degenerate distributions as the Dirac or the vanishing one, namely, between $\mathcal{U}^p(\mu, \Sigma)$, $\mathcal{N}^p(\mu, \Sigma)$, $\mathcal{L}^p(\mu, \Sigma)$ and $\mathcal{D}^p(\mu)$ or $\mathcal{O}^p$ distribution, with density functions given by

$$f_{\mathcal{U}}(x \mid \mu, \Sigma) = \begin{cases} \dfrac{\Gamma\left(\frac{p}{2}+1\right)}{\pi^{p/2}\sqrt{|\det\Sigma|}}, & x \in \mathbb{R}^p \text{ with } Q(x) \le 1, \\ 0, & x \in \mathbb{R}^p \text{ with } Q(x) > 1, \end{cases} \tag{1}$$

$$f_{\mathcal{N}}(x \mid \mu, \Sigma) = \frac{1}{(2\pi)^{p/2}\sqrt{|\det \Sigma|}} \exp\left\{-\frac{1}{2} Q(x)\right\}, \quad x \in \mathbb{R}^p, \tag{2}$$

$$f_{\mathcal{L}}(x \mid \mu, \Sigma) = \frac{\Gamma\left(\frac{p}{2}+1\right)}{\pi^{p/2} p! \sqrt{|\det \Sigma|}} \exp\{-\sqrt{Q(x)}\}, \quad x \in \mathbb{R}^p, \tag{3}$$

$$f_{\mathcal{D}}(x \mid \mu) = \begin{cases} +\infty & x = \mu, \\ 0, & x \neq \mu, \end{cases} \quad x \in \mathbb{R}^p, \tag{4}$$

$$f_{\mathcal{O}}(x) = 0, \quad x \in \mathbb{R}^p, \tag{5}$$

respectively. In Section 2, a detailed study of the above classification is provided. The heavy{tailed behavior of this family is also analyzed in Section 2, while in Section 3, a compact form of the characteristic function of this family of distributions is presented and extensively discussed.

2. The γ-ordered Normal Distribution

The multivariate and elliptically contoured γ-ordered Normal distribution is defined as follows [5].

Definition 2.1. The p-dimensional random variable X follows the γ-ordered Normal $\mathcal{N}_\gamma^p(\mu, \Sigma)$ with mean μ and scale parameter matrix Σ when the density function f_X is of the form

$$f_X(x \mid \mu, \Sigma, \gamma) = C_\gamma^p |\det \Sigma|^{-1/2} \exp\left\{-\frac{\gamma-1}{\gamma} Q(x)^{\frac{\gamma}{2(\gamma-1)}}\right\}, \quad x \in \mathbb{R}^p, \tag{6}$$

with Q the quadratic form $Q(x) = (x-\mu)\Sigma^{-1}(x-\mu)^T$. We shall write $X \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$. The normality factor C_γ^p is defined as

$$C_\gamma^p = \pi^{-p/2} \frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(p\frac{\gamma-1}{\gamma}+1\right)} \left(\frac{\gamma-1}{\gamma}\right)^{p\frac{\gamma-1}{\gamma}}. \tag{7}$$

The following theorem for γ-ordered Normals $\mathcal{N}_\gamma^p(\mu, \Sigma)$ defined for $\gamma \in \mathbb{R} - [0, 1]$ provides a smooth-bridging between the multivariate (and elliptically countered) Uniform, Normal, Laplace as well as the degenerate distributions as the Dirac. That is, the $\mathcal{N}_\gamma^p$ family of distributions with order γ defined outside the open interval (0, 1), not only generalizes the Normal distribution but also includes two other, very significant, distributions as the Uniform and Laplace distributions. In addition, the degenerate distributions also belong to this family. Indeed:

Theorem 2.1. *The multivariate γ-ordered Normal distribution* $\mathcal{N}_\gamma^p(\mu, \Sigma)$, *for order values of* $\gamma = 0, 1, 2, \pm\infty$ *coincides with*

$$\mathcal{N}_\gamma^p(\mu, \Sigma) = \begin{cases} \mathcal{D}^p(\mu), & \gamma = 0, \quad p = 1, 2, \\ \mathcal{O}^p(\mu), & \gamma = 0, \quad p \geq 3, \\ \mathcal{U}^p(\mu, \Sigma), & \gamma = 1, \\ \mathcal{N}^p(\mu, \Sigma), & \gamma = 2, \\ \mathcal{L}^p(\mu, \Sigma), & \gamma = \pm\infty. \end{cases} \tag{8}$$

Proof. From Definition 2.1 of $\mathcal{N}_\gamma^p$, the order γ is defined over $\mathbb{R} - [0, 1]$, i.e., parameter γ is a real number outside the closed interval $[0, 1]$. Let $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$ and denote $\alpha = \frac{\gamma - 1}{\gamma}$. We consider now the following cases:

(i) The limiting case $\gamma = 1$. For $x \in \mathbb{R}^p$ such that $Q(x) \leq 1$, from (6), we have that

$$\lim_{\gamma \to 1^+} f_{X_\gamma}(x) = \frac{\Gamma\left(\frac{p}{2} + 1\right)}{\pi^{p/2}\sqrt{|\det \Sigma|}} \left(\lim_{g \to 0^+} \alpha^\alpha\right)\left(\lim_{g \to 0^+} \exp\{-\alpha Q(x)^{-1/2\alpha}\}\right)$$

$$= \frac{\Gamma\left(\frac{p}{2} + 1\right)}{\pi^{p/2}\sqrt{|\det \Sigma|}} \cdot 1 \cdot e^0,$$

while, for $x \in \mathbb{R}^p$ with $Q(x) > 1$, we have

$$\lim_{\gamma\to 1^+} f_{X_\gamma}(x) = \frac{\Gamma\left(\frac{p}{2}+1\right)}{\pi^{p/2}\sqrt{|\det\Sigma|}}\left(\lim_{g\to 0^+}\alpha^{\alpha}\right)\left(\lim_{g\to 0^+}\exp\{-\alpha Q(x)^{1/2\alpha}\}\right)$$

$$= \frac{\Gamma\left(\frac{p}{2}+1\right)}{\pi^{p/2}\sqrt{|\det\Sigma|}}\cdot 1\cdot 0,$$

due to $\alpha Q(x)^{1/\alpha} \to +\infty$ as $\alpha \to 0^+$. Therefore, it holds that $\mathcal{N}_1^p(\mu, \Sigma) = \lim_{\gamma\to 1^+}\mathcal{N}_\gamma^p(\mu, \Sigma) = \mathcal{U}^p(\mu, \Sigma)$ as $\lim_{\gamma\to 1^+} f_{X_\gamma} = f_{X_1} = f_{\mathcal{U}}$, i.e., the multivariate first-ordered Normals are, in fact, the multivariate (elliptically contoured) uniform distributions:

(ii) The case $\gamma = 2$. It is clear that $\mathcal{N}_2^p(\mu, \Sigma) = \mathcal{N}^p(\mu, \Sigma)$, as f_{X_2} coincides with the multivariate (and elliptically contoured) Normal density $f_{\mathcal{N}}$ as in (2), i.e., the multivariate second-ordered Normals are in fact the multivariate Normal distributions.

