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UNIFORM STABILITY IN TERMS OF TWO MEASURES FOR IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAYS

Xilin Fu, Xuling Wang, Haydar Akca† and Xiaodi Li*

School of Mathematical Sciences
Shandong Normal University
Ji'nan, 250014, P. R. China
e-mail: sodymath@163.com

†Department of Mathematics
College of Arts and Science Faculty
Abu Dhabi University
P. O. Box 59911, Abu Dhabi, U. A. E.

Abstract

This paper studies the uniform stability in terms of two measures for impulsive functional differential equations with infinite delays by using Lyapunov functions and Razumikhin technique. The results obtained improve and extend some recent works. The main advantage is that it can be applied to the stability problem of nonlinear impulsive functional differential equations with infinite delays. Finally, two examples are given to illustrate the effectiveness and advantages of the results obtained.

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*Corresponding author

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1. Introduction

Recently, there has been a significant development in the theory of impulsive functional differential equations, especially in the area of impulsive functional differential equations with finite delays; see ([1-10] and references therein). For example, by employing the Lyapunov functions and Razumikhin technique, several global exponential stability criteria are established for general impulsive functional differential equations with finite delays in [6]. Liu and Fu [11] considered the uniform stability and uniform asymptotic stability of impulsive functional differential equations with finite delays by establishing some comparison theorems. Recently, special interest has been devoted to the stability problem of impulsive functional differential equations with infinite delays, see [12-19] where Zhang and Sun [12] investigated the uniform stability of impulsive functional differential equations with infinite delays by using Lyapunov functions and partial variables method. More recently, Li [13] further developed and explored a new criterion for the uniform stability of impulsive functional differential equations with infinite delays and solved a class of nonlinear problem such as

$$\begin{cases} \dot{x}(t) = -2x(t) + \int_0^{\infty} e^{-s} x^N(t-s)ds, \quad t \geq 0, \quad t \neq k, \\ x(t) - x(t^-), \quad t = k, \quad k \in \mathbb{Z}_+. \end{cases} \tag{A}$$

However, those results can only be applied to some special cases and there are still many cases that cannot be solved to now such as

$$\begin{cases} \dot{x}(t) = \frac{1}{2}x(t) + x(t-\tau(t)) - \frac{x(t)}{1+x^2(t)}(1+t^2), \quad t \geq 0, \quad t \neq t_k, \\ x(t) - x(t^-) = \beta_k x(t^-), \quad t = t_k, \quad k \in \mathbb{Z}_+. \end{cases} \tag{B}$$

Motivated by the above discussions, we further investigate the uniform stability in terms of two measures for impulsive functional differential equations with infinite delays. The results obtained can be used to solve the stability of some nonlinear problems. The rest of this paper is organized as follows: in Section 2, we introduce some basic notations and definitions. In Section 3, we get the main criterion for uniform stability of impulsive infinite delay differential equations. In Section 4, two examples are given to illustrate the effectiveness and advantages of the results obtained.

2. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, $\mathbb{Z}_+$ the set of positive integers and $\mathbb{R}^n$ the n-dimensional real space equipped with the Euclidean norm $\|\cdot\|$. For any $t \geq t_0 \geq 0 > -\alpha \geq -\infty$, let $f(t, x_t(s))$, where $s \in [t-\alpha, t]$ is a Volterra type functional. In the case, where $\alpha = +\infty$, the interval $[t-\alpha, t]$ is understood to be replaced by $(-\infty, t]$.

Consider the impulsive functional differential equations:

$$\begin{cases} \dot{x}(t) = f(t, x_t), \quad t \geq \sigma, \quad t \neq t_k, \\ \Delta x(t_k) = x(t_k) - x(t_k^-) = I_k(t_k, x(t_k^-)), \quad k \in \mathbb{Z}_+, \\ x_\sigma = \phi(s), \quad s \in [-\alpha, 0], \end{cases} \tag{1}$$

