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SOLUTION AND STABILITY OF A QUADRATIC FUNCTIONAL EQUATION IN BANACH SPACE AND ITS APPLICATION

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Abstract

In this paper, we obtain the general solution and generalized Ulam-Hyers stability of a new type of quadratic functional equation of the form

$$f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_0-x_1}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)+f\left(\frac{x_2-x_3}{2}\right)$$

$$=\frac{1}{2}[f(x_0)+f(x_1)+f(x_2)+f(x_3)]$$

in Banach spaces. An application of the above quadratic functional equation is also discussed in this paper.

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1. Introduction

In 1940, the stability of functional equations had been first raised by Ulam [25]. In 1941, Hyers [12] gave an affirmative answer to the question of Ulam for Banach spaces.

In 1950, Aoki [2] was the second author to treat this problem for additive mappings. In 1978, Rassias [20] succeeded in extending Hyers' Theorem by weakening the condition for the Cauchy difference controlled by $(\| x \|^p + \| y \|^p),\ p \in [0, 1),$ to be unbounded.

In 1982, Rassias [19] replaced the factor $\| x \|^p + \| y \|^p$ by $\| x \|^p \| y \|^q$ for $p, q \in R.$ A generalization of all the above stability results was obtained by Gavruta [10] in 1994 by replacing the unbounded Cauchy difference by a general control function $\varphi(x, y).$

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi et al. [23] by considering the summation of both the sum and the product of two p-norms. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 9, 13, 16, 18]) and reference cited therein.

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

and therefore the equation (1.1) is called *quadratic functional equation*.

The Hyers-Ulam stability theorem for the quadratic functional equation (1.1) was proved by Skof [24] for the functions $f : E_1 \to E_2,$ where E_1 is a normed space and E_2 is a Banach space. The result of Skof is still true if the relevant domain E_1 is replaced by an Abelian group and it was dealt by Cholewa [7]. Czerwik [8] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.1). This result was further generalized by Rassias [22], Borelli and Forti [6].

The solution and stability of following quadratic functional equations

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(x+z), \tag{1.2}$$

$$f(x-y-z)+f(x)+f(y)+f(z)=f(x-y)+f(y+z)+f(z-x), \tag{1.3}$$

$$f(x+y+z)+f(x-y)+f(y-z)+f(z-x)=3f(x)+3f(y)+3f(z), \tag{1.4}$$

$$f\left(\sum_{i=1}^{n} x_i\right)+\sum_{1\le i<j\le n} f(x_i-x_j)=n\sum_{i=1}^{n} f(x_i),\ (n\ge 2), \tag{1.5}$$

$$f(2x\pm y\pm z)+2f(y)+2f(z)=2f(x\pm y)+2f(x\pm z)+f(y+z) \tag{1.6}$$

were investigated by Jung [14, 15], Kannappan [17], Bae and Jun [5], Bae [4] and Arunkumar et al. [3].

In this paper, the authors have proved the general solution and generalized Ulam-Hyers stability of a new type of quadratic functional equation of the form

$$f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_0-x_1}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)+f\left(\frac{x_2-x_3}{2}\right)$$

$$=\frac{1}{2}[f(x_0)+f(x_1)+f(x_2)+f(x_3)] \tag{1.7}$$

in Banach spaces.

In Section 2, the general solution of the functional equation (1.7) is given. In Section 3, the generalized Ulam-Hyers stability of the functional equation (1.7) is proved. An application of the quadratic functional equation (1.7) is discussed in Section 4.

2. General Solution of the Functional Equation (1.7)

In this section, the authors investigate the general solution of the functional equation (1.7). Throughout this section, let us consider *X* and *Y* be real vector spaces.

Theorem 2.1. *Let* $f : X \to Y$ *be a function satisfying the functional equation*

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{2.1}$$

for all $x, y \in X$ *if and only if* $f : X \to Y$ *satisfies the functional equation*

$$f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_0-x_1}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)+f\left(\frac{x_2-x_3}{2}\right)$$

$$=\frac{1}{2}[f(x_0)+f(x_1)+f(x_2)+f(x_3)] \tag{2.2}$$

for all $x_0, x_1, x_2, x_3 \in X$.

