THE ECCENTRIC DIGRAPH OF FRIENDSHIP GRAPH AND FIRECRACKER GRAPH

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Abstract

Let G be a graph with a set of vertices $V(G)$ and a set of edges $E(G)$. Then the distance from vertex u to vertex v in G, denoted by $d(u, v)$, is the length of the shortest path from vertex u to v. The eccentricity of vertex u in a graph G is the maximum distance from vertex u to any other vertices in G, denoted by $e(u)$. Vertex v is an eccentric vertex from u if $d(u, v) = e(u)$. The eccentric digraph $ED(G)$ of a graph G is a graph that has the same set of vertices as G, and there is an arc (directed edge) joining vertex u to v if v is an eccentric vertex from u. In this paper, we determine the eccentric digraph of a class of graphs called the friendship graph F_k^n and firecracker graph $F_{n, k}$.

1. Introduction

Most of the notation and terminologies follow that of Chartrand and Oellermann [2]. Let G be a graph with a set of vertices $V(G)$ and a set of edges $E(G)$. Then the distance from vertex u to vertex v in G, denoted by $d(u, v)$, is a length of the shortest path from vertex u to v. If there is not a path joining vertex u and vertex v, then $d(u, v) = \infty$. The eccentricity of vertex u in a graph G is the maximum distance from vertex u to any other vertices in G, denoted by $e(u)$, and so $e(u) = \max\{d(u, v)|v \in V(G)\}$. Radius of a graph G, denoted by $rad(G)$, is the minimum eccentricity of every vertex in G. The diameter of a graph G, denoted by $diam(G)$, is the maximum eccentricity of every vertex in G. If $e(u) = rad(G)$, then vertex u is called *central vertex*. Center of a graph G, denoted by $cen(G)$, is an induced subgraph formed from central vertices of G. Vertex v is an eccentric vertex from u if $d(u, v) = e(u)$. The eccentric digraph $ED(G)$ of a graph G is a graph that has the same set of vertices as G, $V(ED(G)) = V(G)$, and there is an arc (directed edge) joining vertex u to v if v is an eccentric vertex from u. An arc of a digraph D joining vertex u to v and vertex v to u is called a *symmetric arc*.

One of the topics in graph theory is to determine the eccentric digraph of a graph. The eccentric digraph of a graph was initially introduced by Fred Buckley (Boland and Miller [1]). Some authors have investigated the problem of finding the eccentric digraph. For example, Gimbert et al. [5] found the characterization of the eccentric digraphs, while Boland and Miller [1] determined the eccentric digraph of a digraph. Boland and Miller [1] also proposed an open problem to find the eccentric digraph of various classes of graphs.

Some results related to this open problem can be found in Wang and Sun [8] and Kusmayadi and Rivai [6, 7]. In this paper, we tackle the open problem proposed by Boland and Miller [1]. In particular, we determine the

eccentric digraph of a graph called *friendship graph* F_k^n and *firecracker graph* $F_{n,k}$.

2. Breadth First Search (BFS) - Moore Algorithm

The following is breadth first search (BFS) - Moore algorithm to determine the eccentric digraph of a graph. The first step is to determine the distance from vertex u to every vertex v in the graph, denoted by $d(u, v)$, using breadth first search (BFS) - Moore algorithm taken from Chartrand and Oellermann [2] as follows:

(1) Take any vertex, say u, and label 0 stating the distance from u to itself, and other vertices are labeled ∞.

(2) All vertices having label ∞ which are adjacent to u are labeled by 1.

(3) All vertices having label ∞ which are adjacent to 1 are labeled by 2 and so on until the required vertex, say v, is already labeled.

The second step is to find vertex eccentricity u by choosing the maximum distance from the vertex u, and so it results in eccentric vertex v from u if $d(u, v) = e(u)$. The final step is to join an arc from vertex u to its eccentric vertex, so it gives an eccentric digraph from the given graph.