where $\sigma \geq t_0 \geq 0$, the impulse times t_k satisfy $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to+\infty} t_k = +\infty$; $\phi \in \mathbb{C}$, $\mathbb{C}$ is an open set in $PC([-\alpha, 0], \mathbb{R}^n)$, where $PC([-\alpha, 0], \mathbb{R}^n) = \{\psi : [-\alpha, 0] \to \mathbb{R}^n$ is continuous everywhere except at finite number of points t_k, at which $\psi(t_k^+)$ and $\psi(t_k^-)$ exist and $\psi(t_k^+) = \psi(t_k)\}$; $x_t, x_{t^-} \in PC([-\alpha, 0], \mathbb{R}^n)$ are defined by $x_t(s) = x(t+s)$ and $x_{t^-}(s) = x(t^- + s)$ for $-\alpha \leq s \leq 0$, respectively. Define $PCB(t) = \{x_t \in \mathbb{C} :$ x_t is bounded$\}$. For any $\psi \in PCB(t)$, the norm of ψ is defined by $\|\psi\| = \sup_{-\alpha\leq\theta\leq 0} |\psi(\theta)|$, where $\|\cdot\|$ denotes the norm of vector in $\mathbb{R}^n$. For any $\sigma \geq 0$, let $PCB_\delta(\sigma) = \{\psi \in PCB(\sigma) : \|\psi\| < \delta\}$.

For convenience, we define the following classes of functions:

$$K = \{a \in C(\mathbb{R}_+, \mathbb{R}_+) \,|\, a(0) = 0 \text{ and } a(s) > 0 \text{ for } s > 0 \text{ and}$$

$$a \text{ is non-decreasing in } s\}.$$

$$\Gamma = \{h \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+) \,|\, \inf_{(t,x)} h(t, x) = 0\}.$$

$$\Gamma_0 = \{h_0 : \mathbb{R}_+ \times PCB([-\alpha, 0], \mathbb{R}^n) \to \mathbb{R}_+ \,|\, h_0(t, \psi)$$

$$= \sup_{-\alpha \le s \le 0} h^0(t + s, \psi(s)) : h^0 \in \Gamma\}.$$

In order to prove our main results, we need the following lemma and definitions:

Lemma 2.1 [15]. *The initial problem of system* (1) *exists with a unique solution which will be written in the form* $x(t, \sigma, \phi)$ *if the following hypotheses hold*:

(H_1) $f : [t_{k-1}, t_k) \times \mathbb{C} \to \mathbb{R}^n$, $k \in \mathbb{Z}_+$, *is continuous and for all* $k \in \mathbb{Z}_+$ *and for any* $\psi \in \mathbb{C}$, *the* $\lim_{(t,x)\to(t_k^-, \psi)} f(t, x) = (t_k^-, \psi)$ *exist.*

(H_2) $f(t, \psi)$ *is Lipschitzian in* ψ *in each compact set in* $\mathbb{C}$.

(H_3) $I_k(t, x) : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and for any* $\rho > 0$, *there exists a* $\rho_1 \in (0, \rho)$ *such that* $x \in S(\rho_1)$ *implies that* $x + I_k(t_k, x) \in S(\rho)$, *where* $d(\rho) = \{x : |x| < \rho, x \in \mathbb{R}^n\}$.

(H_4) *For any* $\psi \in \mathbb{C}$, $f(t, \psi_t) \in PC([t_0, \infty), \mathbb{R}^n)$.

Throughout this paper, we let (H_1)-(H_4) hold. Furthermore, we assume that $f(t, 0) = 0$, $I_k(t_k, 0) = 0$, $k \in \mathbb{Z}_+$; then $x(t) \equiv 0$ is a solution of system (1), which is called the *trivial solution*. Moreover, we always assume that the solution $x(t, \sigma, \phi)$ of (1) can be continued to ∞ from the right of $-\sigma$.

Definition 2.1 [13]**.** A function $V : [-\alpha, \infty) \times \mathbb{C} \to \mathbb{R}_+$ is said to *belong to class* v_0 if

(i) V is continuous in each of the sets $[-\alpha, \infty) \times \mathbb{C}$ and $\lim_{(t, y) \to (t_k^-, x)} V(t, y) = V(t_k^-, x)$ exists;

(ii) $V(t, x)$ is locally Lipschitzian in x and for all $t \geq t_0$, $V(t, 0) \equiv 0$.

