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COMMON FIXED POINT THEOREM FOR MAPPINGS SATISFYING A CONTRACTIVE CONDITION OF INTEGRAL TYPE IN *G*-CONE METRIC SPACE

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Abstract

In this paper, we prove a common fixed point theorem for two maps defined on a *G*-cone metric space, satisfying generalized contractive condition of integral type.

1. Introduction

In 2005, Mustafa and Sims introduced a new structure of generalized metric spaces [9], which is called a *G-metric space* as generalization of a metric space (X, d) to develop and introduce a new fixed point theory for various mappings in this new structure.

In 2007, Huang and Zhang introduced a cone metric space by substituting an ordered Banach space for the real numbers and proved some

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fixed point theorems in this space [8]. Many authors have studied this subject and many fixed point theorems have been proved.

In 2010, Beg et al. introduced a G-cone metric space that is an extension of cone metric space, and obtained a common fixed point theorem for mappings defined on a G-cone metric space satisfying a generalized contraction condition [2].

In this paper, we introduce a contraction condition of integral type on a G-cone metric space and present a common fixed point theorem for two maps defined on a G-cone metric space with this contraction condition.

Definition 1 [8]**.** Let E be real Banach space. A subset P of E is called a *cone* if and only if the following hold:

(a) P is closed, nonempty, and $P \neq \{0\}$;

(b) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply that $ax + by \in P$;

(c) $x \in P$ and $-x \in P$ imply that $x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in P^\circ$, where P° denotes the interior of P. The cone P is called *normal* if there is a constant $K > 0$ such that $0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$ for all $x, y \in E$. This positive number K is called the *normal constant*.

We suppose that E is a real Banach space, P is a cone in E with $P^\circ \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Definition 2 [8]**.** Let X be a nonempty set. A function $d : X \times X \to X$ is called a *cone metric* on X if it satisfies the following conditions:

(a) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

In 2010, Beg et al. in [2] introduced a G-cone metric space that is an extension of cone metric space.

Definition 3 [2]**.** Let X be a nonempty set. Suppose that the mapping $G : X \times X \times X \to E$ satisfies:

(a) $0 \leq G(x, y, z)$ for all $x, y, z \in X$ and $G(x, y, z) = 0$ if and only if $x = y = z$;

(b) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

(c) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;

(d) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ for all $x, y, z \in X$,

(symmetric in all three variables);

(e) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$,

(rectangle inequality).

Then G is called a *generalized cone metric* on X and (X, G) is called a *G-cone metric space*. The concept of a G-cone metric space is more general than that of G-metric spaces and cone metric spaces.

Definition 4 [2]**.** A G-cone metric space X is said to be *symmetric* if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 5 [3]**.** Let (X, G) be a G-cone metric space. A sequence $\{x_n\}$ in X is said to be

(a) *Cauchy sequence* if for every $c \in E$ with $0 \ll c$, there is $N \in \mathbb{N}$ such that for all $n, m, l > N$, $G(x_n, x_m, x_l) \ll c$.

(b) *Convergent sequence* if for every $c \in E$ with $0 \ll c$, there is $N \in \mathbb{N}$ such that for all $n, m > N$, $G(x_n, x_m, x) \ll c$ for some fixed $x \in X$. Here, x is called the *limit* of a sequence $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n = x$.

A G-cone metric space X is said to be *complete* if every Cauchy sequence in X is convergent in X.

Remark 1. Let (X, G) be G-cone metric space. For $x, y, z \in X$,

(a) if $x \ll y \ll z$, then $x \ll z$.

(b) if $x \ll y \leq z$, then $x \ll z$.

(c) if $x \leq y \ll z$, then $x \ll z$.

(d) if E is a real Banach space with cone P and if $a \leq \lambda a$, where $a \in P$ and $\lambda \in [0, 1)$, then $a = 0$.

Lemma 1 [2]**.** *Let X be a G-cone metric space. Then the following statements are equivalent.*

(a) $\{x_n\}$ *is convergent to* x.

(b) $G(x_n, x_n, x) \ll c$, *as* $n \to \infty$.

(c) $G(x_n, x, x) \ll c$, *as* $n \to \infty$.

(d) $G(x_n, x_m, x) \ll c$, *as* $m, n \to \infty$.

