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ASCENTS AND DESCENTS OF SEMISTAR OPERATIONS AND LOCALIZING SYSTEMS, II

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Abstract

We study strict relations of basic properties of semistar operations and localizing systems.

1. Introduction

This is a continuation of [2] and [3]. Let D be an integral domain with quotient field K, and let $\overline{F}(D)$ be the set of non-zero D-submodules of K. A mapping $\overline{F}(D) \overset{\star}{\to} \overline{F}(D)$, $h \mapsto h^{\star}$ is called a *semistar operation* if for every $x \in K \setminus \{0\}$ and h, $h_1 \in \overline{F}(D)$, $(xh)^{\star} = xh^{\star}$; $h \subseteq h^{\star}$; $(h^{\star})^{\star} = h^{\star}$; and $h \subseteq h_1$ implies $h^{\star} \subseteq h_1^{\star}$. The identity mapping from $\overline{F}(D)$ to $\overline{F}(D)$ is a semistar operation on D, called the *d-semistar operation*, and denoted by d_D. Similarly, we may define the *e-semistar operation* e_D on D: $h^{e_D} := K$ for every $h \in \overline{F}(D)$. The set of semistar operations on D is denoted by $\mathrm{SStar}(D)$.

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Let $\mathcal{F}$ be a non-empty set of ideals of D with $\mathcal{F} \not\ni (0)$ which satisfies the following two conditions for every ideals I, I_1 of D: If $I \in \mathcal{F}$ and $I \subseteq I_1$, then $I_1 \in \mathcal{F}$; If $I \in \mathcal{F}$ and $I_1 :_D iD \in \mathcal{F}$ for every $i \in I$, then $I_1 \in \mathcal{F}$. Then $\mathcal{F}$ is called a *localizing system* of D. The set of localizing systems of D is denoted by LS(D). Let $\mathcal{F}$ be a localizing system of D. Then the mapping $h \mapsto h^{\star(\mathcal{F})} = \bigcup\{(h :_K I) \mid I \in \mathcal{F}\}$ is a semistar operation on D. A localizing system $\mathcal{F}$ is called *finite type* if, for every $I \in \mathcal{F}$, $\mathcal{F}$ contains a finitely generated ideal I_1 of D such that $I_1 \subseteq I$. We set $\mathcal{F}_f := \{I \mid I$ is an ideal of D which contains a finitely generated ideal $I_1 \in \mathcal{F}$ of $D\}$. We refer to Fontana and Huckaba [1] for semistar operations and localizing systems. Thus, let $f(D)$ be the set of elements of $\overline{F}(D)$ which is finitely generated over D. Let $\star$ be a semistar operation on D. Then $\mathcal{F}(\star) = \{I \mid I$ is a non-zero ideal of D with $I^\star \ni 1\}$ is a localizing system of D. The semistar operation $h \mapsto \bigcup\{f^\star \mid f \in f(D)$ with $f \subseteq h\}$ is denoted by $\star_f$. If $\star = \star_f$, then $\star$ is called *finite type*. We define the semistar operation $\star_a$ on D by $f^{\star_a} := \bigcup\{((ff_1)^\star :_K f_1^\star) \mid f_1 \in f(D)\}$ for every $f_1 \in f(D)$ and by $h^{\star_a} := \bigcup\{f^{\star_a} \mid f \in f(D)$ with $f \subseteq h\}$ for every $h \in \overline{F}(D)$. We define the semistar operation $[\star]$ on D by $f^{[\star]} := \bigcup\{((f_1^\star :_K f_1^\star)f)^{\star_f} \mid f_1 \in f(D)\}$ for every $f \in f(D)$ and by $h^{[\star]} := \bigcup\{f^{[\star]} \mid f \in f(D)$ with $f \subseteq h\}$ for every $h \in \overline{F}(D)$.

Let D be a domain with quotient field K, and let T be any extension domain with quotient field L. We use the letter h (resp., f) to denote an element of $\overline{F}(D)$ (resp., $f(D)$), use H (resp., F) to denote an element of $\overline{F}(T)$ (resp., $f(T)$), and use x (resp., y) to denote an element of K (resp., of L). We use the letter I (resp., J) to denote a non-zero ideal of D (resp., a non-zero ideal of T).