3. The Eccentric Digraph of a Friendship Graph

According to Gallian [4], a friendship graph F_k^n can be defined as a graph having a common vertex of n-cycle of length k. We assume that the friendship graph has vertex set $V(F_k^n) = \{u, v_{1,1}, v_{1,2}, ..., v_{1,k-1}, v_{2,1}, v_{2,2}, ..., v_{2,k-1}, ..., v_{n,1}, v_{n,2}, ..., v_{n,k-1}\}$. The friendship graph F_k^n can be described as in Figure 1.
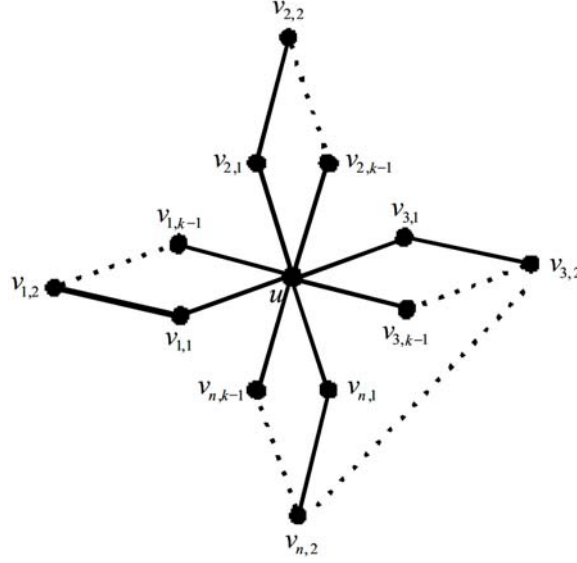
Figure 1. The friendship graph F_k^n.

Lemma 3.1. *Let F_k^n be a friendship graph for k even. Then the eccentric vertex u is $v_{i,\frac{k}{2}}$ for $i \in [1, n]$, and the eccentric vertex of vertex $v_{i,j}$ is $v_{s,\frac{k}{2}}$ for $i, s \in [1, n]$, $j \in [1, k-1]$, $i \neq s$.*

Proof. By using BFS - Moore algorithm, the eccentricity of vertex u is $\frac{k}{2}$, so the eccentric vertex of u is $v_{i,\frac{k}{2}}$ for $i \in [1, n]$. Also, the eccentricity of vertex $v_{i,j}$ is $\frac{k}{2} + j$, for $i, s \in [1, n]$, and $j \in [1, k-1]$, where $j \leq \frac{k}{2}$, and $i \neq s$. So, the eccentric vertex of $v_{i,j}$ is $v_{s,\frac{k}{2}}$. In addition, the eccentricity of vertex $v_{i,j}$ is $\frac{3k}{2} - j$, for $i, s \in [1, n]$, and $j \in [1, k-1]$, where $j > \frac{k}{2}$, and $i \neq s$. So, the eccentric vertex $v_{i,j}$ is $v_{s,\frac{k}{2}}$.

Theorem 3.2. *Let F_k^n be a friendship graph for k even. Then the*

eccentric digraph of F_k^n, $ED(F_k^n)$, *is a digraph having vertex set*

$$V(F_k^n) = \{u, v_{1,1}, v_{1,2}, ..., v_{1,k-1}, v_{2,1}, v_{2,2}, ..., v_{2,k-1}, ..., v_{n,1},$$

$$v_{n,2}, ..., v_{n,k-1}\}$$

and the arc set

$$A(ED(F_k^n)) = \left\{ \overrightarrow{uv_{i,j}} \mid i \in [1, n], \ j = \frac{k}{2} \right\}$$

$$\cup \left\{ \overleftrightarrow{v_{i,\frac{k}{2}} v_{j,\frac{k}{2}}} \mid i, j \in [1, n], \ i \neq j \right\}$$

$$\cup \left\{ \overrightarrow{v_{i,j} v_{s,t}} \mid i, j \in [1, n], \ j \neq \frac{k}{2}, \ s \in [1, n], \ s \neq i, \ t = \frac{k}{2} \right\}.$$

Proof. By Lemma 3.1, the arcs from vertex u to vertex $v_{i,\frac{k}{2}}$ for $i \in [1, n]$, are not symmetric. In addition, the arcs from vertex $v_{i,\frac{k}{2}}$ to vertex $v_{j,\frac{k}{2}}$ for $i, j \in [1, n]$ and $i \neq j$, are symmetric. Now, the arcs from vertex $v_{i,j}$ to vertex $v_{s,t}$ for $i, j \in [1, n]$, $j \neq \frac{k}{2}$ and $s \in [1, n]$, $s \neq i$, $t = \frac{k}{2}$, are not symmetric. So, the obtained digraph is a digraph having the vertex set $V(F_k^n)$ and the arc set $A(ED(F_k^n))$. This completes the proof of the theorem.