Definition 2.2 [13]**.** Given a function $V \in v_0$, the upper right-hand derivative of V along the solution $x(t)$ of system (1) is defined by

$$D^+V(t, x(t)) = \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, x(t) + hf(t, x_t)) - V(t, x(t))].$$

Definition 2.3 [7]**.** Let $h \in \Gamma$, $h_0 \in \Gamma_0$. Then the system (1) is said to be

(S_1) (h_0, h)-*stable*, if for any $\sigma \geq t_0$ and $\varepsilon > 0$, there exists some $\delta = \delta(\varepsilon, \sigma) > 0$ such that $\phi \in PCB_\delta(\sigma)$ implies $|x(t, \sigma, \phi)| < \varepsilon$, $t \geq \sigma$, where $x(t) = x(t, \sigma, \phi)$ is any solution of system (1).

(S_2) (h_0, h)-*uniformly stable*, if the δ in S_1 is independent of σ.

Definition 2.4 [13]**.** The trivial solution $x(t) = 0$ of system (1) is said to be

(P_1) *stable*, if for any $\sigma \geq t_0$ and $\varepsilon > 0$, there exists some $\delta = \delta(\varepsilon, \sigma) > 0$ such that $\phi \in PCB_\delta(\sigma)$ implies $|x(t, \sigma, \phi)| < \varepsilon$, $t \geq \sigma$, where $x(t) = x(t, \sigma, \phi)$ is any solution of system (1).

(P_2) *uniformly stable*, if the δ in (P_1) is independent of σ.

3. Main Results

Theorem 3.1. *Let conditions* (H_1)-(H_4) *be satisfied. Assume that there exist functions* $V \in v_0$, $W_1, W_2, W_3 \in K$ *and constants* $T^* > 0$, $\beta_k \geq 0$,

$k \in \mathbb{Z}_+$ *such that*

(i) $W_1(h(t, \psi(0))) \le V(t, \psi(0)) \ge W_2(h_0(t, \psi(s)))$, *where*

$$(t, \psi) \in [t_0 - \alpha, \infty) \times PC([-\alpha, 0], \mathbb{R}^n);$$

(ii) $V(t, \psi(0)) \le W_3(h(t, \psi(0)))$, *where*

$$(t, \psi) \in [T^*, \infty) \times PC([-\alpha, 0], \mathbb{R}^n);$$

(iii) $D^+V(t, \psi(0)) \le -\lambda(t)V(t, \psi(0)) + g(t)$, $t \in [t_{k-1}, t_k)$, *whenever* $V(t, \psi(0)) \ge V(t+s, \psi(s))$ *for* $s \in [0-\alpha, 0]$, *where*

$$g(t) \in PC([t_0 - \alpha, \infty), \mathbb{R}), \quad \lim_{t \to +\infty} g(t) = 0, \quad \lambda(t) \in PC([t_0 - \alpha, \infty], \mathbb{R}_+)$$

and $\inf_{t \ge t_0} \lambda(t) > 0$;

(iv) $V(t_k, \psi(0) + I_k(t_k, \psi)) \le (1 + \beta_k)V(t_k^-, \psi(0))$ *with* $\sum_{k=1}^{\infty} \beta_k < \infty$.

Then the system of (1) *is* (h_0, h)*-uniformly stable.*

Proof. For any $\sigma \ge t_0$, let $x(t) = x(t, \sigma, \phi)$ be a solution of system (1) through (σ, σ). For any given $\varepsilon > 0$, from the definitions of W_1, W_2, W_3, we can choose $\delta = \delta(\varepsilon) > 0$ and $\varepsilon^* \in (0, \varepsilon)$ such that

$$W_2(\delta) < \beta^{-1}W_1(\varepsilon) \text{ and } W_3(\varepsilon^*) < \beta^{-1}W_1(\varepsilon), \tag{2}$$

where $\prod_{k=1}^{\infty}(1 + \beta_k) \doteq \beta$. Next, we show that $h_0(\sigma, \phi) < \delta$ implies

$$h(t, x(t)) < \varepsilon, \quad t \ge \sigma.$$

Since $\lim_{t \to +\infty} g(t) = 0$, there exists a $T^* > 0$ such that $t > T^*$ implies

$$|g(t)| < \frac{1}{2}\lambda\beta^{-1}W_1(\varepsilon), \text{ where } \lambda \doteq \inf_{t \ge t_0} \lambda(t) > 0. \tag{3}$$

Set $T = \max\{T^*, T^*\}$. Then there are two cases:

Case 1. $\sigma \in [t_0, T]$. We show that

$$h(t, x(t)) < \varepsilon, \quad t \geq \sigma.$$

By the continuity of solutions with respect to the initial values on the interval $[\sigma, T]$, one may choose a small enough $\delta^* \in (0, \delta)$ such that $h_0(\sigma, \phi) < \delta^*$ implies

$$h(t, x(t)) < \varepsilon^* < \varepsilon, \quad t \in [\sigma, T].$$

In particular, $h(T, x(T)) < \varepsilon^*$. Then it follows from (2), condition (ii) and $T > T^*$ that

$$V(T, x(T)) \leq W_3(h(T, x(T))) < W_3(\varepsilon^*) < \beta^{-1}W_1(\varepsilon).$$

Next, we prove that

$$h(t, x(t)) < \varepsilon, \quad t \geq T.$$

Suppose that $T \in [t_{m-1}, t_m)$ for some $m \in \mathbb{Z}_+$, then we show that

$$V(t, x(t)) \leq \beta^{-1}W_1(\varepsilon), \quad t \in [T, t_m). \tag{4}$$

If this assertion is not true, then there exists a $t \in [T, t_m)$ such that

$$V(t, x(t)) > \beta^{-1}W_1(\varepsilon).$$

Set $t^* = \inf\{t \in [T, t_m) : V(t, x(t)) > \beta^{-1}W_1(\varepsilon)\}$. Since

$$V(T, x(T)) < \beta^{-1}W_1(\varepsilon),$$

it holds that

$$t^* > T, V(t^*, x(t^*)) = \beta^{-1}W_1(\varepsilon), V(t, x(t)) \leq \beta^{-1}W_1(\varepsilon), t \in [T, t^*]$$

and $D^+V(t^*, x(t^*)) \geq 0$. Note that

$$V(t^*, x(t^*)) = \beta^{-1}W_1(\varepsilon) \geq V(t^* + s, x(t^* + s)), \quad s \in [-\alpha, 0].$$

By conditions (iii) and (3), we have

$$\begin{aligned} D^+V(t^*, x(t^*)) &\le -\lambda(t^*)V(t^*, x(t^*)) + g(t^*) \\ &\le -\lambda\beta^{-1}W_1(\varepsilon) + \frac{1}{2}\lambda\beta^{-1}W_1(\varepsilon) \\ &\le -\frac{1}{2}\lambda\beta^{-1}W_1(\varepsilon) \\ &< 0, \end{aligned}$$

which is a contradiction with $D^+V(t^*, x(t^*)) \ge 0$ and thus (4) holds. Considering (i) and (4), it can be deduced that

$$h(t, x(t)) < \varepsilon, \quad t \in [T, t_m).$$

Then from (4) and condition (iv), we get

$$\begin{aligned} V(t_m, x(t_m)) &= V(t_m, x(t_m^-) + I_m(t_m, x(t_m^-))) \\ &\le (1 + \beta_m)V(t_m^-, x(t_m^-)) \\ &\le \beta^{-1}(1 + \beta_m)W_1(\varepsilon). \end{aligned}$$

By the same argument, we may prove that

$$V(t, x(t)) \le \beta^{-1}(1 + \beta_m)W_1(\varepsilon), \quad t \in [t_m, t_{m+1}). \tag{5}$$

If (5) does not hold, then there exists a $t \in [t_m, t_{m+1})$ such that

$$V(t, x(t)) > \beta^{-1}(1 + \beta_m)W_1(\varepsilon).$$

Set $t^* = \inf\{t \in [t_m, t_{m+1}) : V(t, x(t)) > \beta^{-1}(1 + \beta_m)W_1(\varepsilon)\}$. Since $V(t_m, x(t_m)) \le \beta^{-1}(1 + \beta_m)W_1(\varepsilon)$, it holds that $t^* > t_m$,

$$V(t^*, x(t^*)) = \beta^{-1}(1 + \beta_m)W_1(\varepsilon),$$

$V(t, x(t)) \le \beta^{-1}(1 + \beta_m)W_1(\varepsilon)$, $t \in [t_m, t^*]$ and $D^+V(t^*, x(t^*)) \ge 0$. Note that

$$V(t^*,\ x(t^*)) = \beta^{-1}(1+\beta_m)W_1(\varepsilon) \geq V(t^* + s,\ x(t^* + s)),\quad s \in [-\alpha,\ 0].$$

