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COMMON FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIVE TYPE AND GENERALIZED QUASI-CONTRACTIVE TYPE MAPPINGS ON CONE METRIC SPACES

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Abstract

In this paper, some new generalized contractive type and generalized quasi-contractive type conditions for a pair of mappings in cone metric spaces are defined and certain common fixed point theorems for these mappings are established.

1. Introduction and Preliminaries

Recently, a fixed point theorem for mappings defined on cone metric spaces satisfying cone integral type contractive condition [2] is proved.

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In this paper, we establish some new generalized contractive type and generalized quasi-contractive type conditions for a pair of mappings defined on cone metric spaces and prove some new common fixed point theorems for these mappings. Our results are the generalizations of results in [1, 2].

Let (E, τ) be a topological vector space and $P \subset E$. Then, P is called a *cone* whenever

(i) P is closed, non-empty and $P \neq \{0\}$;

(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b;

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by

$$x \leq y \text{ if and only if } y - x \in P.$$

We write $x < y$ to indicate that $x \leq y$ but $x \neq y$.

For $x, y \in P$, $x \ll y$ stands for $y - x \in int(P)$, where $int(P)$ is the interior of P.

A cone P is called *regular* if every increasing sequence which is bounded from above is convergent. That is, if $\{u_n\}$ is a sequence such that for some $z \in E$,

$$u_1 \leq u_2 \leq \cdots \leq z,$$

then there exists a $u \in E$ such that

$$\lim_{n \to \infty} \| u_n - u \| = 0.$$

Equivalently, a cone P is regular if and only if every decreasing sequence which is bounded from below is convergent.

If E is a normed space, then a cone P is called *normal* whenever there exists a number $K \geq 1$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$. The least positive number satisfying this norm inequality is called the *normal constant* of P [1]. There are non-normal cones (see [3]).

It is well known [1] that every regular cone is normal (see also [3]).

For a nonempty set *X*, a mapping $d : X \times X \to E$ is called *cone metric* [1] on *X* if the following conditions are satisfied:

(i) $0 \leq d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

From now on, we assume that *E* is a normed space, *P* is a cone in *E* with $int(P) \neq \varnothing$ and $\leq$ is a partial ordering with respect to *P*. And let *X* be a cone metric space with a cone metric *d*.

The following definitions are found in [1].

Let $x \in X$, and let $\{x_n\}$ be a sequence of points of *X*. Then

(1) $\{x_n\}$ *converges* to *x* (denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$) if for any $c \in int(P)$, there exists an *N* such that for all $n > N$, $d(x_n, x) \ll c$.

(2) $\{x_n\}$ is *Cauchy* if for any $c \in \mathrm{int}(P)$, there exists an *N* such that for all $n, m > N$, $d(x_n, x_m) \ll c$.

(3) (X, d) is *complete* if every Cauchy sequence in *X* is convergent.

Note that if *P* is a normal cone, then $\{x_n\}$ converges to *x* if and only if $\lim_{n\to\infty} d(x_n, x) = 0$. Further, in this case, $\{x_n\}$ is a Cauchy sequence in *X* if and only if $\lim_{n, m\to\infty} d(x_n, x_m) = 0$ (see [1]).

A function $f : P \to P$ is called $<$-*increasing* (resp. $\leq$-*increasing*) if, for each $x, y \in P$, $x < y$ (resp. $x \leq y$) if and only if $f(x) < f(y)$ (resp. $f(x) \leq f(y)$).

Let $F : P \to P$ be a function such that

(F1) $F(t) = 0$ if and only if $t = 0$;

(F2) $0 \ll F(t)$ for each $0 \ll t$;

(F3) F is $<$-increasing;

(F4) F is continuous.

We denote by $\Im(P, P)$ the family of functions satisfying (F1)-(F4).

Example 1.1. (1) Let $F(t) = t$ for each $t \in P$. Then $F \in \Im(P, P)$.

(2) Let $a, b \in E$, and let $[a, b] = \{x \in E : x = tb + (1-t)a, t \in [0, 1]\}$.

Suppose that a mapping $\varphi : P \to P$ is non-vanishing cone integrable on each $[a, b] \subset P$ such that for each $c \in int(P)$, $\int_0^c \varphi d_P \in int(P)$ (see [2]).

Let $F(t) = \int_0^t \varphi d_P$. Then $F \in \Im(P, P)$.

Note that if $\varphi : P \to P$ is subadditive, then F is subadditive (see [2]).