Lemma 3.3. *Let* F_k^n *be a friendship graph for k odd. Then the eccentric vertices u are* $v_{i,\frac{k-1}{2}}$ *and* $v_{i,\frac{k+1}{2}}$ *for* $i \in [1, n]$, *and the eccentric vertices of vertex* $v_{i,j}$ *are* $v_{s,\frac{k-1}{2}}$ *and* $v_{s,\frac{k+1}{2}}$ *for* $i, s \in [1, n]$, $j \in [1, k-1]$, $i \neq s$.

Proof. By using BFS - Moore algorithm, the eccentricity of vertex u is $\frac{k-1}{2}$, so the eccentric vertices of u are $v_{i,\frac{k-1}{2}}$ and $v_{i,\frac{k+1}{2}}$ for $i \in [1, n]$.

Also, the eccentricity of vertex $v_{i,j}$ is $\dfrac{k-1}{2} + j$, for $i, s \in [1, n]$, and

$j \in [1, k-1]$, where $j \leq \dfrac{k-1}{2}$, and $i \neq s$. So, the eccentric vertices of

$v_{i,j}$ are $v_{s,\frac{k-1}{2}}$ and $v_{s,\frac{k+1}{2}}$. In addition, the eccentricity of vertex $v_{i,j}$ is

$\dfrac{3k-1}{2} - j$, for $i, s \in [1, n]$, and $j \in [1, k-1]$, where $j \geq \dfrac{k+1}{2}$, and $i \neq s$.

So, the eccentric vertices of $v_{i,j}$ are $v_{s,\frac{k-1}{2}}$ and $v_{s,\frac{k+1}{2}}$.

Theorem 3.4. *Let F_k^n be a friendship graph for k odd. Then the eccentric digraph of F_k^n, $ED(F_k^n)$ is a digraph having the vertex set*

$$V_i = V(\overline{K}_{k-3,i}) = \left\{ v_{i,1}, ..., v_{i,\frac{k-3}{2}}, v_{i,\frac{k+3}{2}}, ..., v_{i,k-1} \right\},$$

$$V_i' = V(\overline{K}_{2,i}) = \left\{ v_{i,\frac{k-1}{2}}, v_{i,\frac{k+1}{2}} \right\}$$

and the arc set

$$A(ED(F_k^n)) = \{ \overrightarrow{u\alpha_i} \mid i \in V_i', i \in [1, n] \}$$

$$\cup \{ \overrightarrow{\alpha_r \alpha_s} \mid \alpha_r \in V_r', \alpha_s \in V_s', r, s \in [1, n], r \neq s \}$$

$$\cup \{ \overrightarrow{\beta_p \beta_q} \mid \beta_p \in V_i, \beta_q \in V_j', i, j \in [1, n], i \neq j \},$$

$$p = 1, ..., \frac{k-3}{2}, \frac{k+3}{2}, ..., k-1, q = \frac{k-1}{2}, \frac{k+1}{2}.$$

Proof. By Lemma 3.3, the arcs from vertex u to vertices $v_{i,\frac{k-1}{2}}$ and

$v_{i,\frac{k+1}{2}}$ for $i \in [1, n]$, are not symmetric. In addition, the arcs from vertex

$v_{r,\frac{k-1}{2}}$ to vertex $v_{s,\frac{k}{2}}$ for $r, s \in [1, n]$ and $r \neq s$, are symmetric. Now, the

arcs from vertex $v_{i,p}$ to vertex $v_{j,q}$ for $i, j \in [1, n]$, and $i \neq j$, where

$$p = 1, ..., \frac{k-3}{2}, \frac{k+3}{2}, ..., k-1, \quad q = \frac{k-1}{2}, \frac{k+1}{2},$$ are not symmetric. So,

the obtained digraph is a digraph having the vertex set $V(F_k^n) = \{V_i, V_i'\}$ and

the arc set $A(ED(F_k^n))$. This completes the proof of the theorem.

