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GENERALIZED LEGENDRE EXPANSION METHODS AND FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract

We use Legendre function expansion methods to find numerical solutions of linear ordinary differential equations on the interval $[a, b]$, and functional differential equations $y'(x) + p(x)y(h(x)) = g(x)$, and $y''(x) + p(x)y'(x) + q(x)y(h(x)) = g(x)$, where $h(x) \geq a$ and $g(x)$ are given, and $b - a > 1$.

1. Introduction

The study of ordinary differential equations plays an important role in physics, engineering and many other areas. It is not always possible to find exact solutions of differential equations, as such one needs to find their

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approximate numerical solutions. There are several ways in finding the numerical solutions for ordinary differential equations. One way is to convert any interval $[a, b]$ to the interval $[0, 1]$, but it is very unstable in many cases. One also encounters the accuracy problem which is usually not so good as expected. In this paper, we extend our method introduced in [1-3] from $[0, 1]$ to the interval $[a, b]$, use an orthogonal basis on $[a, b]$ and apply this orthogonal basis to find the approximate numerical solutions of functional differential equations. We obtain a very good accuracy and it is also easier to implement when we compare with other methods.

2. Legendre Function Expansion Methods

In [1-3], we introduce Legendre polynomial function expansion methods in solving the numerical solutions of functional integral and differential equations on the interval $[0, 1]$. We define an orthogonal basis on the interval $[0, 1]$ as $\{p_i(2x-1)\}_{i=0}^{\infty}$ (see details in [1-3]), i is the order of Legendre polynomial $p_i(x)$, which is defined on the interval $[-1, 1]$, and can be obtained by the following iteration formula:

$$p_0(x) = 1,$$

$$p_1(x) = x,$$

$$p_{n+1}(x) = \frac{2n+1}{n+1} x p_n(x) - \frac{n}{n+1} p_{n-1}(x), \quad n = 1, 2, 3, \ldots .$$

In what follows, we will extend this method to any bounded interval $[a, b]$ by changing the basis $\{p_i(2x-1)\}_{i=0}^{\infty}$ on $[0, 1]$ to the basis $\left\{p_i\left(2\frac{x-a}{b-a} - 1\right)\right\}_{i=0}^{\infty}$ on $[a, b]$. We have developed the following two theorems for our methods.

Theorem 1 (Uniform Convergence Theorem)**.** *If an integrable function* $f(x)$, *defined on* $[a, b]$, *has a bounded second derivative*, *then the function*

$f(x)$ can be expanded as $\sum_{i=0}^{\infty} a_i p_i\left(2\frac{x-a}{b-a}-1\right)$, where

$$a_i = \frac{2i+1}{b-a}\int_a^b f(x)\, p_i\left(2\frac{x-a}{b-a}-1\right)dx,$$

and the series converges to the function $f(x)$ uniformly.

Proof. Let $t = 2\frac{x-a}{b-a}-1$. Then $x = (b-a)\frac{t+1}{2}+a$ and $dx = \frac{b-a}{2}dt$. One has

$$\begin{aligned}
a_i &= \frac{2i+1}{b-a}\int_a^b f(x)\, p_i\left(2\frac{x-a}{b-a}-1\right)dx \\
&= \frac{2i+1}{b-a}\int_{-1}^1 f\left((b-a)\frac{t+1}{2}+a\right)p_i(t)\frac{b-a}{2}dt \\
&= \frac{2i+1}{2}\int_{-1}^1 f\left((b-a)\frac{t+1}{2}+a\right)p_i(t)dt \\
&= \frac{1}{2}\int_{-1}^1 f\left((b-a)\frac{t+1}{2}+a\right)d(p_{i+1}(t)-p_{i-1}(t)) \\
&= \frac{1}{2} f\left((b-a)\frac{t+1}{2}+a\right)(p_{i+1}(t)-p_{i-1}(t))\,\Big|_{-1}^{1} \\
&\quad -\frac{1}{2}\int_{-1}^1 f'\left((b-a)\frac{t+1}{2}+a\right)\frac{b-a}{2}(p_{i+1}(t)-p_{i-1}(t))dt \\
&= -\frac{b-a}{4}\int_{-1}^1 f'\left((b-a)\frac{t+1}{2}+a\right)(p_{i+1}(t)-p_{i-1}(t))dt \\
&= \frac{(b-a)^2}{8}\int_{-1}^1 f''\left((b-a)\frac{t+1}{2}+a\right) \\
&\qquad\qquad \times\left(\frac{p_{i+2}(t)-p_i(t)}{2i+3}-\frac{p_i(t)-p_{i-2}(t)}{2i-1}\right)dt.
\end{aligned}$$

Hence,

$$| a_i |^2 \leq \frac{(b-a)^4}{64} \int_{-1}^{1} \left| f''\left((b-a)\frac{t+1}{2} + a\right)\right|^2 dt$$

$$\times \int_{-1}^{1} \left(\frac{p_{i+2}(t) - p_i(t)}{2i+3} - \frac{p_i(t) - p_{i-2}(t)}{2i-1} \right)^2 dt$$

$$= \frac{M^2(b-a)^4}{32(2i+3)^2(2i-1)^2} \int_{-1}^{1} ((2i-1)^2 p_{i+2}^2(t)$$

$$+ 4(2i+1)^2 p_i^2(t) + (2i+3)^2 p_{i-2}^2(t))\,dt$$

$$= \frac{M^2(b-a)^4}{32(2i+3)^2(2i-1)^2} \left\{ \frac{2(2i-1)^2}{2i+5} + \frac{8(2i+1)^2}{2i+1} + \frac{2(2i+3)^2}{2i-3} \right\}$$

$$< \frac{M^2(b-a)^4}{32(2i+3)^2(2i-1)^2} \frac{12(2i+3)^2}{2i-3}$$

$$= \frac{3M^2(b-a)^4}{8(2i-1)^2(2i-3)}$$

$$\leq \frac{3M^2(b-a)^4}{8(2i-3)^3}.$$

Therefore

$$| a_i | \leq \frac{c}{(2i-3)^{\frac{3}{2}}},$$

where $c = \left(\frac{3M^2(b-a)^4}{8}\right)^{\frac{1}{2}}$. It follows that $\sum_{i=0}^{\infty} a_i$ converges, hence the expansion series converges uniformly.