Let $\Phi(P, P)$ be the family of $\leq$-increasing and continuous functions $\phi : int(P) \cup \{0\} \to int(P) \cup \{0\}$ satisfying the following conditions:

(ϕ1) $\phi(t) = 0$ if and only if $t = 0$;

(ϕ2) for each $t \in int(P)$, $\phi(t) \ll t$;

(ϕ3) either $\phi(t) \leq d(x, y)$ or $d(x, y) \leq \phi(t)$ for $t \in int(P) \cup \{0\}$ and $x, y \in X$.

From now on, let $C(a) = \{p \in P : a \leq p\}$ for $a \in X$, and let $F \in \Im(P, P)$ and $\phi \in \Phi(P, P)$.

Lemma 1.1. *Let E be a topological vector space. Then the following are satisfied*:

(1) *If, for* $u, v, w \in E$, $u \ll v$ *and* $v \leq w$, *then* $u < w$.

(2) *If* $u \in E$ *such that for any* $c \in int(P)$, $0 \leq u \ll c$, *then* $u = 0$.

(3) *Let* $u, v, u_n \in E$ *such that* $u_n \leq v_n$ *for all* $n \geq 0$.

If $\lim_{n\to\infty} u_n = u$ *and* $\lim_{n\to\infty} v_n = v$, *then* $u \leq v$.

(4) *If* $c_n \in E$ *and* $c_n \to 0$, *then for each* $c \in int(P)$, *there exists an* N *such that* $c_n \ll c$ *for all* $n > N$.

Lemma 1.2. *Let* $P \subset E$ *be a normal cone. Suppose that* $\{u_n\}$, $\{v_n\}$ *and* $\{w_n\}$ *are sequences of points in* E *such that* $u_n \leq w_n \leq v_n$ *for all* $n \in \mathbb{N}$. *If* $\lim_{n\to\infty} u_n = u$ *and* $\lim_{n\to\infty} v_n = u$ *for some* $u \in E$, *then* $\lim_{n\to\infty} w_n = u$.

Proof. Since $v_n - u_n - (w_n - u_n) = v_n - w_n \in P$, we have $w_n - u_n \leq v_n - u_n$, and so $\| w_n - u_n \| \leq K\| v_n - u_n \|$, where K is the normal constant of P.

Thus we have

$$\begin{aligned} \| w_n - u \| &\leq \| w_n - u_n \| + \| u_n - u \| \\ &\leq K\| v_n - u_n \| + \| u_n - u \| \\ &\leq K(\| v_n - u \| + \| u - u_n \|) + \| u_n - u \| \to 0. \end{aligned}$$

Hence $\lim_{n\to\infty} w_n = u$. □

Lemma 1.3. *Let E be a topological vector space and* $a \in \{0\} \cup int(P)$.

If $F(a) \leq \phi(F(a))$, *then* $a = 0$.

Lemma 1.4 [1]. *Let* $P \subset E$ *be a normal cone. Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences of points in X and* $x, y \in X$.

If $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$, *then* $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

2. Common Fixed Point Theorems

We prove a generalized contractive type common fixed point theorem for a pair of mappings defined on cone metric space.

Theorem 2.1. *Let* (X, d) *be a complete cone metric space with regular cone* P *such that* $0 \ll d(x, y)$ *for* $x, y \in X$ *with* $x \neq y$. *Suppose that mappings* $S, T : X \to X$ *are satisfying*

$$C(F(d(Tx, Sy))) \cap \phi(F(m(x, y))) \neq \varnothing \tag{2.1}$$

for all $x, y \in X$, *where*

$$m(x, y) = \left\{d(x, y), d(Tx, x), d(Sy, y), \frac{1}{2}\{d(Tx, y) + d(Sy, x)\}\right\}.$$

Then S and T have a unique common fixed point in X. Moreover, for each $x_0 \in X$, *the iterated sequence* $\{x_n\}$ *with* $x_{2n+1} = Tx_{2n}$ *and* $x_{2n+2} = Sx_{2n+1}$ *is convergent to the common fixed point of S and T.*

Proof. We first prove that any fixed point of T is also a fixed point of S, and conversely.

If $Tz = z$, then we have

$$m(z, z) = \left\{d(z, z), d(Tz, z), d(Sz, z), \frac{1}{2}\{d(Tz, z) + d(Sz, z)\}\right\}$$

$$= \left\{0, d(Sz, z), \frac{1}{2}d(Sz, z)\right\}.$$

Thus from (2.1) with $x = z$, $y = z$, we obtain

$$C(F(d(Tz, Sz))) \cap \phi(F(m(z, z))) \neq \varnothing$$

which implies

$$C(F(d(z, Sz))) \cap \phi\left(F\left(\left\{0, d(Sz, z), \frac{1}{2}d(Sz, z)\right\}\right)\right) \neq \varnothing.$$

By Lemma 1.3, $d(z, Sz) = 0$. Hence $z = Sz$.