(iii) The limiting case $\gamma = \pm\infty$. It is $\mathcal{N}_{\pm\infty}^p(\mu, \Sigma) = \lim_{\gamma\pm\infty}\mathcal{N}_\gamma^p(\mu, \Sigma) = \mathcal{L}^p(\mu, \Sigma)$ as $f_{X_{\pm\infty}} = \lim_{\gamma\to\pm\infty} f_{X_\gamma}$ coincides with the multivariate (and elliptically contoured) Laplace density $f_{\mathcal{L}}$ as in (3), i.e., the multivariate infinite-ordered Normals are, in fact, the multivariate (elliptically contoured) Laplace distributions.

(iv) The limiting case $\gamma = 0$. First, we assume that $x = \mu$, i.e., $Q(x) = 0$, or from Definition 2.1,

$$f_{X_\gamma}(\mu) = \pi^{-p/2}\Gamma\left(\frac{p}{2}+1\right)\frac{\alpha^{p\alpha}}{\Gamma(p\alpha+1)}|\det\Sigma|^{-1/2}. \tag{9}$$

The following limiting result hold,

$$\lim_{\gamma\to 0^-} f_{X_\gamma}(\mu) = \lim_{\alpha=\frac{\gamma-1}{\gamma}\to+\infty} f_{X_\gamma}(\mu) = \lim_{k=[p\alpha]\to\infty} f_{X_\gamma}(\mu),$$

where $[x]$ is the integer value of $x \in \mathbb{R}$, and thus

$$\lim_{\gamma\to 0^-} f_{X_\gamma}(\mu) = \frac{\Gamma\left(\frac{p}{2}+1\right)}{\pi^{p/2}\sqrt{|\det\Sigma|}}\left(\lim_{k\to\infty}\frac{k^k}{p^k k!}\right). \tag{10}$$

Utilizing now the Stirling's asymptotic formula $k! \approx \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$ as $k \to \infty$, (10) implies

$$\lim_{\gamma\to 0^-} f_{X_\gamma}(\mu) = \frac{\Gamma\left(\frac{p}{2}+1\right)}{\pi^{p/2}\sqrt{|\det\Sigma|}}\left[\lim_{k\to\infty}\frac{1}{\sqrt{2\pi}\left(\frac{p}{e}\right)^k}\right], \tag{11}$$

and so, for $p \geq 3$, (11) implies $\lim_{\gamma\to 0^-} f_{X_\gamma}(\mu) = 0$ while, for $p = 1$ or $p = 2$, implies $\lim_{\gamma\to 0^-} f_{X_\gamma}(\mu) = +\infty$.

Assuming now $x \neq \mu$ and using (10), we have

$$\lim_{\gamma\to 0^-} f_{X_\gamma}(x) = \lim_{\gamma\to 0^-} f_{X_\gamma}(\mu)\left[\lim_{\alpha\to+\infty}\exp\{-\alpha Q(x)^{1/2\alpha}\}\right], \tag{12}$$

and so, for $p \geq 3$, (12) implies $\lim_{\gamma\to 0^-} f_{X_\gamma}(x) = 0$ for every $x \neq \mu$ while, for $p = 1$ or $p = 2$, applying (11) into (12), we obtain

$$\lim_{\gamma\to 0^-} f_{X_\gamma}(\mu) = \frac{\Gamma\left(\frac{p}{2}+1\right)}{\pi^{p/2}\sqrt{|\det\Sigma|}}\left[\lim_{k\to\infty}\frac{\exp\left\{1-\frac{1}{p}Q(x)^{p/2k}\right\}}{p^k\sqrt{2\pi k}}\right] = 0.$$

Therefore, for $p = 1, 2,$ it is $\mathcal{N}_0^p(\mu, \Sigma) = \lim_{\gamma \to 0^-} \mathcal{N}_\gamma^p(\mu, \Sigma) = \mathcal{D}^p(\mu)$ as $f_{X_0} = \lim_{\gamma \to 0^-} f_{X_\gamma}$ coincides with the multivariate Dirac density $f_{\mathcal{D}}$ as in (4), i.e., the univariate and bivariate zero-ordered Normals are, in fact, the (univariate and bivariate) degenerate Dirac distributions, while the *n*-variate, $n \geq 3,$ zero-ordered Normals are, in fact, the degenerate vanishing distributions.

From the above limiting cases (i), (iii) and (iv), we can then safely extend the defining order values γ in Definition 2.1 to the values of $\gamma = 0, 1, \pm\infty,$ i.e., γ can now be defined outside the open interval (0, 1). Eventually, the family of the γ-ordered Normals conclude the Uniform, Normal, Laplace and also the degenerate distributions as the Dirac or the vanishing ones. □

Corollary 2.1. *The univariate γ-ordered Normals* $\mathcal{N}_\gamma(\mu, \sigma) = \mathcal{N}_\gamma^1(\mu, \sigma)$ *for order values* $\gamma = 0, 1, 2, \pm\infty$ *coincides with the usual* (*univariate*) *Dirac* $\mathcal{D}(\mu)$, *Uniform* $\mathcal{U}(\mu - \sigma, \mu + \sigma)$, *Normal* $\mathcal{N}(\mu, \sigma)$ *and Laplace* $\mathcal{L}(\mu, \sigma)$ *distributions, respectively*.

Proof. From the univariate form of Theorem 2.1, it is $\mathcal{N}_1^1(\mu, \sigma) = \mathcal{U}^1(\mu, \sigma)$ which coincides with the known (continuous) Uniform distribution $\mathcal{U}(a, b),$ i.e., with $\mathcal{U}(\mu - \sigma, \mu + \sigma).$ In fact, for every Uniform distribution expressed with the usual notation $\mathcal{U}(a, b),$ it holds that

$$\mathcal{U}(a, b) = \mathcal{N}_1^1\left(\frac{b-a}{2}, \frac{a+b}{2}\right) = \mathcal{U}^1(\mu, \sigma).$$

Moreover, we have $\mathcal{N}_2(\mu, \sigma^2) = \mathcal{N}(\mu, \sigma^2),$ $\mathcal{N}_{\pm\infty}(\mu, \sigma) = \mathcal{L}(\mu, \sigma)$ and finally $\mathcal{N}_0(\mu, \sigma) = \mathcal{D}(\mu).$ □

The following Figure 1 illustrates Corollary 2.1 by presenting together all the density functions $f_{X_\gamma}(x)$, $X_\gamma \sim \mathcal{N}_\gamma(0, 1)$ for $x \in [-3, 3]$ and $\gamma \in [-10, 0) \cup [1, 10]$, which forms the appeared semi-transparent surface. The known densities of Uniform $(\gamma = 1)$ and Normal $(\gamma = 2)$ distributions are denoted. Also, denoted the densities of $\mathcal{N}_{\gamma=-10,10}(0, 1)$ which approximate the density of Laplace distribution $\mathcal{L}_{\pm\infty}(0, 1)$ as well as the density of $\mathcal{N}_{-0.005}(0, 1)$ which approximates the degenerate Dirac distribution $\mathcal{D}(0)$. Finally, notice the smooth-bringing between all these significant distributions that included, eventually, in the $\mathcal{N}_\gamma^p$ family, as shown in Theorem 2.1.