Let $\star$ be a semistar operation on T. For every $h \in \overline{F}(D)$, set $h^{\delta(\star)} := (hT)^{\star} \cap K$. Then $\delta(\star)$ is a semistar operation on D, called the *descent* of $\star$ to D.

Let $\mathcal{F}$ be a localizing system of D. Set $\alpha(\mathcal{F}) := \{J \mid J$ is an ideal of T with $J \supseteq I$ for some $I \in \mathcal{F}\}$. $\alpha(\mathcal{F})$ is a localizing system of T, called the *ascent* of $\mathcal{F}$ to T.

For every localizing system $\mathcal{F}$ of T, set $\delta(\mathcal{F}) = \{I \mid I$ is an ideal of D with $IT \in \mathcal{F}\}$. $\delta(\mathcal{F})$ is called the *descent* of $\mathcal{F}$ to D.

Let $\star$ be a semistar operation on D. Let $\{\star_\lambda \mid \lambda \in \Lambda\}$ be the set of semistar operations $\star'$ on T such that $\delta(\star') \geq \star$. Then the mapping $\overline{F}(T) \to \overline{F}(T)$, $H \mapsto \bigcap_\lambda H^{\star_\lambda}$ is a semistar operation on T, denoted by $\alpha(\star)$, and called the *ascent* of $\star$.

In this paper, we pursue studying strict relations of basic properties of ascents and descents of semistar operations and localizing systems.

2. Preliminary Results

Each assertion of the following Proposition 1 holds *always*. For instance, in (1) of Proposition 1, for every domain D, for every extension domain T of D, for every localizing system $\mathcal{F}$ of T, we have $\alpha(\delta(\mathcal{F})) \subseteq \mathcal{F}$.

Proposition 1 (cf., [3, Sections 3, 4 and 5]). *We have the following properties*:

(1) $\alpha(\delta(\mathcal{F})) \subseteq \mathcal{F}$.

(2) $\delta(\alpha(\mathcal{F})) \supseteq \mathcal{F}$.

(3) $\delta(\alpha(\star)) \geq \star$.

(4) $\alpha(\delta(\star)) \leq \star$.

(5) $\delta(\alpha(\delta(\star))) = \delta(\star)$.

(6) $\alpha(\delta\alpha(\star)) = \alpha(\star)$.

(7) $\delta(\alpha\delta(\mathcal{F})) = \delta(\mathcal{F})$.

(8) $\alpha(\delta\alpha(\mathcal{F})) = \alpha(\mathcal{F})$.

(9) $\delta(\star)_f = \delta(\star_f)$.

(10) $\star(\delta(\mathcal{F})) \leq \delta(\star(\mathcal{F}))$.

(11) $\mathcal{F}(\delta(\star)) = \delta(\mathcal{F}(\star))$.

(12) $\mathcal{F}(\alpha(\star)) \supseteq \alpha(\mathcal{F}(\star))$.

(13) $\star(\alpha(\mathcal{F})) \geq \alpha(\star(\mathcal{F}))$.

(14) $\alpha(\mathcal{F})_f \supseteq \alpha(\mathcal{F}_f)$.

(15) $\delta(\mathcal{F})_f = \delta(\mathcal{F}_f)$.

(16) $\alpha(\star)_f \geq \alpha(\star_f)$.

(17) $\delta(\mathcal{F})_f = \delta(\mathcal{F}_f)$.

(18) $\delta(\star)_a \leq \delta(\star_a)$.

(19) $[\delta(\star)] \leq \delta([\star])$.

(20) $\delta(\mathcal{F}(\alpha(\star))) \supseteq \mathcal{F}(\star)$.

(21) $\delta(\star(\alpha(\mathcal{F}))) \geq \star(\mathcal{F})$.

(22) $\mathcal{F}(\delta(\star(\mathcal{F}))) = \delta(\mathcal{F})$.

(23) $\mathcal{F}(\star) \supseteq \alpha(\mathcal{F}(\delta(\star)))$.

(24) $\alpha(\star(\delta(\mathcal{F}))) \leq \star(\mathcal{F})$.

(25) $\delta(\alpha(\star)_f) \geq \star_f$.

(26) $\delta(\alpha(\mathcal{F})_f) \supseteq \mathcal{F}_f$.

(27) $\alpha(\delta(\star)_f) \leq \star_f$.

(28) $\alpha(\delta(\mathcal{F})_f) \subseteq \mathcal{F}_f$.