Proof. Let $f : X \to Y$ satisfy (2.1). Setting $x = y = 0$ in (2.1), we get $f(0) = 0$. Let $x = 0$ in (2.1), we obtain $f(-y) = f(y)$ for all $y \in X$. Therefore, f is an even function. Replacing y by x and $2x$ respectively in (2.1), we get $f(2x) = 2^2 f(x)$ and $f(3x) = 3^2 f(x)$ for all $x \in X$. In general, for any positive integer a, we have $f(ax) = a^2 f(x)$ for all $x \in X$. Replacing (x, y) by $\left(\frac{x_0}{2}, \frac{x_1}{2}\right)$ in (2.1), we obtain

$$f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_0-x_1}{2}\right)=\frac{1}{2}[f(x_0)+f(x_1)] \tag{2.3}$$

for all $x_0, x_1 \in X$. Again replacing (x, y) by $\left(\frac{x_2}{2}, \frac{x_3}{2}\right)$ in (2.1), we obtain

$$f\left(\frac{x_2+x_3}{2}\right)+f\left(\frac{x_2-x_3}{2}\right)=\frac{1}{2}[f(x_2)+f(x_3)] \tag{2.4}$$

for all $x_2, x_3 \in X$. Adding (2.3) and (2.4), we arrive (1.7) as desired.

Conversely assume that $f : X \to Y$ satisfies (2.2). Setting (x_0, x_1, x_2, x_3) by $(0, 0, 0, 0)$ in (2.2), we get $f(0) = 0$. Letting (x_0, x_1, x_2, x_3) by $(2x, 0, 0, 0)$ in (2.2), we obtain $f(2x) = 2^2 f(x)$ for all $x \in X$. Replacing

(x_0, x_1, x_2, x_3) by $(0, x, 0, 0)$ in (2.2), we arrive $f(x) = f(-x)$ for all $x \in X$. Therefore, f is an even function. Again replacing (x_0, x_1, x_2, x_3) by $(2x, x, 0, 0)$ in (2.2), we get $f(3x) = 3^2 f(x)$ for all $x \in X$. In general, for any positive integer b, we have $f(bx) = b^2 f(x)$ for all $x \in X$. Replacing (x_0, x_1, x_2, x_3) by $(x + y, x - y, 0, 0)$ in (2.2), we obtain (2.1) as desired. □

3. Generalized Ulam-Hyers Stability of the Quadratic Functional Equation (1.7)

In this section, the authors presented the generalized Ulam-Hyers stability of the functional equation (1.7).

Throughout this section, let X be a normed space and Y be a Banach space, respectively. Define a mapping $Df : X^4 \to Y$ by

$$Df(x_0, x_1, x_2, x_3)$$

$$= f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_0 - x_1}{2}\right) + f\left(\frac{x_2 + x_3}{2}\right) + f\left(\frac{x_2 - x_3}{2}\right)$$

$$- \frac{1}{2}[f(x_0) + f(x_1) + f(x_2) + f(x_3)]$$

for all $x_0, x_1, x_2, x_3 \in X$.

Theorem 3.1. *Let* $j \in \{-1, 1\}$. *Assume* $\alpha : X^4 \to [0, \infty)$ *is a function such that*

$$\sum_{n=0}^{\infty} \frac{\alpha(2^{nj} x_0, 2^{nj} x_1, 2^{nj} x_2, 2^{nj} x_3)}{4^{nj}}$$

converges in $\mathbb{R}$ *and*

$$\lim_{n \to \infty} \frac{\alpha(2^{nj} x_0, 2^{nj} x_1, 2^{nj} x_2, 2^{nj} x_3)}{4^{nj}} = 0, \tag{3.1}$$

for all $x_0, x_1, x_2, x_3 \in X$. *Let* $f : X \to Y$ *be a function satisfying the* *inequality*

$$\| Df(x_0, x_1, x_2, x_3) \| \leq \alpha(x_0, x_1, x_2, x_3) \tag{3.2}$$

for all $x_0, x_1, x_2, x_3 \in X$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\| f(x) - Q(x) \| \leq \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(2^{(k+1)j} x, 0, 0, 0)}{4^{kj}} \tag{3.3}$$

for all $x \in X$. *The mapping* $Q(x)$ *is defined by*

$$Q(x) = \lim_{n\to\infty} \frac{f(2^{nj} x)}{4^{jn}} \tag{3.4}$$

for all $x \in X$.