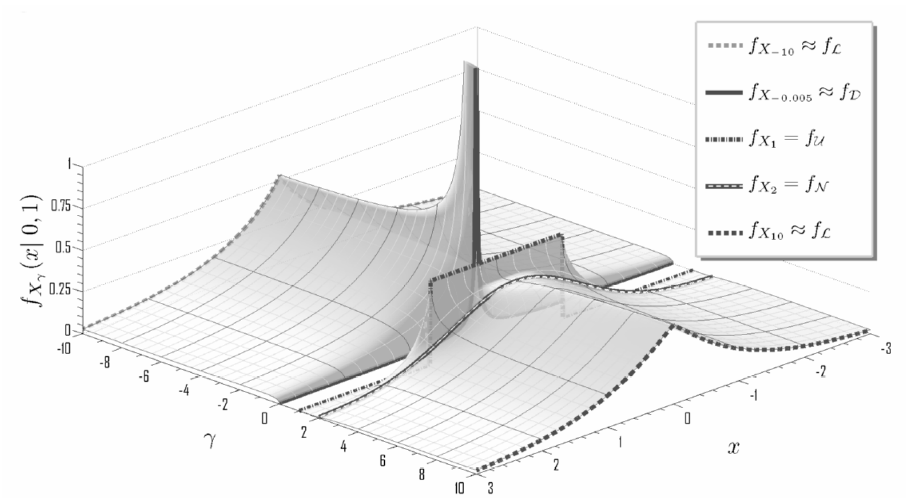

Figure 1. Graph of all the densities $f_{X_\gamma}(x)$, $X_\gamma \sim \mathcal{N}_\gamma(0, 1)$ along x and γ.

From Definition 2.1, the height $h(X_\gamma) = h^p(X_\gamma)$ of the density function f_{X_γ} of the density function f_{X_γ} of a random variable X_γ following the γ-ordered Normal distribution $\mathcal{N}_\gamma^p(\mu, \Sigma)$ is achieved for $x = \mu$, i.e., $h(X_\gamma) = \max_{x \in \mathbb{R}^p} f_{X_\gamma}(x) = f_{X_\gamma}(\mu) = C_\gamma^p |\det \Sigma|^{-1/2}$.

For the multivariate normally distributed $X \sim \mathcal{N}^p(\mu, \Sigma) = \mathcal{N}_2^p(\mu, \Sigma)$ it is clear, from (2), that the height $h(X)$ decreases as dimension $p \in \mathbb{N}$ rises, providing "flattened" probability densities. This is also true for the multivariate Laplace distributed $X \sim \mathcal{L}^p(\mu, \Sigma) = \mathcal{N}_{\pm\infty}^p(\mu, \Sigma)$. In fact, from (3), we have that $h(X_{\pm\infty}) = \pi^{-p/2}\frac{1}{p!}\Gamma\left(\frac{p}{2}+1\right)|\det\Sigma|^{-1/2}$ and therefore, the high-dimensional Laplace distributions densities are "flattened", since the height values decrease as $p \in \mathbb{N}$ increases. This is true because, for dimensions $2p$ as

$$h(X_{\pm\infty}) = \pi^{-p/2}\frac{1}{(p+1)(p+2)\cdots 2p}|\det\Sigma|^{-1/2}.$$

Hence, as in the Normal distribution case, they provide, in general, heavy tails as the dimension increases.

This is not the case for the multivariate Uniform distributed $X \sim \mathcal{U}^p(\mu, \Sigma) = \mathcal{N}_1^p(\mu, \Sigma)$, because the volume of the corresponding p-elliptical-cylinder shape of their density functions, as in (1), may always equal to 1, however, they have no tails to "absorb" probability mass when dimension increases, as the Normal or the Laplace distributions does.

Considering the above remark, the following proposition proves that, in fact, among all elliptical multivariate Uniform distributions $\mathcal{U}^p(\mu, \Sigma)$ with fixed scale matrix Σ, $\mathcal{U}^5(\mu, \Sigma)$ has the minimum height h.

Proposition 2.1. *For the elliptically contoured Uniformly distributed* $X \sim \mathcal{U}^p(\mu, \Sigma)$, *we have*

$$\min_{p\in\mathbb{N}}\{h^p(X)\} = \frac{15}{6\pi^2}|\det\Sigma|^{-1} = h^5(X),$$

i.e., the five-dimensional Uniform distribution $\mathcal{U}^p(\mu, \Sigma)$ *provides the minimum height value among all* $\mathcal{U}^p(\mu, \Sigma)$ *with fixed scale matrix* Σ.

Proof. Let $w(p) = \pi^{-\frac{p}{2}}\Gamma\left(\frac{p}{2}+1\right)$, $p \geq 1$. Differentiating *h*, we get

$$w'(p) = \frac{1}{2}\pi^{-\frac{p}{2}}\Gamma\left(\frac{p}{2}+1\right)\left[\log\pi - \psi\left(\frac{p}{2}+1\right)\right].$$

There is a unique real value, $p = 2\psi^{-1}(\log\pi) - 2$, for which $w'(p) = 0$. Computing numerically this value, we obtained $p \approx 5.5269$. This is the unique extreme point for *h*. As a result, there is a unique extreme integer value $p \in \mathbb{N}$ for the peak of all *p*-variate Uniform distributions $U^p(\mu, \Sigma) = \mathcal{N}_1^p(\mu, \Sigma)$, as $h^p(X) = C_1^p|\det\Sigma|^{-1} = w(p)|\det\Sigma|^{-1}$. The corresponding dimension *p*, evaluated above, is $p = 5$ as $p \in \mathbb{N}$. In fact, $p = 5$ is the minimum value for $h^p(X)$, because

$$h^p(X) = \frac{15}{6}\pi^{-2}|\det\Sigma|^{-1} < \frac{1}{2}|\det\Sigma|^{-1} = h^1(X)$$

and

$$\lim_{p\to\infty} h^p(X) = \lim_{p\to\infty} h(p)|\det\Sigma|^{-1} = \lim_{2p\to\infty}\frac{p!}{\pi^p}|\det\Sigma|^{-1} = +\infty.$$