From this and condition (iii), we have

$$\begin{aligned} D^+(t^*,\ x(t^*)) &\leq -\lambda(t^*)V(t^*,\ x(t^*)) + g(t^*) \\ &\leq -\lambda\beta^{-1}(1+\beta_m)W_1(\varepsilon) + \frac{1}{2}\lambda\beta^{-1}W_1(\varepsilon) \\ &\leq -\lambda\beta^{-1}(1+\beta_m)W_1(\varepsilon) + \frac{1}{2}\lambda\beta^{-1}(1+\beta_m)W_1(\varepsilon) \\ &\leq -\frac{1}{2}\lambda\beta^{-1}(1+\beta_m)W_1(\varepsilon) \\ &< 0, \end{aligned}$$

which is a contradiction with $D^+V(t^*,\ x(t^*)) \geq 0$ and thus (5) holds. Considering (i) and (5), it can be deduced that

$$h(t,\ x(t)) < \varepsilon,\quad t \in [t_m,\ t_{m+1}).$$

By simple induction, we can prove that for $t \in [T,\ t_m) \cup [t_k,\ t_{k+1})$, $k \geq m$,

$$V(t,\ x(t)) \leq \beta^{-1}(1+\beta_m)\cdots(1+\beta_k)W_1(\varepsilon) < W_1(\varepsilon),$$

which implies that

$$W_1(h(t,\ x(t))) \leq V(t,\ x(t)) < W_1(\varepsilon),\quad t \geq T.$$

So $h(t,\ x(t)) < \varepsilon,\ t \geq T$.

Case 2. $\sigma \in [T,\ \infty)$. We show that

$$h(t,\ x(t)) < \varepsilon,\quad t \geq \sigma.$$

From condition (i) and (2), we have

$$V(\sigma,\ x(\sigma)) \leq W_2(h_0(\sigma,\ \phi)) \leq W_2(\delta) < \beta^{-1}W_1(\varepsilon).$$

Then, by the same argument, it can be deduced that $h(t, x(t)) < \varepsilon$, $t \geq \sigma$. The proof of Theorem 3.1 is complete.

Remark 3.1. Theorem 3.1 presents a new criterion for the uniform stability in terms of two measures for impulsive functional differential equations with infinite delays. It can be applied to some nonlinear problems, which is different from the existing results given in [13]. The criterion is more general than several recent works. Furthermore, a few choices of two measures (h_0, h) can be given and the corresponding theory of stability would be established. In particular, we choose $h(t, x(t)) = h_0(t, x(t)) = |x(t)|$, then we have the following results by Theorem 3.1.

Corollary 3.1. *Let conditions* (H_1)-(H_4) *be satisfied. Assume that there exist functions* $V \in v_0$, $W_1, W_2 \in K$ *and constant* $\beta_k \geq 0$, $k \in \mathbb{Z}_+$ *such that*

(i) $W_1(|x(t)|) \leq V(t, x(t)) \leq W_2(|x(t)|)$, $(t, x) \in [t_0 - \alpha, \infty) \times \mathbb{R}^n$;

(ii) $D^+V(t, \psi(0)) \leq -\lambda(t)V(t, \psi(0)) + g(t)$, $t \in [t_{k-1}, t_k)$, *whenever* $V(t, \psi(0)) \geq V(t + s, \psi(s))$ *for* $s \in [-\alpha, 0]$, *where*

$$g(t) \in PC([t_0 - \alpha, \infty), \mathbb{R}), \quad \lim_{t \to +\infty} g(t) = 0, \quad \lambda(t) \in PC([t_0 - \alpha, \infty], \mathbb{R}_+)$$

and $\inf_{t \geq t_0} \lambda(t) > 0$;

(iii) $V(t_k, \psi(0) + I_k(t_k, \psi)) \leq (1 + \beta_k)V(t_k^-, \psi(0))$ *with* $\sum_{k=1}^{\infty} \beta_k < \infty$.

Then the trivial solution of system (1) *is uniformly stable.*

Proof. Choose $h(t, x(t)) = h_0(t, x(t)) = |x(t)|$. By Theorem 3.1, we can obtain the above corollary.