Lemma 2 [2]**.** *Let X be a G-cone metric space.*

(a) *If* $\{x_n\}$, $\{y_m\}$ *and* $\{z_l\}$ *are sequences in X such that* $x_n \to x$, $y_m \to y$ *and* $z_l \to z$, *then* $G(x_n, y_m, z_l) \to G(x, y, z)$ *as* $n, m, l \to \infty$.

(b) *If* $\{x_n\}$ *is a sequence in X and* $x, y \in X$, *such that* $\{x_n\}$ *converges to x, y, then* $x = y$.

(c) *If* $\{x_n\}$ *is a sequence in X,* $x \in X$ *and* $\{x_n\}$ *converges to x, then* $G(x_m, x_n, x) \to 0$, *as* $m, n \to \infty$.

(d) *If* $\{x_n\}$ *is a sequence in X,* $x \in X$ *and* $\{x_n\}$ *converges to* $x \in X$, *then* $\{x_n\}$ *is Cauchy sequence.*

(e) *If* $\{x_n\}$ *is a Cauchy sequence in X, then* $G(x_n, x_m, x_l) \to 0,$ *as* $n, m, l \to \infty.$

Definition 6. Let X be a G-cone metric space and f, g be self mappings on X. If $y = fx = gx$ for some $x, y \in X,$ then x is called a *coincidence point* of f and g, and y is called a *point of coincidence* of f and g.

Definition 7. Let (X, G) be a G-cone metric space. The self mappings f and g defined on X are said to be *compatible* if

$$\lim_{n\to\infty} G(fgx_n - gfx_n,\ fgx_n - gfx_n,\ 0) = 0$$

or equivalently for all $c \in E$ and $0 \ll c,$ there exists an $N \in \mathbb{N}$ such that for all $n > N,$

$$G(fgx_n - gfx_n,\ fgx_n - gfx_n,\ 0) \ll c,$$

where $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$ $t \in X.$

Definition 8 [5]**.** Let X be a G-cone metric space and f, g be two self mappings of X. Then f and g are said to be *weakly compatible* if $ft = gt$ imply $fgt = gft$ for all $t \in X.$

Remark 2. Compatible self mappings are weakly compatible.

Lemma 3 [1]**.** *Let f and g be weakly compatible self maps of a set X. If f and g have a unique coincidence point* $y = fx = gx,$ *then y is the unique common fixed point of f and g.*

In 2002, Branciari [4] introduced a general contractive condition of integral type as follows.

Theorem 1. *Let* (X, d) *be a complete metric space,* $\alpha \in (0, 1)$ *and* $f : X \to X$ *be a mapping such that, for all* $x, y \in X,$

$$\int_0^{d(fx,\, fy)} \varphi(t)dt \le \alpha \int_0^{d(x,\, y)} \varphi(t)dt,$$

where $\varphi : [0, \infty) \to [0, \infty)$ *is a Lebesgue integrable mapping which is summable* (*i.e., with finite integral*) *on each compact subset of* $[0, \infty)$, *such that for each* $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t) dt > 0$. *Then f has a unique fixed point* $a \in X$ *such that for each* $x \in X$, $\lim_{n \to \infty} f^n x = a$.

Khojasteh et al. in [6] introduced the concept of integrability of the function $\varphi : [a, b] \to P$ with respect to a cone. We generalize this concept and result in *G*-cone metric spaces and obtain coincidence and common fixed point theorems for mappings defined on a *G*-cone metric space satisfying a generalized contractive condition of integral type.

Definition 9 [6]**.** Suppose that *P* is a normal cone in *E*. Let $a, b \in E$ and $a < b$. We define

$$[a, b] := \{x \in E : x = tb + (1 - t)a \text{ for some } t \in [0, 1]\},$$

$$[a, b) := \{x \in E : x = tb + (1 - t)a \text{ for some } t \in [0, 1)\}.$$

Definition 10 [6]**.** The set $\mathcal{P} = \{a = x_0, x_1, ..., x_n = b\}$ is called a *partition* for $[a, b]$ if and only if the sets $\{[x_{i-1}, x_i)\}_{i=1}^{n}$ are pairwise disjoint and $[a, b] = \bigcup_{i=1}^{n} [x_{i-1}, x_i) \cup \{b\}$.

Definition 11 [6]**.** For each partition $\mathcal{Q}$ of $[a, b]$ and each increasing function $\varphi : [a, b] \to P$, we define *cone lower summation* and *cone upper summation* as

$$L_n^{Con}(\varphi, \mathcal{Q}) = \sum_{i=0}^{n-1} \varphi(x_i) \| x_i - x_{i-1} \|,$$

$$U_n^{Con}(\varphi, \mathcal{Q}) = \sum_{i=0}^{n-1} \varphi(x_{i+1}) \| x_i - x_{i-1} \|,$$

respectively.