4. The Eccentric Digraph of a Firecracker Graph

Chen et al. [3] and Gallian [4] defined a firecracker graph $F_{n,k}$ as a graph obtained by the concatenation of n k-stars by linking one leaf from each k-stars. We assume that the firecracker graph has the vertex set $V(F_{n,k})$ $= \{u_{1,1}, u_{1,2}, ..., u_{n,k-2}, v_1, v_2, ..., v_n, w_1, w_2, ..., w_n\}$. The firecracker graph $F_{n,k}$ can be described as in Figure 2.
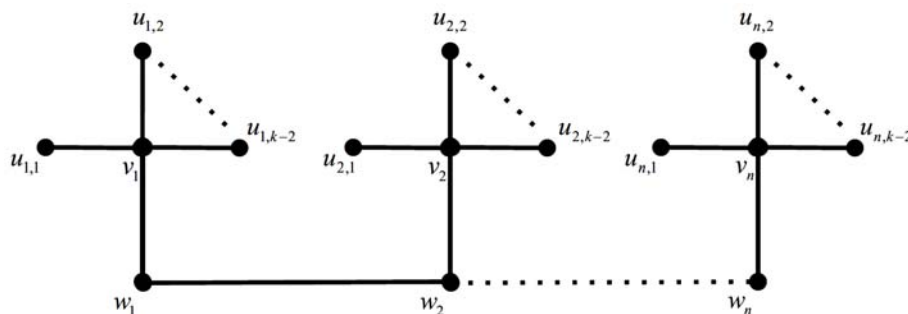

Figure 2. The firecracker graph $F_{n,k}$.

Lemma 4.1. *Let $F_{n,k}$ be a firecracker graph. Then the eccentric vertex of*

$$e(u_{i,j})$$

$$= \begin{cases} u_{n,1}, u_{n,2}, ..., u_{n,k-2} & i = 1, 2, ..., \left\lceil \dfrac{n}{2} \right\rceil, \quad j = 1, 2, ..., k-2, \\[4mm] u_{1,1}, u_{1,2}, ..., u_{1,k-2} & i = \left\lfloor \dfrac{n}{2} \right\rfloor + 1, \left\lfloor \dfrac{n}{2} \right\rfloor + 2, ..., n, \quad j = 1, 2, ..., k-2, \end{cases}$$

$$e(v_i) = \begin{cases} u_{n,1}, u_{n,2}, ..., u_{n,k-2} & i = 1, 2, ..., \left\lceil \dfrac{n}{2} \right\rceil, \\ u_{1,1}, u_{1,2}, ..., u_{1,k-2} & i = \left\lfloor \dfrac{n}{2} \right\rfloor + 1, \left\lfloor \dfrac{n}{2} \right\rfloor + 2, ..., n, \end{cases}$$

and

$$e(w_i) = \begin{cases} u_{n,1}, u_{n,2}, ..., u_{n,k-2} & i = 1, 2, ..., \left\lceil \dfrac{n}{2} \right\rceil, \\ u_{1,1}, u_{1,2}, ..., u_{1,k-2} & i = \left\lfloor \dfrac{n}{2} \right\rfloor + 1, \left\lfloor \dfrac{n}{2} \right\rfloor + 2, ..., n. \end{cases}$$

Proof. By using BFS-Moore algorithm, the farthest distance from vertex $u_{i,j}$ is $n + 4 - i$ for $i = 1, 2, ..., \left\lceil \dfrac{n}{2} \right\rceil$, $j = 1, 2, ..., k - 2$, so the eccentric vertex of $u_{i,j}$ is $u_{n,1}, u_{n,2}, ..., u_{n,k-2}$. Also, the farthest distance from vertex $u_{i,j}$ is $i + 3$ for $i = \left\lfloor \dfrac{n}{2} \right\rfloor + 1, \left\lfloor \dfrac{n}{2} \right\rfloor + 2, ..., n$, $j = 1, 2, ..., k - 2$, so the eccentric vertex of $u_{i,j}$ is $u_{1,1}, u_{1,2}, ..., u_{1,k-2}$. In addition, the farthest distance from vertex v_i is $n + 3 - i$ for $i = 1, 2, ..., \left\lceil \dfrac{n}{2} \right\rceil$, so the eccentric vertex of v_i is $u_{n,1}, u_{n,2}, ..., u_{n,k-2}$. Now, the farthest distance from vertex v_i is $i + 2$ for $i = \left\lfloor \dfrac{n}{2} \right\rfloor + 1, \left\lfloor \dfrac{n}{2} \right\rfloor + 2, ..., n$, so the eccentric vertex of v_i is $u_{1,1}, u_{1,2}, ..., u_{1,k-2}$. Again the farthest distance from vertex w_i is $n + 2 - i$ for $i = 1, 2, ..., \left\lceil \dfrac{n}{2} \right\rceil$, so the eccentric vertex of w_i is $u_{n,1}, u_{n,2}, ..., u_{n,k-2}$. Also, the farthest distance from vertex w_i is $i + 1$ for $i = \left\lfloor \dfrac{n}{2} \right\rfloor + 1, \left\lfloor \dfrac{n}{2} \right\rfloor + 2, ..., n$, so the eccentric vertex of w_i is $u_{1,1}, u_{1,2}, ..., u_{1,k-2}$.