Proof. Assume $j = 1$. Replacing (x_0, x_1, x_2, x_3) by $(2x, 0, 0, 0)$ in (3.2) and dividing by 4, we get

$$\left\| f(x) - \frac{f(2x)}{4} \right\| \leq \frac{\alpha(2x, 0, 0, 0)}{4} \tag{3.5}$$

for all $x \in X$. Now replacing *x* by 2*x* and dividing by 4 in (3.5), we get

$$\left\| \frac{f(2x)}{4} - \frac{f(2^2 x)}{4^2} \right\| \leq \frac{\alpha(2^2 x, 0, 0, 0)}{4^2} \tag{3.6}$$

for all $x \in X$. From (3.5) and (3.6), we obtain

$$\begin{aligned} \left\| f(x) - \frac{f(2^2 x)}{4^2} \right\| &\leq \left\| f(x) - \frac{f(2x)}{4} \right\| + \left\| \frac{f(2x)}{4} - \frac{f(2^2 x)}{4^2} \right\| \\ &\leq \frac{1}{4} \left[\alpha(2x, 0, 0, 0) + \frac{\alpha(2^2 x, 0, 0, 0)}{4} \right] \end{aligned} \tag{3.7}$$

for all $x \in X$. In general, for any positive integer *n*, we get

$$\left\| f(x) - \frac{f(2^n x)}{4^n} \right\| \le \frac{1}{4} \sum_{k=0}^{n-1} \frac{\alpha(2^{(k+1)} x, 0, 0, 0)}{4^k}$$

$$\le \frac{1}{4} \sum_{k=0}^{\infty} \frac{\alpha(2^{(k+1)} x, 0, 0, 0)}{4^k} \tag{3.8}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{ \frac{f(2^n x)}{4^n} \right\}$, replacing x by $2^m x$ and dividing by 4^m in (3.8), for any $m, n > 0$, we deduce

$$\left\| \frac{f(2^m x)}{4^m} - \frac{f(2^{n+m} x)}{4^{(n+m)}} \right\| = \frac{1}{4^m} \left\| f(2^m x) - \frac{f(2^n \cdot 2^m x)}{4^n} \right\|$$

$$\le \frac{1}{4} \sum_{k=0}^{\infty} \frac{\alpha(2^{k+m+1} x, 0, 0, 0)}{4^{k+m}}$$

$$\to 0 \quad \text{as} \quad m \to \infty$$

for all $x \in X$. Hence, the sequence $\left\{ \frac{f(2^n x)}{4^n} \right\}$ is Cauchy sequence. Since Y is complete, there exists a mapping $Q : X \to Y$ such that

$$Q(x) = \lim_{n \to \infty} \frac{f(2^n x)}{4^n}, \quad \forall x \in X.$$

Letting $n \to \infty$ in (3.8), we see that (3.3) holds for all $x \in X$. In order to prove, Q satisfies (1.7), replacing (x_0, x_1, x_2, x_3) by $(2^n x_0, 2^n x_1, 2^n x_2, 2^n x_3)$ and dividing by 4^n in (3.2), we obtain

$$\frac{1}{4^n} \left\| f\left(\frac{2^n (x_0 + x_1)}{2} \right) + f\left(\frac{2^n (x_0 - x_1)}{2} \right) + f\left(\frac{2^n (x_2 + x_3)}{2} \right) \right.$$

$$+ f\left(\frac{2^n(x_2 - x_3)}{2}\right) - \frac{1}{2}[f(2^n x_0) + f(2^n x_1) + f(2^n x_2) + f(2^n x_3)]\Bigg\|$$

$$\leq \frac{1}{4^n}\alpha(2^n x_0, 2^n x_1, 2^n x_2, 2^n x_3)$$

for all $x_0, x_1, x_2, x_3 \in X$. Letting $n \to \infty$ in the above inequality and using the definition of $Q(x)$, we see that