Theorem 2 (Error Estimation)**.** *If an integrable function* $f(x)$, *defined on* $[a, b]$, *has a bounded second derivative*, *then we have the following error* *estimation for the expansion of the function* $f(x)$ *on* $[a, b]$,

$$\sigma_n \le \left(\sum_{i=n+1}^{\infty} \frac{3M^2(b-a)^5}{8(2i-3)^4} \right)^{\frac{1}{2}},$$

where

$$\sigma_n = \left\{ \int_a^b \left(f(x) - \sum_{i=0}^{n} a_i p_i \left(2\frac{x-a}{b-a} - 1 \right) \right)^2 dx \right\}^{\frac{1}{2}}.$$

We omit the proof here.

3. Numerical Examples

In this section, we will present two examples for ODE problems on the extended domain $[a, b]$. Our approach is to choose uniform partition points for x on $[a, b]$, and substitute $f(x) \approx \sum_{i=0}^{n} a_i p_i \left(2\frac{x-a}{r} - 1 \right)$ into the equation to form a linear system and get its approximate solution by least squares method. We also compute error σ_n for each of the following examples and list the results in the following tables, respectively, where $r = \max_{a \le x \le b} \{b - a, |h(x) - a|\}$. The absolute error in $L^2([a, b])$ is defined as follows:

$$\sigma_n = \left\{ \int_a^b \left(f(x) - \sum_{i=0}^{n} a_i p_i \left(2\frac{x-a}{b-a} - 1 \right) \right)^2 dx \right\}^{\frac{1}{2}}$$

$$\approx \left\{ \frac{1}{n} \sum_{i=0}^{n} e_n^2(x_i) \right\}^{\frac{1}{2}},$$

where $e_n(x_i) = y_{exact}(x_i) - y_n(x_i)$. In order to test efficiency of our methods, we select the following functions for $h(x) = x$, xe^{-x}, $\log(x + 4)$, $1.6\sin(x)$ and some quadratic functions.

Example 1.

$$y''(x) + xy' - x^2 y(h(x)) = g(x),$$

$$g(x) = -\sin(x) + x\cos(x) - x^2 \sin(h(x)),$$

$$-1 \le x \le 2,$$

$$y(-1) = -\sin(1),$$

$$y(2) = \sin(2).$$

The exact solution is $y(x) = \sin(x)$. We calculate errors for four different functions $h(x)$ as follows.

Table 1. Computed errors σ_n on $L^2([-1, 2])$

n	$h(x) = x$	$h(x) = xe^{-x}$	$h(x) = 1.6\sin(x)$	$h(x) = \log(x + 5)$
3	$5.92e-02$	$6.86e-02$	$5.37e-02$	$6.21e-02$
4	$1.48e-02$	$1.63e-02$	$1.45e-02$	$1.52e-02$
5	$2.08e-03$	$2.38e-03$	$1.95e-03$	$2.07e-03$
6	$4.22e-04$	$4.92e-04$	$4.17e-04$	$4.61e-04$
7	$4.57e-05$	$5.17e-05$	$4.52e-05$	$4.30e-05$
8	$7.48e-06$	$1.03e-05$	$7.61e-06$	$7.99e-06$
9	$6.32e-07$	$1.10e-06$	$6.46e-07$	$5.72e-07$

Example 2.

$$y'' + \sin(x)y'(x) + \cos(x)y(h(x)) = g(x),$$

$$g(x) = -\sin(x) - \sin(x)\cos(x) + \cos(x)\sin(h(x)),$$

$$-1 \le x \le 2,$$

$$y(-1) = \sin(-1),$$

$$y'(-1) = \cos(-1).$$

The exact solution is $y(x) = \sin(x)$. We calculate errors for four different functions $h(x)$ as follows.

Table 2. Computed errors σ_n on $L^2([-1, 2])$

n	$h(x) = x$	$h(x) = \frac{1}{6}(x+1)^2 + \frac{1}{2}$	$h(x) = 1.6\sin(x)$	$h(x) = \log(x+4)$
3	$3.79e-01$	$3.10e-01$	$4.28e-01$	$2.13e-01$
4	$1.25e-01$	$1.03e-01$	$1.83e-01$	$1.19e-01$
5	$2.00e-02$	$1.40e-02$	$2.72e-02$	$1.56e-02$
6	$6.94e-03$	$5.01e-03$	$9.30e-03$	$5.63e-03$
7	$6.31e-04$	$4.65e-04$	$7.27e-04$	$5.27e-04$
8	$1.66e-04$	$1.25e-04$	$1.84e-04$	$1.41e-04$
9	$1.16e-05$	$8.84e-06$	$1.31e-05$	$1.00e-05$
10	$2.43e-06$	$1.91e-06$	$2.71e-06$	$2.18e-06$

Remark. This generalized method can be applied to more general domain and has good accuracy results. It would be interesting to apply the method in this paper to solve two dimensional problems on the extended domain.

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