Theorem 4.2. *Let $F_{n,k}$ be a firecracker graph. Then, for n is even, the eccentric digraph $ED(F_{n,k})$ is a 4-partite digraph*

$$F_{k-2,\,k-2,\,\frac{nk-2k+4}{2},\,\frac{nk-2k+4}{2}}$$

having vertex set

$$V(F_{n,k}) = \begin{cases} V_1 = \{u_{1,1},\, u_{1,2},\, ...,\, u_{1,k-2}\}, \\[2mm] V_2 = \{u_{n,1},\, u_{n,2},\, ...,\, u_{n,k-2}\}, \\[2mm] V_3 = \left\{u_{2,1},\, ...,\, u_{\frac{n}{2},k-2},\, v_1,\, ...,\, v_{\frac{n}{2}},\, w_1,\, ...,\, w_{\frac{n}{2}}\right\}, \\[3mm] V_4 = \left\{u_{\frac{n}{2}+1,1},\, ...,\, u_{n,k-2},\, v_{\frac{n}{2}+1},\, ...,\, v_n,\, w_{\frac{n}{2}+1},\, ...,\, w_n\right\} \end{cases}$$

and the arc set

$$A(ED(F_{n,k}))$$

$$= \left\{ \overrightarrow{\alpha_i \alpha_j} \,|\, \alpha_i \in V_4,\, \alpha_j \in V_1,\, i = 1,\, 2,\, ...,\, \frac{nk-2k+4}{2},\, j = 1,\, 2,\, ...,\, k-2 \right\}$$

$$\cup \left\{ \overrightarrow{\alpha_s \alpha_t} \,|\, \alpha_s \in V_3,\, \alpha_t \in V_2,\, s = 1,\, 2,\, ...,\, \frac{nk-2k+4}{2},\, t = 1,\, 2,\, ...,\, k-2 \right\}$$

$$\cup \left\{ \overrightarrow{\beta_i \beta_j} \,|\, \beta_i \in V_1,\, \beta_j \in V_2,\, i = j = 1,\, 2,\, ...,\, k-2 \right\}.$$

Proof. By Lemma 4.1, the arcs are from vertex $u_{i,j}$ to the vertex $u_{n,1},\, u_{n,2},\, ...,\, u_{n,k-2}$ for every $i = 1,\, 2,\, ...,\, \frac{n}{2},\, j = 1,\, 2,\, ...,\, k-2,$ and to the vertex $u_{1,1},\, u_{1,2},\, ...,\, u_{1,k-2}$ for every $i = \frac{n}{2}+1,\, \frac{n}{2}+2,\, ...,\, n,\, j = 1,\, 2,$ $...,\, k-2.$ Also, the arcs are from vertex v_i to the vertex $u_{n,1},\, u_{n,2},\, ...,$ $u_{n,k-2}$ for every $i = 1,\, 2,\, ...,\, \frac{n}{2},$ and to vertex $u_{1,1},\, u_{1,2},\, ...,\, u_{1,k-2}$ for every $i = \frac{n}{2}+1,\, \frac{n}{2}+2,\, ...,\, n.$ Also, the arcs are from vertex w_i to the vertex