(29) $\alpha(\delta(\star)_a) \leq \star_a$.

(30) $\alpha([\delta(\star)]) \leq [\star]$.

(31) $\delta([\alpha(\star)]) \geq [\star]$.

(32) $\alpha([\star]) \leq [\alpha(\star)]$.

(33) $\star_a \leq \delta(\alpha(\star)_a)$.

(34) $\alpha(\star_a) \leq \alpha(\star)_a$.

Let n be a positive integer. In the following Remark 2, the assertion (n') corresponds to Proposition 1 (n).

Remark 2 (cf., [3, Remark 3.4, Examples 4.4, 4.5, 5.2, 5.4, 5.7, 5.11 and 5.14])**.**

(1′) There is an example such that $\alpha(\delta(\mathcal{F})) \underset{\neq}{\subseteq} \mathcal{F}$.

(2′) There is an example such that $\delta(\alpha(\mathcal{F})) \underset{\neq}{\supseteq} \mathcal{F}$.

(3′) There is an example such that $\delta(\alpha(\star)) \underset{\neq}{\geq} \star$.

(4′) There is an example such that $\alpha(\delta(\star)) \underset{\neq}{\leq} \star$.

(10′) There is an example such that $\star(\delta(\mathcal{F})) \underset{\neq}{\leq} \delta(\star(\mathcal{F}))$.

(12′) There is an example such that $\mathcal{F}(\alpha(\star)) \underset{\neq}{\supseteq} \alpha(\mathcal{F}(\star))$.

(20′) There is an example such that $\delta(\mathcal{F}(\alpha(\star))) \supsetneqq \mathcal{F}(\star)$.

(21′) There is an example such that $\delta(\star(\alpha(\mathcal{F}))) \gneqq \star(\mathcal{F})$.

(23′) There is an example such that $\mathcal{F}(\star) \supsetneqq \alpha(\mathcal{F}(\delta(\star)))$.

(26′) There is an example such that $\delta(\alpha(\mathcal{F})_f) \supsetneqq \mathcal{F}_f$.

(28′) There is an example such that $\alpha(\delta(\mathcal{F})_f) \subsetneqq \mathcal{F}_f$.

3. Remarks

In this section, we give examples as in Remark 2 for each of (14), (16), (24), (25), (27), (29), (30), (31), (32), (33) and (34) of Proposition 1. We give propositions for each of (13), (18) and (19) of Proposition 1.

Remark 3. There is an example such that $\alpha(\mathcal{F})_f \supsetneqq \alpha(\mathcal{F}_f)$.

Example. We have $\alpha(\mathcal{F})_f \supseteq \alpha(\mathcal{F}_f)$ by Proposition 1 (14). We have $\alpha(\mathcal{F})_f = \{J \mid$ there are $I \in \mathcal{F}$, and $F \in f(T)$ such that $I \subseteq F \subseteq J\}$, and $\alpha(\mathcal{F}_f) = \{J \mid$ there is a finitely generated ideal $I \in \mathcal{F}$ of D such that $I \subseteq J\}$.

Let $D = V$ be a valuation domain whose value group is the set $\boldsymbol{R}$ of the real numbers, and let M be its maximal ideal. Then $\mathcal{F} = \{D, M\}$ is a localizing system of D. Let X be an indeterminate, and let T be the extension domain of D generated by the subset $\left\{\frac{p}{X}, X \mid p \in M\right\}$ of $K(X)$. Since $XT \supseteq M$, we have $XT \in \alpha(\mathcal{F})_f$. Since $\frac{1}{X} \notin T$, we have $XT \notin \alpha(\mathcal{F}_f)$.

Remark 4. There is an example such that $\star(\mathcal{F}) \gneqq \alpha(\star(\delta(\mathcal{F})))$.

Example. We have $\star(\mathcal{F}) \geq \alpha(\star(\delta(\mathcal{F})))$ by Proposition 1 (24). Let $D = K$, $T = K[X]$, and let $\mathcal{F} = \{J \mid J$ is a non-zero ideal of $T\}$. Then

$\delta(\mathcal{F}) = \{K\}$, $\star(\delta(\mathcal{F})) = d_D$, and $\alpha(\star(\delta(\mathcal{F}))) = d_T$. We have $\frac{1}{X} \notin T^{d_T}$, but $\frac{1}{X} \in T^{\star(\mathcal{F})}$. Hence $\star(\mathcal{F}) \gneqq \alpha(\star\delta(\mathcal{F}))$.