Remark 3.2. In [1], by using Lyapunov functions and Razumikhin technique, the author obtained some sufficient conditions for the uniform stability of system (1). But it can only be applied to solve the stability of some linear problems. Note that in our result, we require that the function

D^+V satisfies the assumption (ii) to obtain the desirable stability. Therefore, our development result can be widely used in solving the stability problem of nonlinear impulsive functional differential equations with infinite delays. Later, we present an example to illustrate that our conditions are more feasible than that given in the earlier.

Theorem 3.2. *Let conditions* (H_1)-(H_4) *be satisfied. Assume that there exist functions* $V \in v_0, W_1, W_2 \in K$ *and constant* $\beta_k \geq 0,\ k \in \mathbb{Z}_+$ *such that*

(i) $W_1(|x(t)|) \leq V(t, x(t)) \leq W_2(|x(t)|),\ (t, x) \in [t_0 - \alpha, \infty) \times \mathbb{R}^n$;

(ii) $D^+V(t, \psi(0)) \leq \lambda(t)V(t, \psi(0)) - g(t), \quad t \in [t_{k+1}, t_k), \quad$ *whenever* $V(t, \psi(0)) \geq V(t+s, \psi(s))$ *for* $s \in [-\alpha, 0]$, *where*

$$g(t) \in PC([t_0 - \alpha, \infty), \mathbb{R}_+), \quad \inf_{t \geq t_0} g(t) > 0, \quad \lambda(t) \in PC([t_0 - \alpha, \infty), \mathbb{R})$$

and $\sup_{t \geq t_0} \lambda(t) < \infty$;

(iii) $V(t_k, \psi(0) + I_k(t_k, \psi)) \leq (1 + \beta_k)V(t_k^-, \psi(0))$ *with* $\sum_{k=1}^{\infty} \beta_k < \infty$.

Then the trivial solution of system (1) *is uniformly stable.*

Proof. For any $\sigma \geq t_0$, let $x(t) = x(t, \sigma, \phi)$ be a solution of system (1) through (σ, ϕ). For any given $\varepsilon > 0$, from the definitions of W_1, W_2, we can choose $\delta = \delta(\varepsilon) > 0$ such that

$$W_2(\delta) < \beta^{-1}W_1(\varepsilon),$$

where $\prod_{k=1}^{\infty}(1 + \beta_k) \doteq \beta$. Next, we show that $\phi \in PCB_\delta(\sigma)$ implies

$$|x(t)| < \varepsilon, \quad t \geq \sigma.$$

First, it is obvious that

$$W_1(|x(t)|) \leq V(t, x(t)) \leq W_2(|x(t)|) \leq W_2(\delta) < \beta^{-1}W_1(\varepsilon), \ t \in [\sigma - \alpha, \sigma].$$

Then suppose that $\sigma \in [t_{m-1}, t_m)$ for some $m \in \mathbb{Z}_+$, next we show that

$$V(t, x(t)) \leq \beta^{-1} W_1(\varepsilon), \quad t \in [\sigma, t_m). \tag{6}$$

If this assertion is not true, then there exists a $t \in [\sigma, t_m)$ such that

$$V(t, x(t)) > \beta^{-1} W_1(\varepsilon).$$

Set $t^* = \inf\{t \in [\sigma, t_m) : V(t, x(t)) > \beta^{-1} W_1(\varepsilon)\}$. Since $V(\sigma, x(\sigma)) < \beta^{-1} W_1(\varepsilon)$, it holds that

$$t^* > \sigma, V(t^*, x(t^*)) = \beta^{-1} W_1(\varepsilon), \;\; V(t, x(t)) \leq \beta^{-1} W_1(\varepsilon), \;\; t \in [\sigma, t^*]$$

and $D^+ V(t^*, x(t^*)) \geq 0$. Note that

$$V(t^*, x(t^*)) = \beta^{-1} W_1(\varepsilon) \geq V(t^* + s, x(t^* + s)), \quad s \in [-\alpha, 0].$$