$u_{n,1}, u_{n,2}, ..., u_{n,k-2}$ for every $i = 1, 2, ..., \dfrac{n}{2}$, and to the vertex $u_{1,1}$, $u_{1,2}, ..., u_{1,k-2}$ for every $i = \dfrac{n}{2} + 1, \dfrac{n}{2} + 2, ..., n$, not all of the arcs are symmetric. Based on the arc set, the vertex set $V(ED(F_{n,k}))$ can be partitioned into four subsets of vertices $V_1 = \{u_{1,1}, u_{1,2}, ..., u_{1,k-2}\}$, $V_2 = \{u_{n,1}, u_{n,2}, ..., u_{n,k-2}\}$,

$$V_3 = \left\{ u_{2,1}, ..., u_{\frac{n}{2}, k-2}, v_1, ..., v_{\frac{n}{2}}, w_1, ..., w_{\frac{n}{2}} \right\}$$

and

$$V_4 = \left\{ u_{\frac{n}{2}+1, 1}, ..., u_{n, k-2}, v_{\frac{n}{2}+1}, ..., v_n, w_{\frac{n}{2}+1}, ..., w_n \right\}.$$

All arcs from vertices of V_3 are incident to the vertices of V_2, while all arcs from vertices of V_4 are incident to the vertices of V_1. The arcs from V_1 and V_2 are symmetric arcs. From these partitions, there is no arc from the same subsets. Therefore, the digraph can be formed to be 4-partite digraph. It is easy to observe that the eccentric digraph of $ED(F_{2,3})$ is a digraph $F_{1,1,2,2}$, $ED(F_{4,3})$ is a digraph $F_{1,1,5,5}$, $ED(F_{4,4})$ is a digraph $F_{2,2,6,6}$, and $ED(F_{4,5})$ is a digraph $F_{3,3,7,7}$. Hence, the eccentric digraph of $ED(F_{n,k})$ is a 4-partite $F_{k-2, k-2, \frac{nk-2k+4}{4}, \frac{nk-2k+4}{2}}$ digraph, for n even.

Theorem 4.3. *Let $F_{n,k}$ be a firecracker graph. Then, for n is odd, the eccentric digraph $ED(F_{n,k})$ is a 5-partite digraph*

$$F_{k-2, k-2, k, \frac{nk-3k+4}{2}, \frac{nk-3k+4}{2}}$$

having vertex set

$$V(F_{n,k}) = \begin{cases} V_1 = \{u_{1,1},\, u_{1,2},\, ...,\, u_{1,k-2}\}, \\[2mm] V_2 = \{u_{n,1},\, u_{n,2},\, ...,\, u_{n,k-2}\}, \\[2mm] V_3 = \left\{u_{\frac{n+1}{2},1},\, ...,\, u_{\frac{n+1}{2},k-2},\, v_1,\, ...,\, v_{\frac{n-1}{2}},\, w_1,\, ...,\, w_{\frac{n-1}{2}}\right\}, \\[2mm] V_4 = \left\{u_{2,1},\, ...,\, u_{\frac{n-1}{2},k-2},\, v_1,\, ...,\, v_{\frac{n-1}{2}},\, w_1,\, ...,\, w_{\frac{n-1}{2}}\right\}, \\[2mm] V_5 = \left\{u_{\frac{n+3}{2}+1,1},\, ...,\, u_{n,k-2},\, v_{\frac{n+3}{2}},\, ...,\, v_n,\, w_{\frac{n+3}{2}},\, ...,\, w_n\right\} \end{cases}$$

and the arc set

$$A(ED(F_{n,k}))$$

$$= \left\{\overrightarrow{\alpha_i \alpha_j} \,|\, \alpha_i \in V_5,\, \alpha_j \in V_1,\, i = 1, 2, ...,\, \frac{nk - 3k + 4}{2},\, j = 1, 2, ...,\, k - 2\right\}$$

$$\cup \left\{\overrightarrow{\alpha_s \alpha_t} \,|\, \alpha_s \in V_4,\, \alpha_t \in V_2,\, s = 1, 2, ...,\, \frac{nk - 3k + 4}{2},\, t = 1, 2, ...,\, k - 2\right\}$$

$$\cup \{\overrightarrow{\beta_i \beta_j} \,|\, \beta_i \in V_1,\, \beta_j \in V_2,\, i = j = 1, 2, ...,\, k - 2\}$$

$$\cup \{\overrightarrow{\gamma_i \gamma_j} \,|\, \gamma_i \in V_3,\, \gamma_j \in V_1,\, i = 1, 2, ...,\, k,\, j = 1, 2, ...,\, k - 2\}$$

$$\cup \{\overrightarrow{\mu_i \mu_j} \,|\, \mu_i \in V_3,\, \mu_j \in V_2,\, i = 1, 2, ...k,\, j = 1, 2, ...,\, k - 2\}.$$