Moreover, $w''(5) > 0$. Figure 3 illustrates clearly the above proposition. □

Tables 1 and 2 provide evaluations for the probability mass of a univariate and a bivariate random variable X_γ following the γ-ordered Normals $\mathcal{N}_\gamma(0, 1)$ and $\mathcal{N}_\gamma^2(0, \mathbb{I}_2)$, respectively, for various positive order values. Notice that, for the positive-ordered case $(\gamma > 1)$ heavy-tailed distributions obtained as order γ increases approaching Laplace, while heavier-tailed ones obtained for the negative orders and especially for orders γ close to 0. Figure 2 confirms the above as the depicted heights of the probability densities of the univariate and the bivariate $\mathcal{N}_\gamma^p(\mu, \mathbb{I}_p)$ increases rapidly as γ tends to 0, while for higher dimensions $p \geq 3$ falls even more rapidly to 0. This is so because, in both dimensional cases, the multivariate γ-ordered Normals reach the degenerate Dirac or the vanishing distributions as proved in Theorem 2.1, and therefore their probability tails grows heavily.

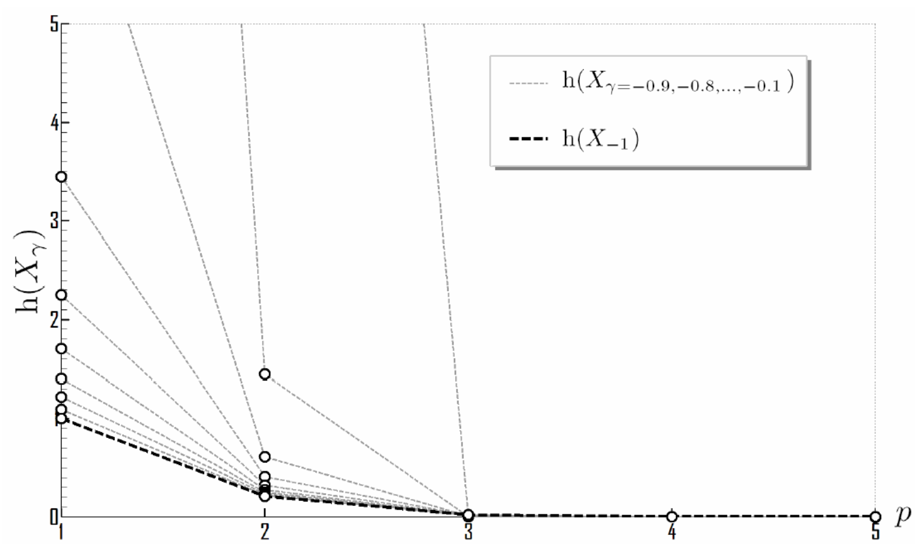

Figure 2. Graphs of $h(X_\gamma)$ for various negative-ordered $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \mathbb{I}_p)$ along dimensions $p \in \mathbb{N}$ (and for any $\mu \in \mathbb{R}$).

The γ-ordered Normal distribution $\mathcal{N}_\gamma^p(\mu, \Sigma)$ is an elliptically contoured distribution, and therefore every $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$ admits a stochastic representation $X_\gamma = \mu + \sqrt{V}\Sigma^{-1/2}U$, where U is uniformly distributed on a unit sphere of $\mathbb{R}^p$ and V is independent of U.

Table 1. Probability mass values for various $X_\gamma \sim \mathcal{N}_\gamma(0, 1)$

γ	$\Pr(\lvert X_\gamma \rvert \le 1)$	$\Pr(\lvert X_\gamma \rvert \le 2)$	$\Pr(\lvert X_\gamma \rvert \le 3)$
–100	0.6315	0.8633	0.9491
–10	0.6262	0.8516	0.9392
–2	0.6084	0.8100	0.8995
–1	0.5940	0.7737	0.8603
–0.05	0.5290	0.5889	0.6233
1	**1.0000**	**1.0000**	**1.0000**
2	**0.6827**	**0.9545**	**0.9973**
5	0.6470	0.8953	0.9724
10	0.6390	0.8792	0.9614
100	0.6328	0.8669	0.9513
$\pm\infty$	**0.632**	**0.866**	**0.951**

Table 2. Probability mass values for various $X_\gamma \sim \mathcal{N}_\gamma^2(0, \mathbb{I}_2)$

γ	$\Pr(\lvert X_\gamma \rvert \le 1)$	$\Pr(\lvert X_\gamma \rvert \le 2)$	$\Pr(\lvert X_\gamma \rvert \le 3)$
–100	0.2624	0.5898	0.7965
–10	0.2467	0.5538	0.7576
–2	0.1912	0.4253	0.6032
–1	0.1429	0.3144	0.4556
–0.05	0.9999	0.9999	0.9999
1	**1.0000**	**1.0000**	**1.0000**
2	**0.3935**	**0.8647**	**0.9889**
5	0.3057	0.6873	0.8889
10	0.2837	0.6383	0.8451
100	0.2661	0.5982	0.8052
$\pm\infty$	**0.2642**	**0.5940**	**0.8009**

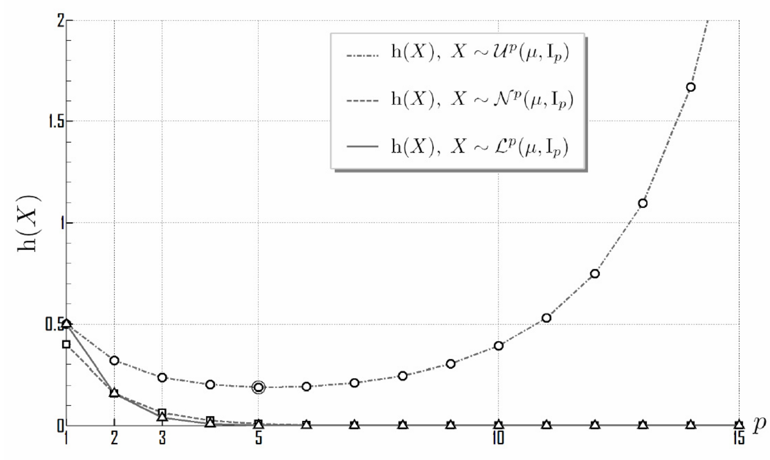

Figure 3. Graphs of $h^p(X_\gamma)$ with $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \mathbb{I}_p)$ for $\gamma = 1, 2, +\infty$ along dimensions $p \in \mathbb{N}$ (and for any $\mu \in \mathbb{R}$).