Definition 12 [6]**.** Let P be a normal cone in E. The function $\varphi : [a, b] \to P$ is called an *integrable function on* $[a, b]$ *with respect to cone P* or to simplicity, *cone integrable function* if and only if for all partition $\mathcal{Q}$ of $[a, b]$,

$$\lim_{n\to\infty} L_n^{Con}(\varphi, \mathcal{Q}) = S^{Con} = \lim_{n\to\infty} U_n^{Con}(\varphi, \mathcal{Q}),$$

where S^{Con} must be unique. We show that the common value S^{Con} is given by

$$\int_a^b \varphi(x)\,dP(x) \text{ or to simplicity, } \int_a^b \varphi dP.$$

We denote the set of all cone integrable functions by $\pounds^1([a, b], P)$.

Lemma 4 [6]**.** *For each* $f, g \in \pounds^1(X, P)$ *and* $\alpha, \beta \in \mathbb{R}$, *we have*:

(a) *If* $[a, b] \subset [a, c]$, *then* $\int_a^b f dP \leq \int_a^c f dP$;

(b) $\int_a^b (\alpha f + \beta g) dP = \alpha \int_a^b f dP + \beta \int_a^b g dP$.

Proof. See [6]. □

In 2010, Beg et al. in [3] obtained a common fixed point theorem for mappings defined on a G-cone metric space, satisfying a generalized contractive condition:

Theorem 2 [3]**.** *Let* X *be a* G*-cone metric space and mappings* $f, g : X \to X$ *satisfying*

$$G(fx, fy, fz) \leq h u_{(f,g)}(x, y, z),$$

where

$$u_{(f,g)}(x, y, z) \in \Big\{ G(gx, gy, gz), G(gx, fx, fx), G(gy, fy, fy), G(gz, fz, fz), G(gy, fz, fz), \frac{G(gx, fy, fy) + G(gz, fx, fx)}{2} \Big\},$$

for all $x, y, z \in X$ *and* $0 \leq h < 1$. *If the range of g contains the range of f and* $g(X)$ *is a complete subspace of X, then f and g have a coincidence point in X. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point.*

2. Main Results

In this section, several coincidence and common fixed point theorems for mappings defined on a *G*-cone metric space, satisfying generalized contractive conditions of integral type are obtained.

Theorem 3. *Let f and g be weakly compatible self mappings of G-cone metric space* (X, G) *satisfying the following conditions*:

(a) $f(X) \subset g(X)$ *and* $g(X)$ *is complete,*

(b) $\int_0^{G(fx, fy, fz)} \varphi dP \leq h \int_0^{u_{(f,g)}(x, y, z)} \varphi dP,$

for all $x, y, z \in X$ *and* $h \in [0, 1)$, *where*

$$u_{(f,g)}(x, y, z) \in \Big\{ G(gx, gy, gz), G(gx, fx, fx), G(gy, fy, fy), G(gz, fz, fz), \frac{G(gx, fy, fy) + G(gy, fx, fx)}{2}, \frac{G(gx, fz, fz) + G(gz, fx, fx)}{2}, \frac{G(gy, fz, fz) + G(gz, fy, fy)}{2} \Big\}$$

and $\varphi : P \to P$ *is a nonvanishing map, cone integrable on each* $[a, b] \subset P$ *such that for all* $0 \ll \varepsilon$, $0 \ll \int_0^{\varepsilon} \varphi dP$. *Then f and g have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Since the range of g contains range of f, choose $x_1 \in X$ such that $fx_0 = gx_1$. Continuing this process, having choose $x_n \in X$, we obtain $gx_{n+1} = fx_n$. Then, from (b), we have

$$\int_0^{G(gx_n, gx_{n+1}, gx_{n+1})} \varphi dP = \int_0^{G(fx_{n-1}, fx_n, fx_n)} \varphi dP$$

$$\leq h \int_0^{u_{(f,g)}(x_{n-1}, x_n, x_n)} \varphi dP,$$

where

$$u_{(f,g)}(x_{n-1}, x_n, x_n)$$

$$\in \Big\{ G(gx_{n-1}, gx_n, gx_n), G(gx_{n-1}, fx_{n-1}, fx_{n-1}),$$

$$G(gx_n, fx_n, fx_n), \frac{G(gx_{n-1}, fx_n, fx_n) + G(gx_n, fx_{n-1}, fx_{n-1})}{2} \Big\}$$