Proof. By Lemma 4.1, the arcs are from vertex $u_{i,j}$ to the vertex $u_{n,1},\, u_{n,2},\, ...,\, u_{n,k-2}$ for every $i = 1, 2, ...,\, \frac{n+1}{2},\, j = 1, 2, ...,\, k - 2$, and to the vertex $u_{1,1},\, u_{1,2},\, ...,\, u_{1,k-2}$ for every $i = \frac{n+1}{2},\, \frac{n+1}{2} + 1, ...,\, n$,

$j = 1, 2, ..., k - 2$. Also, the arcs are from vertex v_i to the vertex $u_{n,1}$,
$u_{n,2}, ..., u_{n,k-2}$ for every $i = 1, 2, ..., \dfrac{n+1}{2}$, and to vertex $u_{1,1}, u_{1,2}, ...,$
$u_{1,k-2}$ for every $i = \dfrac{n+1}{2}, \dfrac{n+1}{2} + 1, ..., n$. Also, the arcs are from vertex
w_i to the vertex $u_{n,1}, u_{n,2}, ..., u_{n,k-2}$ for every $i = 1, 2, ..., \dfrac{n+1}{2}$, and to
the vertex $u_{1,1}, u_{1,2}, ..., u_{1,k-2}$ for every $i = \dfrac{n+1}{2}, \dfrac{n+1}{2} + 1, ..., n$, not all
of the arcs are symmetric. Based on the arc set, the vertex set $V(ED(F_{n,k}))$
can be partitioned into five subsets of vertices $V_1 = \{u_{1,1}, u_{1,2}, ..., u_{1,k-2}\}$,
$V_2 = \{u_{n,1}, u_{n,2}, ..., u_{n,k-2}\}$,

$$V_3 = \left\{ u_{\frac{n+1}{2},1}, ..., u_{\frac{n+1}{2},k-2}, v_1, ..., v_{\frac{n-1}{2}}, w_1, ..., w_{\frac{n-1}{2}} \right\},$$

$$V_4 = \left\{ u_{2,1}, ..., u_{\frac{n-1}{2},k-2}, v_1, ..., v_{\frac{n-1}{2}}, w_1, ..., w_{\frac{n-1}{2}} \right\}$$

and

$$V_5 = \left\{ u_{\frac{n+3}{2}+1,1}, ..., u_{n,k-2}, v_{\frac{n+3}{2}}, ..., v_n, w_{\frac{n+3}{2}}, ..., w_n \right\}.$$

All arcs from vertices of V_5 are incident to the vertices of V_1, while all arcs
from vertices of V_4 are incident to the vertices of V_2. All arcs from the
vertices V_3 are incident to the vertices of V_1 and V_2. The arcs from V_1 and
V_2 are symmetric arcs. From these partitions, there is no arc from the same
subsets. Therefore, the digraph can be formed to be 5-partite digraph. Based
on the observation, we obtain the eccentric digraph of $ED(F_{3,3})$ is a digraph
$F_{1,1,3,2,2}$, $ED(F_{5,3})$ is a digraph $F_{1,1,3,5,5}$, $ED(F_{3,4})$ is a digraph
$F_{2,2,4,2,2}$, and $ED(F_{3,5})$ is a digraph $F_{3,3,5,2,2}$. Hence, the eccentric

digraph of $ED(F_{n,k})$ is a 5-partite

$$F_{k-2,\,k-2,\,k,\,\frac{nk-3k+4}{2},\,\frac{nk-3k+4}{2}}$$

digraph, for n odd.

5. Concluding Remarks

As mentioned in the previous section, the main goal of this paper is to find the eccentric digraph of a given class of graphs. Some authors have conducted research on this problem. Most of them have left some open problems on their paper for the future research. We suggest the readers to investigate the problem proposed by Boland and Miller [1] by considering other classes of graphs.

Acknowledgement

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