Proposition 2.2. *For the random variable* $X_\gamma = \mu + \sqrt{V_\gamma}\Sigma^{-1/2}U \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$, *the* $2t$*-th moments of* V_γ *are given by*

$$E(V_\gamma^{2t}) = \frac{\Gamma\left((p+2t)\frac{\gamma-1}{\gamma}\right)}{\Gamma\left(p\frac{\gamma-1}{\gamma}\right)}\left(\frac{\gamma}{\gamma-1}\right)^{2t\frac{\gamma-1}{\gamma}}. \tag{13}$$

Proof. The random variable V_γ is distributed on $\mathbb{R}_+$ with density function having $f_{V_\gamma^2}$ the form

$$f_{V_\gamma}(v) = \frac{\pi^{p/2}}{\Gamma(p/2)} v^{\frac{p}{2}-1} g_{X_\gamma}(v), \quad v \in \mathbb{R}_+,$$

where g_{X_γ} is the generating function of X_γ, i.e.,

$$f_{X_\gamma}(v) = C_\gamma^p \exp\left\{-\frac{\gamma-1}{\gamma} v^{\frac{\gamma}{2(\gamma-)}}\right\}, \quad v \in \mathbb{R}_+.$$

Consequently, setting $s = \frac{\gamma}{2(\gamma - 1)}$, the multivariate $2t$-th raw moments of V_γ are

$$m_{2t}(V_\gamma) = E(V_\gamma^{2t})$$

$$= \frac{\pi^{p/2}}{\Gamma(p/2)} C_\gamma^p \int_{\mathbb{R}_+} v^{t+p/2-1} \exp\left\{-\frac{1}{2s} v^s\right\} dv$$

$$= \frac{\pi^{p/2}}{s\Gamma(p/2)} C_\gamma^p \int_{\mathbb{R}_+} v^{t+p/2-1} \exp\left\{-\frac{1}{2s} v^s\right\} dv^s$$

$$= \frac{\pi^{p/2}}{s\Gamma(p/2)} C_\gamma^p \int_{\mathbb{R}_+} (v^s)^{\frac{p+2t}{2s}-1} \exp\left\{-\frac{1}{2s} v^s\right\} dv^s$$

$$= \frac{2\pi^{p/2}(2s)^{\frac{p+2t}{3s}-1}}{\Gamma(p/2)} C_\gamma^p \int_{\mathbb{R}_+} \left(\frac{1}{2s} v^s\right)^{\frac{p+2t}{2s}-1} \exp\left\{-\frac{1}{2s} v^s\right\} d\left(\frac{1}{2s} v^s\right)$$

$$= \frac{2\pi^{p/2}(2s)^{\frac{p+2t}{2s}-1}}{\Gamma(p/2)} C_\gamma^p \Gamma\left(\frac{p+2t}{2s}\right),$$

and substituting C_γ^p as in (7) we finally obtain (13). □

Corollary 2.2. *The odd moments of* V_γ *are vanished, while for the second moment, we have*

$$E(V_\gamma^2) = \frac{\Gamma\left((2+p)\frac{\gamma-1}{\gamma}\right)}{\Gamma\left(p\frac{\gamma-1}{\gamma}\right)} \left(\frac{\gamma}{\gamma-1}\right)^{2\frac{\gamma-1}{\gamma}}.$$

Notice that, for the "normal" order value $\gamma = 2$, it is $E(V_\gamma^2) = p$.

Using Theorem 2.8 in [2], we obtain the product moments of X, i.e.,

$$E(X_1^{2t_1} \cdots X_p^{2t_p}) = \frac{E(V_\gamma^{2t})}{\pi^{p/2}} \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p}{2} + t\right)} \prod_{k=1}^{p} \Gamma\left(\frac{1}{2} + t_k\right)$$

$$= \pi^{-p/2}\left(\frac{\gamma - 1}{\gamma}\right)^{2t\frac{\gamma-1}{\gamma}} \frac{\Gamma\left((p + 2t)\frac{\gamma - 1}{\gamma}\right)}{\Gamma\left(p\frac{\gamma - 1}{\gamma}\right)\Gamma\left(\frac{p}{2} + t\right)} \prod_{k=1}^{p} \Gamma\left(\frac{1}{2} + t_k\right),$$

where $t \geq 1,\ i = 1, \ldots, p$ are integers and $t_1 + t_2 + \cdots + t_p = t$.

Consequently, the expected value and the covariance of $X_\gamma = \sqrt{V_\gamma}\Sigma^{-1/2}U$ are, respectively, $E(X_\gamma) = \mu$ for every order values $\gamma \in \mathbb{R}\setminus[0, 1]$, and

$$\operatorname{Cov}(X_\gamma) = \frac{\Gamma\left((p + 2)\frac{\gamma - 1}{\gamma}\right)}{\Gamma\left(p\frac{\gamma - 1}{\gamma}\right)} \left(\frac{\gamma}{\gamma - 1}\right)^{2\frac{\gamma-1}{\gamma}} (\operatorname{rank}\Sigma)^{-1}\Sigma. \tag{14}$$

Corollary 2.3. *For the "normal-ordered"* $X \sim \mathcal{N}_2^p(\mu, \Sigma)$, *the scale parameter matrix* Σ *is in fact the covariance of X, as it is expected.*

Corollary 2.4. *If* $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$, *then*

$$d + CX \sim \mathcal{N}_\gamma^p(d + C\mu, C\Sigma C^T),$$

where d is a vector of constants and C is a constant matrix. In particular, any subset of the $X_{\gamma, i}$ *having a marginal distribution, is also* $\mathcal{N}_\gamma^p(\mu, \Sigma)$.

Corollary 2.5. *If* $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$, *then* $cV \sim \mathcal{N}_\gamma^p(c \cdot \mu, c^T\Sigma c)$, *where c is a constant vector of the same length as X and* $\cdot$ *indicates a vector product.*

Proof. This follows when

$$C = (c_{ij}) = \begin{cases} c_j, & i = 1, \\ 0, & i = 2, 3, \ldots, p, \end{cases} \qquad j = 1, 2, \ldots, p,$$

considering only the first component of the product (the first row of C is the vector c). $\square$

Example 2.1. The skewness of $\mathcal{N}_\gamma(\mu, \sigma^2)$ is zero because the odd moments are zero, see Corollary 2.2, while the kurtosis of $X_\gamma \sim \mathcal{N}_\gamma(\mu, \sigma^2)$ is

$$\operatorname{Kurt}(X_\gamma) = \frac{\gamma}{\gamma - 1}\Gamma\left(\frac{\gamma - 1}{\gamma} + 1\right)\Gamma\left(5\frac{\gamma - 1}{\gamma}\right)\Gamma^{-2}\left(3\frac{\gamma - 1}{\gamma}\right) - 3,$$

see also [7]. Therefore, for the "normal" case $\gamma = 2,$ it is

$$\operatorname{Kurt}(X) = 2\Gamma\left(\frac{1}{2} + 1\right)\Gamma\left(\frac{5}{2}\right)\Gamma^{-2}\left(\frac{3}{2}\right) - 3 = 0,$$

i.e., the kurtosis is vanished, as it was expected for $\mathcal{N}(\mu, \sigma^2)$.