$$= \Big\{ G(gx_{n-1}, gx_n, gx_n), G(gx_n, gx_{n+1}, gx_{n+1})$$

$$\frac{G(gx_{n-1}, gx_{n+1}, gx_{n+1}) + G(gx_n, gx_n, gx_n)}{2} \Big\}$$

$$= \Big\{ G(gx_{n-1}, gx_n, gx_n), G(gx_n, gx_{n+1}, gx_{n+1})$$

$$\frac{G(gx_{n-1}, gx_{n+1}, gx_{n+1})}{2} \Big\}.$$

Case 1. If, for some n, $u_{(f,g)}(x_{n-1}, x_n, x_n) = G(gx_{n-1}, gx_n, gx_n)$, then

$$\int_0^{G(gx_n, gx_{n+1}, gx_{n+1})} \varphi dP \leq h \int_0^{G(gx_{n-1}, gx_n, gx_n)} \varphi dP. \tag{2.1}$$

Case 2. If, for some n, $u_{(f,g)}(x_{n-1}, x_n, x_n) = G(gx_n, gx_{n+1}, gx_{n+1})$, then

$$\int_0^{G(gx_n, gx_{n+1}, gx_{n+1})} \varphi dP \leq h \int_0^{G(gx_n, gx_{n+1}, gx_{n+1})} \varphi dP.$$

Since $h \in [0, 1)$, we have $G(gx_n, gx_{n+1}, gx_{n+1}) = 0$, which implies that $gx_n = gx_{n+1} = fx_n$ for each n, and f, g have x_n as a coincidence point.

Case 3. If, for some n, $u_{(f,g)}(x_{n-1}, x_n, x_n) = \dfrac{G(gx_{n-1}, gx_{n+1}, gx_{n+1})}{2}$, then using Definition 3(e),

$$\frac{G(gx_{n-1}, gx_{n+1}, gx_{n+1})}{2}$$

$$\leq \frac{G(gx_{n-1}, gx_n, gx_n) + G(gx_n, gx_{n+1}, gx_{n+1})}{2}$$

$$\leq \max\{G(gx_{n-1}, gx_n, gx_n), G(gx_n, gx_{n+1}, gx_{n+1})\}.$$

Therefore, we have

$$\int_0^{G(gx_n, gx_{n+1}, gx_{n+1})} \varphi dP$$

$$\leq h \int_0^{\frac{G(gx_{n-1}, gx_{n+1}, gx_{n+1})}{2}} \varphi dP$$

$$\leq h \int_0^{\max\{G(gx_{n-1}, gx_n, gx_n), G(gx_n, gx_{n+1}, gx_{n+1})\}} \varphi dP$$

$$= h \max\left\{\int_0^{G(gx_{n-1}, gx_n, gx_n)} \varphi dP, \int_0^{G(gx_n, gx_{n+1}, gx_{n+1})} \varphi dP\right\}$$

and Case 3 reduces to either Case 1 or Case 2.

Suppose that there exist two distinct common coincidence points u and v; i.e., $fu = gu$ and $fv = gv$, but $gu \neq gv$.

$$u_{(f,g)}(u, v, v) \in \Big\{G(gu, gv, gv), G(gu, fu, fu), G(gv, fv, fv),$$

$$G(gv, fv, fv), \frac{G(gu, fv, fv) + G(gv, fu, fu)}{2},$$

$$\left.\frac{G(gv,\ fv,\ fv)+G(gv,\ fu,\ fu)}{2}\right\}$$

$$= \left\{G(gu,\ gv,\ gv),\ \frac{G(gu,\ gv,\ gv)+G(gv,\ gu,\ gu)}{2}\right\}$$

$$= \{q_1,\ q_2\} \text{ say.}$$

Case I. Suppose that $u_{(f,g)}(u,\ v,\ v) = q_1$. Then we have

$$\int_0^{G(fu,\, fv,\, fv)} \varphi dP = \int_0^{G(gu,\, gv,\, gv)} \varphi dP \leq h \int_0^{G(gu,\, gv,\, gv)} \varphi dP,$$

which implies that $gu = gv$, a contradiction.