For any given $\varepsilon \in \left(0, W_1^{-1}\left(\frac{\beta\eta}{\lambda}\right)\right)$, by condition (ii), we have

$$\begin{aligned} D^+ V(t^*, x(t^*)) &\leq \lambda(t^*) V(t^*, x(t^*)) - g(t^*) \\ &\leq \lambda \beta^{-1} W_1(\varepsilon) - \eta \\ &< 0, \end{aligned}$$

where $\eta \doteq \inf_{t \geq t_0} g(t) > 0$ and $\lambda \doteq \sup_{t \geq t_0} \lambda(t) < \infty$, which is a contradiction with $D^+ V(t^*, x(t^*)) \geq 0$ and thus (6) holds. Considering (i) and (6), it can be deduced that

$$|x(t)| < \varepsilon, \quad t \in [\sigma, t_m).$$

Then, from (6) and condition (iii), we get

$$\begin{aligned} V(t_m, x(t_m)) &= V(t_m, x(t_m^-) + I_m(t_n, x(t_m^-))) \\ &\leq (1 + \beta_m) V(t_m^-, x(t_m^-)) \\ &\leq \beta^{-1}(1 + \beta_m) W_1(\varepsilon). \end{aligned}$$

By the same argument, one may prove that

$$V(t,\ x(t)) \le \beta^{-1}(1+\beta_m)W_1(\varepsilon), \quad t \in [t_m,\ t_{m+1}). \tag{7}$$

If (7) does not hold, then there exists a $t \in [t_m,\ t_{m+1})$ such that

$$V(t,\ x(t)) > \beta^{-1}(1+\beta_m)W_1(\varepsilon).$$

Set $t^* = \inf\{t \in [t_m, t_{m+1}) : V(t, x(t)) > \beta^{-1}(1+\beta_m)W_1(\varepsilon)\}$. Since $V(t_m, x(t_m)) \le \beta^{-1}(1+\beta_m)W_1(\varepsilon)$, it holds that $t^* > t_m$, $V(t^*, x(t^*)) = \beta^{-1}(1+\beta_m)W_1(\varepsilon)$, $V(t, x(t)) \le \beta^{-1}(1+\beta_m)W_1(\varepsilon)$, $t \in [t_m, t^*)$ and $D^+V(t^*, x(t^*)) \ge 0$. Note that

$$V(t^*,\ x(t^*)) = \beta^{-1}(1+\beta_m)W_1(\varepsilon) \ge V(t^*+s,\ x(t^*+s)), \quad s \in [-\alpha,\ 0].$$

By the choice of ε and condition (ii), we have

$$\begin{aligned} D^+V(t^*,\ x(t^*)) &\le \lambda(t^*)V(t^*,\ x(t^*)) - g(t^*) \\ &\le \lambda\beta^{-1}(1+\beta_m)W_1(\varepsilon) - \eta \\ &< 0, \end{aligned}$$

which is a contradiction with $D^+V(t^*,\ x(t^*)) \ge 0$ and thus (7) holds. Considering (i) and (7), we get

$$|\,x(t)\,| < \varepsilon, \quad t \in [t_m,\ t_{m+1}).$$

By simple induction, we can prove that for $t \in [\sigma,\ t_m) \cup [t_k,\ t_{k+1})$, $k \ge m$,

$$V(t,\ x(t)) \le \beta^{-1}(1+\beta_m)\cdots(1+\beta_k)W_1(\varepsilon) < W_1(\varepsilon),$$

which implies that

$$W_1(|\,x(t)\,|) \le V(t,\ x(t)) < W_1(\varepsilon), \quad t \ge \sigma.$$

So $|\,x(t)\,| < \varepsilon$, $t \ge \sigma$. The proof of Theorem 3.2 is complete.

Remark 3.3. By Theorem 3.2, it can be easy to check that system (B) is uniformly stable. Also, this assertion can be shown by following Example 4.2.

4. Applications

In this section, we shall give two examples to show the effectiveness and advantages of our results.

Example 4.1. Consider the following impulsive functional differential equations:

$$\begin{cases} \dot{x}(t) = -a(t)x(t) + b(t)x(t-\tau(t)) + \dfrac{x(t)}{1+x^2(t)}h(t), \\ x(t_k) - x(t_k^-) = \gamma_k x(t_k^-), \quad k \in \mathbb{Z}_+, \end{cases} \tag{8}$$

where $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b \in C(\mathbb{R}_+, \mathbb{R})$, $\tau(t) \in C([0, \infty), [0, \infty))$, $\tau(t) < t$, $t - \tau(t) \to \infty (t \to \infty)$, $h(t) \in C([0, \infty), \mathbb{R})$, $\lim_{t \to +\infty} h(t) = 0$, γ_k are some nonnegative constants which satisfy $\sum_{k=1}^{\infty} \gamma_k < \infty$.