3. Characteristic Function

We need the following lemma.

Lemma 3.1. *For the spherically contoured random variable* $X_\gamma \sim \mathcal{N}_\gamma^p(0, \mathbb{I}_p),$ *the characteristic function* φ_{X_γ} *is given by*

$$\varphi_{X_\gamma}(t) = K_\gamma^p \| t \|^{-p} \Psi_\gamma^p\left(\| t \|^{-\frac{\gamma}{\gamma-1}}\right), \quad t \in \mathbb{R}^p, \tag{15}$$

where

$$K_\gamma^p = 2^{\frac{p}{2}-1} \frac{\gamma}{\gamma - 1} \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(p\frac{\gamma - 1}{\gamma}\right)},$$

and

$$\Psi_\gamma^p(x) = \int_0^\infty J_{\frac{p}{2}-1}(s)s^{\frac{p}{2}} \exp\left\{-s^{\frac{\gamma}{\gamma-1}}x\right\}ds, \quad x > 0, \tag{16}$$

with $J_{\frac{p}{2}-1}$ *being the Bessel function of the first kind of the order* $\frac{p}{2} - 1.$

Proof. Let $F_{X_\gamma}(E) = \int_E f_{X_\gamma}(x)dx_1 \cdots dx_p,$ for $E \subset \mathbb{R}^p.$ Then

$$\sigma_{F_{X_\gamma}}(u) = \int_{\|x\| \le u} f_{X_\gamma}(x)dx_1 \cdots dx_p$$

$$= C_\gamma^p \int_{\|x\| \le u} \exp\left\{-\frac{\gamma-1}{\gamma}\|x\|^{\frac{\gamma}{\gamma-1}}\right\}dx_1 \cdots dx_p,$$

and switching to hyperspherical coordinates,

$$\sigma_{F_{X_\gamma}}(u) = \frac{2\pi^{p/2}}{\Gamma(p/2)} C_\gamma^p \int_0^u \rho^{p-1} \exp\left\{-\frac{\gamma-1}{\gamma}\rho^{\frac{\gamma}{\gamma-1}}\right\}d\rho$$

$$= \frac{2\pi^{p/2}}{\Gamma(p/2)}\left(\frac{\gamma}{\gamma-1}\right)^{p\frac{\gamma-1}{\gamma}} C_\gamma^p \int_0^u \rho^{p-1} \exp\left\{-\rho^{\frac{\gamma}{\gamma-1}}\right\}d\rho. \tag{17}$$

The Fourier transform φ of a spherically contoured distribution with cumulative density F is given by

$$\varphi(t) = 2^{\frac{p}{2}-1}\Gamma\left(\frac{p}{2}\right)\int_0^\infty (\|t\|u)^{1-\frac{p}{2}} J_{\frac{p}{2}-1}(\|t\|u)d\sigma_F(u), \quad t \in \mathbb{R}^p,$$

where $\sigma_F(u) = F(\{x \in \mathbb{R}^p \mid \|x\| \le u\}),$ see [13] for details. Utilizing this for the spherically contoured $\mathcal{N}_\gamma^p(0, \mathbb{I}_p),$ i.e., for (17), we obtain

$$\varphi_{X_\gamma}(t) = (2\pi)^{p/2}\left(\frac{\gamma}{\gamma-1}\right)^{p\frac{\gamma-1}{\gamma}} C_\gamma^p \| t \|^{1-\frac{p}{2}}$$

$$\times \int_0^\infty J_{\frac{p}{2}-1}(\| t \| u) u^{p-1} \exp\left\{-u^{\frac{\gamma}{\gamma-1}}\right\} du$$

$$= (2\pi)^{p/2}\left(\frac{\gamma}{\gamma-1}\right)^{p\frac{\gamma-1}{\gamma}} C_\gamma^p \| t \|^{-p}$$

$$\times \int_0^\infty J_{\frac{p}{2}-1}(s) s^{p/2} \exp\left\{-(\| t \|^{-1} s)^{\frac{\gamma}{\gamma-1}}\right\} ds,$$

and applying (7), we finally derive (15) and lemma has been proved. □

Theorem 3.1. *The characteristic function* $\varphi_{X_\gamma}(t) = \gamma_{X_\gamma}(t \mid \mu, \Sigma),\ t \in \mathbb{R}^p$ *of the elliptically contoured positive-ordered Normal distribution, i.e., for* $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$ *with* $\gamma > 1,$ *has the form*

$$\varphi_{X_\gamma}(t) = e^{-it^T\mu}\left(\frac{\gamma}{\gamma-1}\right)^{p\frac{\gamma-1}{\gamma}} C_\gamma^p Q(t)^{-p/2} \phi\left(\frac{\gamma-1}{\gamma} Q(t)^{-\frac{\gamma}{2(\gamma-1)}}\right), \quad t \in \mathbb{R}^p, \tag{18}$$

where $\phi(z)$ *is an entire function of z and Q the quadratic form* $Q(t) = t^T\Sigma^{-1}t$. *Moreover,* $|\varphi_{X_\gamma}|$ *is bounded from*

$$2^{1-\frac{p}{2}}\sqrt{\frac{\gamma-1}{\gamma}} B(t) \geq |\varphi_{X_\gamma}(t)| \geq 2^{1-\frac{p}{2}} B(t), \quad t \in \mathbb{R}^p, \tag{19}$$

for order values $\gamma \geq 2,$ *while for order values* $1 < \gamma < 2$ *is bounded from*

$$2^{1-p\frac{\gamma-1}{\gamma}}\sqrt{\frac{\gamma-1}{\gamma}} B(t) < |\varphi_{X_\gamma}(t)| < 2^{\frac{1}{2}-p\frac{\gamma-1}{\gamma}} B(t), \quad t \in \mathbb{R}^p, \tag{20}$$

where

$$B(t) = \frac{1}{2\pi^{p/2}}\left(\frac{\gamma}{\gamma-1}\right)^{p\frac{\gamma-1}{\gamma}+\frac{1}{2}}\left(\frac{p}{e}\right)^{p\frac{2-\gamma}{2\gamma}} Q(t)^{-p/2}\left| e^{-it^T\mu}\phi\left(\frac{\gamma-1}{\gamma} Q(t)^{-\frac{\gamma}{2(\gamma-1)}}\right)\right|.$$