Remark 5. There is an example such that $\star_f \gneqq \alpha(\delta(\star)_f)$.

Example. We have $\star_f \geq \alpha(\delta(\star)_f)$ by Proposition 1 (27). Let $D = K$, $T = K[X]$, and let $\star = e_T$. Then $\star_f = e_T$, and $\alpha(\delta(\star)_f) = \alpha(d_D) = d_T$.

Remark 6. There is an example such that $\star_a \gneqq \alpha(\delta(\star)_a)$.

Example. We have $\star_a \geq \alpha(\delta(\star)_a)$ by Proposition 1 (29). Let $D = K$, $T = K[X^2, X^3, X^4, \ldots]$, and let $\star = d_T$. Since D is a field, we have $\delta(\star) = \delta(\star)_a = d_D$. Hence $\alpha(\delta(\star)_a) = \alpha(d_D) = d_T$, and $X \notin T^{\alpha(\delta(\star)_a)} = T$. Let $F = K[X] \in f(T)$. Since $XF \subseteq F$, we have $X \in T^{\star_a}$. Hence $T^{\star_a} \supsetneqq T^{\alpha(\delta(\star)_a)}$.

Remark 7. There is an example such that $[\star] \gneqq \alpha([\delta(\star)])$.

Example. We have $[\star] \geq \alpha([\delta(\star)])$ by Proposition 1 (30). Let $D = K$, $T = K[X^2, X^3, X^4, \ldots]$, and let $\star = d_T$. We have $\delta(\star) = d_D$, $[\delta(\star)] = d_D$, and $\alpha([\delta(\star)]) = d_T$. Hence $T^{\alpha([\delta(\star)])} = T$, hence $X \notin T^{\alpha([\alpha(\star)])}$. Let $F_1 = K[X]$. Then $XF_1 \subseteq F_1$. Hence $X \in T^{[\star]}$, hence $T^{\alpha[\delta(\star)]} \subsetneqq T^{[\star]}$.

Remark 8. There is an example such that $\alpha(\star)_a \gneqq \alpha(\star_a)$.

Example. We have $\alpha(\star)_a \geq \alpha(\star_a)$ by Proposition 1 (34). Let $D = K$, $T = K[X^2, X^3, X^4, \ldots]$, and let $\star = d_D$. Since D is a field, we have $\star_a = d_D$, $\alpha(\star) = d_T$, and $\alpha(\star_a) = d_T$. Hence $X \notin T^{\alpha(\star_a)}$. Let $F = K[X]$. Since $XF \subseteq F$, we have $X \in T^{\alpha(\star)_a}$.

Remark 9. There is an example such that $[\alpha(\star)] \gneqq \alpha([\star])$.

Example. We have $[\alpha(\star)] \geq \alpha([\star])$ by Proposition 1 (32). Let $D = K$, $T = K[X^2, X^3, X^4, ...]$, and let $\star = d_D$. Then $[\star] = d_D$, $\alpha(\star) = d_T$, and $\alpha([\star]) = d_T$, hence $T^{\alpha([\star])} = T \not\ni X$. Let $F_1 = K[X]$. Then $XF_1 \subseteq F_1$, hence $X \in T^{[\alpha(\star)]}$.

Remark 10. There is an example such that $\delta(\alpha(\star)_a) \gneqq \star_a$.

Example. We have $\delta(\alpha(\star)_a) \geq \star_a$ by Proposition 1 (33). Let D be a valuation domain with $D \subsetneqq K$, let $T = K$, and let $\star = d_D$. Then $\star_a = d_D$, and let $\alpha(\star) = \alpha(\star)_a = d_T$. We have $D^{\delta(\alpha(\star)_a)} = D^{\delta(d_T)} = K$, and $D^{\star_a} = D$.

Remark 11. There is an example such that $\delta([\alpha(\star)]) \gneqq [\star]$.

Example. We have $\delta([\alpha(\star)]) \geq [\star]$ by Proposition 1 (31). Let D be a valuation domain with $D \subsetneqq K$, $T = K$, and $\star = d_D$. Then $[\star] = d_D$, and $\alpha(\star) = [\alpha(\star)] = d_T$. Then we have $D^{[\star]} = D$, and $D^{\delta([\alpha(\star)])} = K$.