Case II. Suppose that $u_{(f,g)}(u,\ v,\ v) = q_2$. Then we have, using Definition 3(c),

$$\int_0^{G(gu,\, gv,\, gv)} \varphi dP \leq h \int_0^{\frac{G(gu,\, gv,\, gv)+G(gv,\, gu,\, gu)}{2}} \varphi dP$$

$$\leq \int_0^{G(gu,\, gv,\, gv)} \varphi dP,$$

which yields $gu = gv$, a contradiction, and the common coincidence point is unique.

Since Case 2 immediately yields the uniqueness of the coincidence point, we shall assume that Case 1 is true for all n.

Using (1), with $d_n := G(gx_n,\ gx_{n+1},\ gx_{n+1})$,

$$\int_0^{G(gx_n,\, gx_{n+1},\, gx_{n+1})} \varphi dP \leq h \int_0^{G(gx_{n-1},\, gx_n,\, gx_n)} \varphi dP$$

$$\leq \cdots$$

$$\leq h^n \int_0^{G(gx_0,\, gx_1,\, gx_1)} \varphi dP,$$

and therefore $\lim_n G(gx_n,\ gx_{n+1},\ gx_{n+1}) = 0$.

We shall now show that $\{gx_n\}$ is G-Cauchy. From [9, Proposition 9], a sequence $\{x_n\}$ is G-Cauchy if and only if for each $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for each $n, m \geq N$.

Suppose that $\{gx_n\}$ is not G-Cauchy. Then there exists $\varepsilon > 0$ and subsequences $m(k)$ and $n(k)$, with $m(k) < n(k) < m(k+1)$ such that

$$G(gx_{m(k)}, gx_{n(k)}, gx_{n(k)}) \geq \varepsilon, \text{ and } G(gx_{m(k)}, gx_{n(k)-1}, gx_{n(k)-1}) < \varepsilon. \tag{2.2}$$

From (2.2), and using Definition 3(c),

$$\begin{aligned}\varepsilon &\leq G(gx_{m(k)}, gx_{n(k)}, gx_{n(k)})\\ &\leq G(gx_{m(k)}, gx_{n(k)-1}, gx_{n(k)-1}) + d_{m(k)-1}.\end{aligned} \tag{2.3}$$

Taking the limit of (2.3) as $n \to \infty$ yields

$$\lim_k G(gx_{m(k)}, gx_{n(k)}, gx_{n(k)}) = \varepsilon.$$

From (b),

$$\int_0^{G(gx_{m(k)}, gx_{n(k)}, gx_{n(k)})} \varphi dP \leq h \int_0^{G(gx_{m(k)-1}, gx_{n(k)-1}, gx_{n(k)-1})} \varphi dP. \tag{2.4}$$

But, using Definition 3(e),

$$\begin{aligned}G(gx_{m(k)-1}, gx_{n(k)-1}, gx_{n(k)-1}) &\leq G(gx_{m(k)-1}, gx_{m(k)}, gx_{m(k)})\\ &\quad + G(gx_{m(k)}, gx_{n(k)-1}, gx_{n(k)-1})\\ &\leq d_{m(k)-1} + G(gx_{m(k)}, gx_{n(k)}, gx_{n(k)})\\ &\quad + G(gx_{n(k)}, gx_{n(k)-1}, gx_{n(k)-1})\\ &= d_{m(k)-1} + G(gx_{m(k)}, gx_{n(k)}, gx_{n(k)})\\ &\quad + d_{n(k)-1}.\end{aligned}$$

Substituting into (2.4) and taking the limit as $n \to \infty$ yields

$$\int_0^{\varepsilon} \varphi dP \leq h \int_0^{\varepsilon} \varphi dP,$$

a contradiction. Therefore, $\{gx_n\}$ is G-Cauchy, hence convergent to some point $u \in X$. Since $gx_n = fx_{n-1}$, the G-limit of fx_n is also u. Using hypothesis (a), there exists a point $v \in X$ such that $gv = u$.