Proof. We assume, without loss of generality, that $X_\gamma \sim \mathcal{N}_\gamma^p(0, \mathbb{I}_p)$ with order value $\gamma \in \mathbb{R}\backslash[0, 1]$ arbitrary and fixed. Considering Lemma 3.1, it is sufficient to prove that for positive *x*, the function $\Psi_\gamma^p(x)$ as in (16) is the restriction to a ray $\{x | x > 0\}$ of an entire function. Then (16) is equivalent to

$$\Psi_\gamma^p(x) = \int_0^\beta J_{\frac{p}{2}-1}(s)s^{p/2}\exp\left\{-xs^{\frac{\gamma}{\gamma-1}}\right\}ds$$

$$+\int_\beta^\infty J_{\frac{p}{2}-1}(s)s^{p/2}\exp\left\{-xs^{\frac{\gamma}{\gamma-1}}\right\}ds$$

$$= I_1(x) + I_2(x),$$

where $\beta = \dfrac{\gamma}{2(\gamma-1)} > 0.$

The integral I_1 converges absolutely and uniformly on any compact set in the complex *z*-plane, thus is an entire function on *z*. So, this integral is the restriction of an entire function to $x > 0$. We will rewrite the integral I_2, using the modified Bessel function of the third kind *K*, as follows:

$$I_2(x) = \frac{2}{\pi}\Re\left\{e^{-i\pi p/4}\int_\beta^\infty K_{\frac{p}{2}-1}(-is)s^{p/2}\exp\left\{-xs^{\frac{\gamma}{\gamma-1}}\right\}ds\right\}, \quad x > 0, \quad (21)$$

where $\Re(z)$ being the real part of $z \in \mathbb{C}$. It is known that the modified Bessel function $K_{\frac{p}{2}-1}(z)$ is analytic in the right half-plane $\{z | \Re z \geq 0\}\backslash 0.$ Thus, applying the Cauchy theorem to the boundary $L_{R,a}$ of the circular region $\{z | 0 < \beta \leq |z| \leq R, -\pi/2 \leq \arg z \leq 0\}$, we obtain

$$\oint_{L_{R,\beta}} K_{\frac{p}{2}-1}(z) z^{p/2} \exp\left\{-x(ze^{i\pi/2})^{\frac{\gamma}{\gamma-1}}\right\} dz = 0, \tag{22}$$

where the functions $z^{p/2}$ and $z^{\frac{\gamma}{\gamma-1}}$ are chosen to be

$$z^{p/2} = |z|^{p/2} e^{i\omega p/2} \quad \text{and} \quad z^{\frac{\gamma}{\gamma-1}} = |z|^{\frac{\gamma}{\gamma-1}} e^{i\omega\frac{\gamma}{\gamma-1}}, \quad -\frac{\pi}{2} \le \omega \le 0,$$

which are positive along the ray $\{z \,|\, z > 0\}$. Thus, (22) can be written as

$$\begin{aligned} 0 = &-i\int_{\beta}^{R} K_{\frac{p}{2}-1}(-is) s^{p/2} e^{-i\pi p/2} \exp\left\{-xs^{\frac{\gamma}{\gamma-1}}\right\} ds \\ &- \int_{\beta}^{R} K_{\frac{p}{2}-1}(s) s^{p/2} \exp\left\{-x(se^{i\pi/2})^{\frac{\gamma}{\gamma-1}}\right\} ds \\ &- i\int_{-p/2}^{0} K_{\frac{p}{2}-1}(\beta e^{i\omega}) \beta^{p/2} e^{i\omega\left(\frac{p}{2}+1\right)} \exp\left\{-x\beta^{\frac{\gamma}{\gamma-1}} e^{i\left(\omega+\frac{p}{2}\right)\frac{\gamma}{\gamma-1}}\right\} \beta d\omega \\ &+ i\int_{-p/2}^{0} K_{\frac{p}{2}-1}(\mathrm{Re}^{i\omega}) R^{p/2} e^{i\omega\left(\frac{p}{2}+1\right)} \exp\left\{-xR^{\frac{\gamma}{\gamma-1}} e^{i\left(\omega+\frac{p}{2}\right)\frac{\gamma}{\gamma-1}}\right\} R d\omega, \end{aligned} \tag{23}$$

where the last integral of (23) is bounded for R sufficiently large, as $K_{(p/2)-1}(z)$ in (22) is bounded for large enough $|z|$ due to the known asymptotic expression,

$$K_{(p/2)-1}(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z}, \quad z \to \infty, \tag{24}$$

and, consequently, this integral tends to 0 as $R \to \infty$.

Therefore, we can write (21) in the form

$$I_2(x) = \frac{2}{\pi} \Re\{I_2^*(x) - \beta^{\frac{p}{2}+1} I_2^{**}(x)\}, \quad x > 0,$$

where

$$I_2^*(x) = \int_\beta^\infty K_{\frac{p}{2}-1}(s) s^{p/2} \exp\left\{-x(se^{i\pi/2})^{\frac{\gamma}{\gamma-1}}\right\} ds,$$

and

$$I_2^{**}(x) = \int_{-\pi/2}^{0} K_{\frac{p}{2}-1}(\beta e^{i\omega}) e^{i\omega\left(\frac{p}{2}+1\right)} \exp\left\{-x\beta^{\frac{\gamma}{\gamma-1}} e^{i\left(\omega+\frac{p}{2}\right)\frac{\gamma}{\gamma-1}}\right\} \beta d\omega.$$

With the help of (24), the integrand in I_2^* majorized for all values of x by the function

$$ce^{-s} s^{p/2} \exp\left\{s^{\frac{\gamma}{\gamma-1}} \cos \frac{\pi\gamma}{2(\gamma-1)} |\, x \,|\right\},$$

with $c > 0$ constant. Assuming now $2\beta = \dfrac{\gamma}{\gamma-1} < 1$, i.e., $\gamma > 0$, we conclude that $I_2^*(x)$ is absolutely and uniformly convergent on any compact set of complex values of $x \in \mathbb{C}$. Thus, I_2^* is an entire function. Similarly, this argument applies also to $I_2^* *$ which is a proper integral. Thus, $I_2(x)$ is the real part of an entire function for $x > 0$. Consequently, $I_2(x)$ is the restriction to the ray $\{x \,|\, x > 0\}$ of an entire function $\varepsilon(x) = \sum_{k=0}^\infty a_k x^k$, say, because $I_2(x) = \sum_{k=0}^\infty (\Re a_k) x^k$, as shown above.