Similarly, we have that if $Sz = z$, then $Tz = z$.

We now show that if S and T have a common fixed point, then the common fixed point is unique.

If $Su = Tu = u$ and $Sv = Tv = v$, then $m(u, v) = \{0, d(u, v)\}$.

From (2.1) with $x = u$ and $y = v$, we have

$$C(F(d(u, v))) \cap \phi(F(\{0, d(u, v)\})) \neq \varnothing.$$

By Lemma 1.3, $u = v$.

Let $x_0 \in X$ be fixed, and let $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Sx_{2n+1}$ for all $n \geq 0$.

If there exists an $n \in \mathbb{N}$ such that $x_{n+1} = x_n$, then S or T has a fixed point, and so S and T have a common fixed point. Hence the proof is completed.

Thus we assume that, for any $n \geq 0$, $x_{n+1} \neq x_n$.

From (2.1) with $x = x_{2n}$ and $y = x_{2n-1}$, we have

$$C(F(d(Tx_{2n}, Sx_{2n-1}))) \cap \phi(F(m(x_{2n}, x_{2n-1}))) \neq \varnothing, \tag{2.2}$$

where

$$\begin{aligned} m(x_{2n}, x_{2n-1}) = \Big\{ & d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n}), \\ & \frac{1}{2}\{d(x_{2n+1}, x_{2n-1}) + d(x_{2n}, x_{2n})\}\Big\} \\ = \Big\{ & d(x_{2n}, x_{2n-1}), d(x_{2n+1}, x_{2n}), \frac{1}{2} d(x_{2n+1}, x_{2n-1})\Big\}. \end{aligned}$$

Thus we have

$$\begin{aligned} & C(F(d(x_{2n+1}, x_{2n}))) \\ & \cap \phi\left(F\left(\left\{d(x_{2n}, x_{2n-1}), d(x_{2n+1}, x_{2n}), \frac{1}{2} d(x_{2n+1}, x_{2n-1})\right\}\right)\right) \neq \varnothing \end{aligned}$$

which implies

$$\begin{aligned} & C(F(d(x_{2n+1}, x_{2n}))) \\ & \cap \phi\left(F\left(\left\{d(x_{2n}, x_{2n-1}), \frac{1}{2} d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1})\right\}\right)\right) \neq \varnothing. \end{aligned} \tag{2.3}$$

If

$$F(d(x_{2n+1}, x_{2n})) \leq \phi\left(F\left(\frac{1}{2}\{d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n})\}\right)\right), \quad (2.4)$$

then

$$F(d(x_{2n+1}, x_{2n})) \ll F\left(\frac{1}{2}\{d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n})\}\right),$$

and so $F(d(x_{2n+1}, x_{2n})) < F\left(\frac{1}{2}\{d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n})\}\right)$. Since F is $<$-increasing, $d(x_{2n+1}, x_{2n}) < \frac{1}{2}\{d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n})\}$. Thus, $d(x_{2n+1}, x_{2n}) < d(x_{2n}, x_{2n-1})$.

From (2.4), we obtain $F(d(x_{2n+1}, x_{2n})) \leq \phi(F(d(x_{2n}, x_{2n-1})))$. Thus (2.3) implies $F(d(x_{2n+1}, x_{2n})) \leq \phi(F(d(x_{2n}, x_{2n-1})))$.

Similarly, we have $F(d(x_{2n+2}, x_{2n+1})) \leq \phi(F(d(x_{2n+1}, x_{2n})))$.

Thus, we have

$$F(d(x_{n+1}, x_n)) \leq \phi(F(d(x_n, x_{n-1}))) \text{ for all } n \in \mathbb{N}.$$

By (ϕ2),

$$F(d(x_{n+1}, x_n)) \ll F(d(x_n, x_{n-1})) \text{ for all } n \in \mathbb{N}$$

and so

$$F(d(x_{n+1}, x_n)) < F(d(x_n, x_{n-1})) \text{ for all } n \in \mathbb{N}.$$

Thus,

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}) \text{ for all } n \in \mathbb{N}.$$

Since P is regular, there exists an $r(0 \leq r)$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$.

Letting $n \to \infty$ in (2.3), we obtain $F(r) \leq \phi(F(r)) \ll F(r)$, which is a contradiction unless $r = 0$.

Thus,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.5}$$

We now show that $\{x_n\}$ is a Cauchy sequence in X.

Assume that $\{x_n\}$ is not a Cauchy sequence.