We claim that $fu = gv$. Using (b),

$$\int_0^{G(gx_n,\, fv,\, fv)} \varphi dP = \int_0^{G(fx_{n-1},\, fv,\, fv)} \varphi dP \leq h \int_0^{u_{(f,g)}(x_{n-1}, v, v)} \varphi dP,$$

where

$$u_{(f,g)}(x_{n-1}, v, v) \in \Big\{ G(gx_{n-1}, gv, gv), G(gx_{n-1}, gx_n, gx_n), G(gv, fv, fv), \frac{G(gx_{n-1}, fv, fv) + G(gv, gx_n, gx_n)}{2} \Big\}.$$

Case 1. If, for some n, $u_{(f,g)}(x_{n-1}, v, v) = G(gx_{n-1}, gv, gv)$, then

$$\int_0^{G(gx_n,\, fv,\, fv)} \varphi dP \leq h \int_0^{G(gx_{n-1},\, gv,\, gv)} \varphi dP. \tag{2.5}$$

Case 2. If, for some n, $u_{(f,g)}(x_{n-1}, v, v) = G(gx_{n-1}, gx_n, gx_n)$, then

$$\int_0^{G(gx_n,\, fv,\, fv)} \varphi dP \leq h \int_0^{G(gx_{n-1},\, gx_n,\, gx_n)} \varphi dP. \tag{2.6}$$

Case 3. If, for some n, $u_{(f,g)}(x_{n-1}, v, v) = G(gv, fv, fv)$, then

$$\int_0^{G(gx_n,\, fv,\, fv)} \varphi dP \leq h \int_0^{G(gv,\, fv,\, fv)} \varphi dP. \tag{2.7}$$

Case 4. If, for some n,

$$u_{(f,g)}(x_{n-1}, v, v) = \frac{G(gx_{n-1}, fv, fv) + G(gv, gx_n, gx_n)}{2},$$

then

$$\int_0^{G(gx_n,\, fv,\, fv)} \varphi dP \le h \int_0^{\frac{G(gx_{n-1},\, fv,\, fv) + G(gv,\, gx_n,\, gx_n)}{2}} \varphi dP$$

$$\le h \int_0^{G(gx_{n-1},\, fv,\, fv) + G(gv,\, gx_n,\, gx_n)} \varphi dP$$

$$\le h \int_0^{G(gx_{n-1},\, fv,\, fv)} \varphi dP + h \int_0^{G(gv,\, gx_n,\, gx_n)} \varphi dP. \quad (2.8)$$

At least one of (2.5)-(2.8) must occur for an infinite number of times. Taking the limit as $n \to \infty$ of each of these inequalities yields

$$\int_0^{G(gv,\, fv,\, fv)} \varphi dP \le h \int_0^{G(gv,\, fv,\, fv)} \varphi dP,$$

which implies that $gv = fv$.

It has already been shown that the coincidence point is unique. The result now follows from Remark 2 and Lemma 3. $\square$

Corollary 1. *Let f and g be compatible self mappings of G-cone metric space* (X, G), *such that for* $m \in \mathbb{N}$ *satisfying the following conditions*:

(a) $f(X) \subset g(X)$, $g(X)$ *is complete,*

(b) $\int_0^{G(f^m x,\, f^m y,\, f^m z)} \varphi dP \le h \int_0^{u_{(f,g)}(x,\, y,\, z)} \varphi dP$,

for all $x, y, z \in X$, $h \in [0, 1)$, *where*

$$u_{(f,g)}(x, y, z) \in \Bigg\{ G(g^m x,\, g^m y,\, g^m z),\, G(g^m x,\, f^m x,\, f^m x),$$

$$G(g^m y,\, f^m y,\, f^m y),\, G(g^m z,\, f^m z,\, f^m z),$$

$$\frac{G(g^m x,\, f^m y,\, f^m y) + G(g^m y,\, f^m x,\, f^m x)}{2},$$

$$\left.\frac{G(g^m x,\ f^m z,\ f^m z) + G(g^m z,\ f^m x,\ f^m x)}{2},\quad \frac{G(g^m y,\ f^m z,\ f^m z) + G(g^m z,\ f^m y,\ f^m y)}{2}\right\}$$

and $\varphi : P \to P$ *is a nonvanishing subadditive cone integrable on each* $[a, b] \subset P$ *such that for all* $0 \ll \varepsilon,\ 0 \ll \int_0^{\varepsilon} \varphi(t)\,dt$. *Then f and g have a unique common fixed point.*

Proof. It follows from Theorem 3, that f^m, g^m have a unique common fixed point $p \in X$. Therefore, $fp = f(f^m p) = f^{m+1}p = f^m(fp)$, and $gp = g(g^m p) = g^{m+1}p = g^m(fp)$ implies that fp and gp are also fixed point for f^m and g^m. Hence $fp = gp = p$. □

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