Remark 12. There is an example such that $\delta(\alpha(\star)_f) \gneqq \star_f$.

Example. We have $\delta(\alpha(\star)_f) \geq \star_f$ by Proposition 1 (25). Let $D \subsetneqq K$, $T = K$, and let $\star = d_D$. Then $\star_f = d_D$, $\alpha(\star) = d_T$, and $\alpha(\star)_f = d_T$. We have $D^{\star_f} = D$, and $D^{\delta(\alpha(\star)_f)} = K$.

Set $F(D) = \{h \in \overline{F}(D) |$ we have $ah \subseteq D$ for some $a \in D \setminus \{0\}\}$.

Remark 13. There is an example such that $\alpha(\star)_f \gneqq \alpha(\star_f)$.

Example. Let k be a field, let $D = k[X_1, X_2, X_3, ...]$, and let $T = k[X_1, X_1^{-1}, X_2, X_2^{-1}, ...]$. For every $g \in F(D)$, set $g^\star = g$, and for every $h \in \overline{F}(D) \setminus F(D)$, set $h^\star = K$. Then $\star$ is a semistar operation on D. We have $T^{\star f} = T$. Since $T \in \overline{F}(D) \setminus F(D)$, we have $T^{\star} = K$. On the other hand, $T^{\alpha(\star)f} = T^{\star} = K$, and $T^{\alpha(\star_f)} = T^{\star f} = T$.

Proposition 14. *We have* $\star(\alpha(\mathcal{F})) = \alpha(\star(\mathcal{F}))$.

Proof. Let $\star_\lambda$ be a semistar operation on *T* such that $\delta(\star_\lambda) \geq \star(\mathcal{F})$.

(i) Let $I \in \mathcal{F}$. We have $I^{\delta(\star_\lambda)} \supseteq I^{\star(\mathcal{F})}$. Since $1 \in I^{\star(\mathcal{F})}$, we have $1 \in I^{\delta(\star_\lambda)}$. Hence $1 \in (IT)^{\star\lambda}$.

(ii) Let $y \in H^{\star(\alpha(\mathcal{F}))}$. There is $I \in \mathcal{F}$ such that $yI \subseteq H$. Then $yIT \subseteq H$ and $y(IT)^{\star\lambda} \subseteq H^{\star\lambda}$. By (i), it follows that $y \in H^{\star\lambda}$. Hence $H^{\star(\alpha(\mathcal{F}))} \subseteq H^{\star\lambda}$, and hence $\star(\alpha(\mathcal{F})) \leq \star_\lambda$. It follows that $\star(\alpha(\mathcal{F})) \leq \alpha(\star(\mathcal{F}))$.

A subring *T* of *K* with $D \subseteq T$ is called an *overring* of *D*.

Proposition 15. *Let T be an overring of D. Then* $\delta(\star_a) = \delta(\star)_a$.

Proof. We have $\delta(\star_a) \geq \delta(\star)_a$ by Proposition 1 (18). Let $x \in f^{\delta(\star_a)}$ for $f \in f(D)$. Then there is $F_1 \in f(T)$ such that $xF_1 \subseteq (fF_1)^{\star}$. There is $f_1 \in f(D)$ such that $F_1 = f_1 T$. Then $xf_1 \subseteq (ff_1 T)^{\star}$, hence $x \in f^{\delta(\star)_a}$.

Proposition 16. *Let T be an overring of D. Then* $\delta([\star]) = [\delta(\star)]$.

Proof. We have $\delta([\star]) \geq [\delta(\star)]$ by Proposition 1 (19). Let $x \in f^{\delta([\star])}$ for $f \in f(D)$. There are $F_1 \in f(T)$ and $y_1, ..., y_n \in L$ such that $y_i F_1 \subseteq F_1^{\star}$ for every *i* and $x \in (y_1 fT, ..., y_n fT)^{\star}$. There is $f_1 \in f(D)$ such that $f_1 T = F_1$. Then $y_i f_1 T \subseteq (f_1 T)^{\star}$, and $x \in (y_1 fT, ..., y_n fT)^{\star}$, hence $x \in f^{[\delta(\star)]}$.

Note. All remarks and propositions in this paper hold for grading monoids (or, *g*-monoids).

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