Then there exists a $c \in int(P)$ such that for all $k \in \mathbb{N}$, there exists an $m_k > n_k > k$ satisfying

(i) n_k is even, and m_k is odd,

(ii) $d(x_{n_k}, x_{m_{k-1}}) \leq \phi(c)$,

(iii) $\phi(c) \leq d(x_{n_k}, x_{m_k})$.

Then we have

$$\phi(c) \leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{m_k}) \leq d(x_{m_k}, x_{m_{k-1}}) + \phi(c).$$

By Lemma 1.2 and (2.5),

$$\lim_{n\to\infty} d(x_{n_k}, x_{m_k}) = \phi(c). \tag{2.6}$$

We obtain

$$d(x_{n_k-1}, x_{m_k-1}) \leq d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k-1}) \leq d(x_{n_k-1}, x_{n_k}) + \phi(c)$$

and

$$\phi(c) \leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}).$$

Thus we have

$$\begin{aligned} &\phi(c) - d(x_{n_k}, x_{n_k-1}) - d(x_{m_k-1}, x_{m_k}) \\ &\leq d(x_{n_k-1}, x_{m_k-1}) \\ &\leq d(x_{n_k-1}, x_{n_k}) + \phi(c). \end{aligned}$$

By Lemma 1.2 and (2.5), we have

$$\lim_{n\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \phi(c). \tag{2.7}$$

We have

$$\begin{aligned} & d(x_{n_k-1}, x_{m_k-1}) - d(x_{m_k}, x_{m_k-1}) \\ & \leq d(x_{m_k}, x_{n_k-1}) \\ & \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}) \\ & \leq d(x_{m_k}, x_{m_k-1}) + d(x_{n_k}, x_{n_k-1}) + \phi(c). \end{aligned}$$

By Lemma 1.2, (2.5) and (2.7),

$$\lim_{n\to\infty} d(x_{m_k}, x_{n_k-1}) = \phi(c).$$

Also, we obtain

$$\begin{aligned} & \phi(c) - d(x_{m_k}, x_{m_k-1}) \\ & \leq d(x_{n_k}, x_{m_k}) - d(x_{m_k}, x_{m_k-1}) \\ & \leq (x_{n_k}, x_{m_k-1}) \\ & \leq \phi(c). \end{aligned}$$

By Lemma 1.2 and (2.5),

$$\lim_{n\to\infty} d(x_{n_k}, x_{m_k-1}) = \phi(c).$$

From (2.1) with $x = x_{n_{k-1}}$ and $y = x_{m_{k-1}}$, we obtain

$$C(F(d(Tx_{n_{k-1}}, Sx_{m_{k-1}}))) \cap \phi(F(m(x_{n_{k-1}}, x_{m_{k-1}}))) \neq \emptyset,$$

where

$$m(x_{n_{k-1}}, x_{m_{k-1}}) = \Big\{ d(x_{n_{k-1}}, x_{m_{k-1}}), d(x_{n_k}, x_{n_{k-1}}), d(x_{m_k}, x_{m_{k-1}}), \\ \frac{1}{2} d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_k}, x_{m_{k-1}}) \Big\}.$$

Letting $k \to \infty$ in above inequality, we obtain

$$C(F(\phi(c))) \cap \phi(F(\{0, \phi(c)\})) \neq \emptyset.$$

By Lemma 1.3, $\phi(c) = 0$, and hence $c = 0$, which is a contradiction.

Therefore, $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete, there exists an $a \in X$ such that $\lim_{n\to\infty} x_n = a$.

From (2.1) with $x = x_{2n}$ and $y = a$, we obtain

$$C(F(d(x_{2n+1}, Sa))) \cap \phi(F(m(x_{2n}, a))) \neq \emptyset, \tag{2.8}$$

where

$$m(x_{2n}, a)$$

$$= \left\{d(x_{2n}, a), d(x_{2n+1}, x_{2n}), d(Sa, a), \frac{1}{2}\{d(x_{2n+1}, a) + d(Sa, x_{2n})\}\right\}.$$

Letting $n \to \infty$ in (2.8), we obtain

$$C(F(d(a, Sa))) \cap \phi\left(F\left(\left\{0, d(Sa, a), \frac{1}{2} d(Sa, a)\right\}\right)\right) \neq \emptyset.$$

By Lemma 1.3, $d(a, Sa) = 0$. Thus, $Sa = a$.