$$Q\left(\frac{x_0 + x_1}{2}\right) + Q\left(\frac{x_0 - x_1}{2}\right) + Q\left(\frac{x_2 + x_3}{2}\right) + Q\left(\frac{x_2 - x_3}{2}\right)$$

$$= \frac{1}{2}[Q(x_0) + Q(x_1) + Q(x_2) + Q(x_3)].$$

Hence, Q satisfies (1.7) for all $x_0, x_1, x_2, x_3 \in X$. To prove that Q is unique, we let $R(x)$ be another mapping satisfying (1.7) and (3.3). Then

$$\| Q(x) - R(x) \| = \frac{1}{4^n}\| Q(2^n x) - R(2^n x) \|$$

$$\leq \frac{1}{4^n}\{\| Q(2^n x) - f(2^n x) \| + \| f(2^n x) - R(2^n x) \|\}$$

$$\leq \frac{1}{2}\sum_{k=0}^{\infty}\frac{\alpha(2^{k+n+1}x, 0, 0, 0)}{4^{(k+n)}}$$

$$\to 0 \quad \text{as} \quad n \to \infty$$

for all $x \in X$. Hence, Q is unique.

For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.7).

Corollary 3.2. *Let λ and s be nonnegative real numbers. Let a function*

$f : X \to Y$ satisfy the inequality

$$\| Df(x_0, x_1, x_2, x_3) \| \le \begin{cases} \lambda, \\ \lambda\left(\sum_{i=0}^{3} \| x_i \|^s\right), & s < 2 \text{ or } s > 2; \\ \lambda\left\{\prod_{i=0}^{3} \| x_i \|^s + \left(\sum_{i=0}^{3} \| x_i \|^{4s}\right)\right\}, & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; \end{cases} \tag{3.9}$$

for all $x_0, x_1, x_2, x_3 \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \le \begin{cases} \frac{2\lambda}{3}, \\ \frac{2^{(s+1)}\lambda \| x \|^s}{| 4 - 2^s |}, \\ \frac{2^{(4s+1)}\lambda \| x \|^{4s}}{| 4 - 2^{4s} |}, \end{cases} \tag{3.10}$$

for all $x \in X$.

4. Application of the Functional Equation (1.7)

Consider the quadratic functional equation (1.7), that is

$$f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_0 - x_1}{2}\right) + f\left(\frac{x_2 + x_3}{2}\right) + f\left(\frac{x_2 - x_3}{2}\right)$$

$$= \frac{1}{2}[f(x_0) + f(x_1) + f(x_2) + f(x_3)]. \tag{4.1}$$

This functional equation can be used to express *Every Positive Integer can be Expressed as the Sum of Squares of Four Integers*.

A well-known classical theorem of Lagrange in Algebra [11]: According to old Trick of Euler's formula, if

$$2a = x_0^2 + x_1^2 + x_2^2 + x_3^2, \tag{4.2}$$

where a, x_0, x_1, x_2, x_3 are integers, then

$$a = y_0^2 + y_1^2 + y_2^2 + y_3^2 \tag{4.3}$$

for some positive integers a, y_0, y_1, y_2, y_3.

To see this, if $2a$ is even, then x_i's are all even (or) all odd (or) two even and two odd.

At any rate in all the above three cases, one can renumber the x_i's and pair them in such a way that

$$y_0 = \frac{x_0 + x_1}{2},\ y_1 = \frac{x_0 - x_1}{2},\ y_2 = \frac{x_2 + x_3}{2},\ y_3 = \frac{x_2 - x_3}{2}$$

are all integers. Also

$$\begin{aligned} &y_0^2 + y_1^2 + y_2^2 + y_3^2 \\ &= \left(\frac{x_0 + x_1}{2}\right)^2 + \left(\frac{x_0 - x_1}{2}\right)^2 + \left(\frac{x_2 + x_3}{2}\right)^2 + \left(\frac{x_2 - x_3}{2}\right)^2 \\ &= \frac{1}{2}(x_0^2 + x_1^2 + x_2^2 + x_3^2) \\ &= \frac{1}{2}(2a) \\ &= a. \end{aligned}$$

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