Considering Lemma 3.1 which holds for the reduced case of $X_\gamma \sim \mathcal{N}_\gamma^p(0, \mathbb{I}_p)$, we obtain, through $\Psi_\gamma^p = I_1 + I_2$, that

$$\varphi_{X_\gamma}(t) = e^{-it'\mu}\left(\frac{\gamma}{\gamma-1}\right)^{p\frac{\gamma-1}{\gamma}} C_\gamma^p \| t \|^{-p} \phi\left(\frac{\gamma-1}{\gamma}(\| t \|)^{-\frac{\gamma}{\gamma-1}}\right), \quad t \in \mathbb{R}^p,$$

where ϕ is an entire function. Therefore, (18) indeed holds, for the general elliptically contoured case. □

Corollary 3.1. *For* $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$, $|\varphi_{X_\gamma}|$ *is bounded from*

$$|\varphi_{X_\gamma}(t)| \geq 2^{1-\frac{p}{2}} B(t) \quad \text{and} \quad |\varphi_{X_\gamma}(t)| \leq 2^{1-\frac{p}{2}} \sqrt{\frac{\gamma-1}{\gamma}} B(t), \quad t \in \mathbb{R}^p, \quad (25)$$

for order values $\gamma \geq 2$, *while for order values* $1 < \gamma < 2$ *is bounded from*

$$2^{1-p\frac{\gamma-1}{\gamma}} \sqrt{\frac{\gamma-1}{\gamma}} B(t) < |\varphi_{X_\gamma}(t)| < 2^{\frac{1}{2}-p\frac{\gamma-1}{\gamma}} B(t), \quad t \in \mathbb{R}^p, \quad (26)$$

where

$$B(t) = \frac{1}{2\pi^{p/2}} \left(\frac{\gamma}{\gamma-1}\right)^{p\frac{\gamma-1}{\gamma}+\frac{1}{2}} \left(\frac{p}{e}\right)^{p\frac{2-\gamma}{2\gamma}} Q(t)^{-p/2} \left| e^{-it^T\mu} \phi\left(\frac{\gamma-1}{\gamma} Q(t)^{-\frac{\gamma}{2(\gamma-1)}}\right)\right|.$$

Proof. Assuming $\frac{p}{2} > p^{\frac{\gamma-1}{\gamma}}$, or equivalently, $1 < \gamma < 2$, from the gamma function ratio boundaries [1],

$$\frac{a^{a-1}}{b^{b-1}} e^{b-a} < \frac{\Gamma(a)}{\Gamma(b)} < \frac{a^{a-\frac{1}{2}}}{b^{b-\frac{1}{2}}} e^{b-a}, \quad 0 < b < a, \quad (27)$$

we have

$$\left(\frac{1}{2}\frac{\gamma}{\gamma-1}\right)^{p-\frac{\gamma-1}{\gamma}-\frac{1}{2}} \left(\frac{p}{e}\right)^{p\frac{2-\gamma}{2\gamma}} < \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(p\frac{\gamma-1}{\gamma}\right)}$$

$$< \left(\frac{1}{2}\frac{\gamma}{\gamma-1}\right)^{p\frac{\gamma-1}{\gamma}-1} \left(\frac{p}{e}\right)^{p\frac{2-\gamma}{2\gamma}}, \quad (28)$$

while for $\frac{p}{2} < p^{\frac{\gamma-1}{\gamma}}$, or equivalently, $\gamma > 2$, it is

$$2^{1-\frac{p}{2}}\left(\frac{\gamma}{\gamma-1}\right)^{p-\frac{\gamma-1}{\gamma}-\frac{1}{2}}\left(\frac{e}{p}\right)^{p\frac{\gamma-2}{2\gamma}} \geq \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(p\,\frac{\gamma-1}{\gamma}\right)}$$

$$\geq 2^{1-\frac{p}{2}}\left(\frac{\gamma}{\gamma-1}\right)^{p\frac{\gamma-1}{\gamma}-1}\left(\frac{e}{p}\right)^{p\frac{\gamma-2}{2\gamma}}, \tag{29}$$

where the equalities in (29) hold for $\gamma = 2,$ as the boundaries in (27) end up as equalities when $a = b.$ Therefore, substituting C_γ^p as in (7) into (18) and then utilizing (29) and (28), we obtain (25) and (26), respectively. □

Theorem 3.2. *An explicit analytic form of the characteristic function* φ_{X_γ} *of* $X_\gamma \sim \mathcal{N}_\gamma^p(0, \mathbb{I}_p)$ *is given by*

$$\varphi_{X_\gamma}(t) = e^{-it^T\mu}\,\frac{\frac{\gamma}{2(\gamma-1)}\Gamma(p/2)}{\Gamma\left(p\,\frac{\gamma-1}{\gamma}\right)}\sum_{k=0}^{\infty}(-1)^k\left(\frac{\gamma-1}{\gamma}\right)^{k+p\frac{\gamma-1}{\gamma}} q_k \| t \|^{\frac{k\gamma}{\gamma-1}-p}, \tag{30}$$

where

$$q_k = \frac{2^{p+k\frac{\gamma}{\gamma-1}}}{\pi k!}\,\Gamma\left(k\,\frac{\gamma}{2(\gamma-1)}+\frac{p}{2}\right)\Gamma\left(\frac{k\gamma}{2(\gamma-1)}+1\right)\sin\left(\pi\left(1+\frac{k}{2}\,\frac{\gamma}{\gamma-1}\right)\right). \tag{31}$$

The series in (30) *is absolutely convergent for any* $t \in \mathbb{R}^p \backslash 0.$

Proof. Recall (16). It can be proved that

$$\Psi_\gamma^p(x) = \sum_{k=0}^{\infty}(-1)^{k+1}\psi_k(\gamma,\ p)x^k, \quad x > 0, \tag{32}$$

where

$$\psi_k(\gamma,\ p) = \frac{1}{\pi k!}2^{k\frac{\gamma}{\gamma-1}+\frac{p}{2}}\Gamma\left(\frac{k\gamma}{2(\gamma-1)}+\frac{p}{2}\right)\Gamma\left(\frac{k\gamma}{2(\gamma-1)}+1\right)\sin\left(\frac{k\pi\gamma}{2(\gamma-1)}\right),$$

see [10] for details $\left(\text{where } \beta = \dfrac{\gamma}{\gamma - 1} \text{ and } r = \dfrac{1}{\beta}\right)$. As $\Psi_{\gamma}^{p}(x)$ is defined by (16), then the left hand side of (32) is analytic in $\mathbb{R}_{+}$. Moreover, $\psi_k(\gamma,\ p)$ is an analytic function for every defined order values $\gamma \in \mathbb{R} \backslash [0, 1]$ as the gamma functions are having poles at the negative integers. Using Stirling's formula, we obtain the inequality

$$|\Psi_k(\gamma,\ p)| > ck^{\delta k},$$

where c and δ are positive constants, see also [10]. □

4. Conclusion

This paper introduced a generalization of the multivariate Normal distribution, i.e., the multivariate γ-ordered Normal distribution, extending [7]. Moreover:

1. It proves that the well known distributions, as the multivariate Uniform, Normal, Laplace and the degenerated Dirac distributions are special cases of this generalization.

2. It discusses the influence of the shape parameter γ within this family of distributions, while a theoretical insight for the multivariate Uniform distribution is presented (Proposition 2.1).

3. The heavy-tailed members of this family, i.e., for order values $\gamma > 2$ and for $\gamma < 0$, are also discussed, see also Tables 1 and 2.

4. The moments and the characteristic function of this family are extensively studied.

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