Property 4.1. The trivial solution of system (8) is uniformly stable if $\inf_{t \ge t_0} \{a(t) - |b(t)|\} > 0$.

Proof. Choose $V(t, x) = x^2(t)$, then we have

$$\begin{aligned} D^+V(t, x) &\le 2x(t)\left\{-a(t)x(t) + b(t)x(t-\tau(t)) + \frac{x(t)}{1+x^2(t)}h(t)\right\} \\ &\le -2a(t)x^2(t) + |b(t)|\{x^2(t) + x^2(t-\tau(t))\} + \frac{2x^2(t)}{1+x^2(t)}h(t) \\ &\le -2a(t)V(t, x) + 2|b(t)|V(t, x) + 2h(t) \\ &\le -2\{a(t) - |b(t)|\}V(t, x) + 2h(t) \\ &\le -2\lambda V(t, x) + 2h(t), \end{aligned}$$

where $\lambda \doteq \inf_{t \ge t_0} \{a(t) - |b(t)|\}$. In addition, note that

$$V(t_k, x) = x^2(t_k) = (1+\gamma_k)^2 x^2(t_k^-) = (1+\gamma_k)^2 V(t_k^-, x) = (1+\beta_k)V(t_k^-, x),$$

where $\beta_k = (1+\gamma_k)^2 - 1$. Obviously, $\sum_{k=1}^{\infty}\beta_k < \infty$ if $\sum_{k=1}^{\infty}\gamma_k < \infty$. By Corollary 3.1, we can obtain the above property easily.

Remark 4.1. In particular, we choose $h(t) = \dfrac{1}{1+t^2}$, then the trivial solution of (8) is uniformly stable.

Example 4.2. Consider the following impulsive functional differential equations:

$$\begin{cases} \dot{x}(t) = a(t)x(t) + b(t)x(t-\tau(t)) - \operatorname{sgn}(x(t))H(t), \\ x(t_k) - x(t_k^-) = \beta_k x(t_k^-), \quad k \in \mathbb{Z}_+, \end{cases} \tag{9}$$

where $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b \in C(\mathbb{R}_+, \mathbb{R})$, $\tau(t) \in C([0, \infty), [0, \infty))$, $\tau(t) < t$, $t - \tau(t) \to \infty (t \to \infty)$, $h(t) \in C([0, \infty), \mathbb{R})$, $\inf_{t \ge t_0} H(t) > 0$, β_k are some nonnegative constants which satisfy $\sum_{k=1}^{\infty}\beta_k < \infty$.

Property 4.2. The trivial solution of system (9) is uniformly stable if $\sup_{t\ge t_0}\{a(t) + |b(t)|\} < \infty$.

Proof. Choose $V(t, x) = |x(t)|$, then we have

$$\begin{aligned} D^+V(t, x) &\le a(t)|x(t)| + |b(t)||x(t-\tau(t))| - H(t) \\ &\le a(t)V(t, x) + |b(t)|V(t, x) - H(t) \\ &\le \{a(t) + |b(t)|\}V(t, x) - H(t) \\ &\le \lambda V(t, x) - H(t), \end{aligned}$$

where $\lambda \doteq \sup_{t\ge t_0}\{a(t) + |b(t)|\} < \infty$. In addition, note that

$$V(t_k, x) = |x(t_k)| = (1+\beta_k)|x(t_k^-)| = (1+\beta_k)V(t_k^-, x),$$

where $\sum_{k=1}^{\infty}\beta_k < \infty$. By Theorem 3.2, we can obtain the above property easily.

Remark 4.2. From the above examples, we can find that the results obtained are very simple and effective for implementing in real problems. We believe that this provides some theoretical guidelines for investigation of uniformly stable in terms of two measures for impulsive functional differential equations with infinite delays.

5. Conclusion

In this paper, in terms of two measures for impulsive functional differential equations with infinite delays is considered. We obtain a new sufficient criterion ensuring uniform stability of the trivial solution for such impulsive functional differential equations by using Lyapunov functions and Razumikhin technique. Also, the results here can solve some stability problems of nonlinear impulsive functional differential equations with infinite delays. Our results can be applied to the cases not covered in some earlier references. Finally, two examples have been given to illustrate the effectiveness and advantages of the results obtained.

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