By the proof above, a is a unique common fixed point of S and T. □

Corollary 2.2. *Let (X, d) be a complete cone metric space with regular cone P such that $0 \ll d(x, y)$ for $x, y \in X$ with $x \neq y$. Suppose that a mapping $\varphi : P \to P$ is non-vanishing cone integrable on each $[a, b] \subset P$ such that for each $c \in int(P)$, $\int_0^c \varphi d_P \in int(P)$.*

If mappings $S, T : X \to X$ are satisfying

$$C\left(\int_0^{d(Tx, Sy)} \varphi d_P\right) \cap \phi\left(\left\{\int_0^u \varphi d_P : u \in m(x, y)\right\}\right) \neq \emptyset$$

for all $x, y \in X$, then S and T have a unique common fixed point in X.

Moreover, for each $x_0 \in X$, *the iterated sequence* $\{x_n\}$ *with* $x_{2n+1} = Tx_{2n}$ *and* $x_{2n+2} = Sx_{2n+1}$ *is convergent to the common fixed point of S and T.*

Theorem 2.3. *Let* (X, d) *be a complete cone metric space with normal cone P. Suppose that mappings* $S, T : X \to X$ *are satisfying*

$$C(F(d(Tx, Sy))) \cap kF(m(x, y)) \neq \emptyset \tag{2.9}$$

for all $x, y \in X$, *where* $k \in [0, 1)$, *and* $F : P \to P$ *is satisfying* (F1), (F3) *and* (F4) *such that*

(1) *F is subadditive*;

(2) *if, for* $\{c_n\} \subset P$, $\lim_{n\to\infty} F(c_n) = 0$, *then* $\lim_{n\to\infty} c_n = 0$.

Then S and T have a unique common fixed point in X. Moreover, for each $x_0 \in X$, *the iterated sequence* $\{x_n\}$ *with* $x_{2n+1} = Tx_{2n}$ *and* $x_{2n+2} = Sx_{2n+1}$ *is convergent to the common fixed point of S and T.*

Proof. As in the proof of Theorem 2.1, we have that any fixed point of *T* is also a fixed point of *S*, and conversely. Also, if *S* and *T* have a common fixed point, then the common fixed point is unique.

Let $x_0 \in X$ be fixed, and let $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Sx_{2n+1}$ for all $n \geq 0$.

Then we may assume that for any $n \geq 0$, $x_{n+1} \neq x_n$.

Let $x = x_{2n}$ and $y = x_{2n-1}$ in (2.9).

Then, as in proof of Theorem 2.1, we have

$$C(F(d(x_{2n+1}, x_{2n})))$$
$$\cap\, kF\left(\left\{d(x_{2n}, x_{2n-1}), \frac{1}{2}\{d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1})\}\right\}\right) \neq \emptyset. \tag{2.10}$$

If

$$F(d(x_{2n+1}, x_{2n})) \le kF\left(\frac{1}{2}\{d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n})\}\right), \quad (2.11)$$

then $F(d(x_{2n+1}, x_{2n})) < F\left(\frac{1}{2}\{d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n})\}\right).$

Since F is <-increasing,

$$d(x_{2n+1}, x_{2n}) < \frac{1}{2}\{d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n})\}.$$

Thus, $d(x_{2n+1}, x_{2n}) < d(x_{2n}, x_{2n-1})$.

From (2.11), we obtain $F(d(x_{2n+1}, x_{2n})) \le kF(d(x_{2n}, x_{2n-1}))$. Thus (2.10) implies $F(d(x_{2n+1}, x_{2n})) \le kF(d(x_{2n}, x_{2n-1}))$.

Similarly, we have $F(d(x_{2n+2}, x_{2n+1})) \le kF(d(x_{2n+1}, x_{2n}))$.

Thus, we have

$$F(d(x_{n+1}, x_n)) \le kF(d(x_n, x_{n-1})) \text{ for all } n \in \mathbb{N}.$$

Hence

$$F(d(x_{n+1}, x_n)) \le kF(d(x_n, x_{n-1})) \le \cdots \le k^n F(d(x_1, x_0)).$$

We now show that $\{x_n\}$ is a Cauchy sequence in X.

For $m > n$, we have that

$$F(d(x_n, x_m)) \le F(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m))$$

$$\le F(d(x_n, x_{n+1})) + F(d(x_{n+1}, x_{n+2})) + \cdots + F(d(x_{m-1}, x_m))$$

$$\le k^n F(d(x_1, x_0)) + k^{n+1} F(d(x_1, x_0)) + \cdots + k^{m-1} F(d(x_1, x_0))$$

$$\le \frac{k^m}{1-k} F(d(x_1, x_0)) \to 0.$$

Hence $\lim_{n, m\to\infty} d(x_n, x_m) = 0$. By Lemma 1.1, for any $c \in int(P)$, there exists an N such that for all $n > N$, $d(x_n, x_m) \ll c$.

Hence $\{x_n\}$ is a Cauchy sequence in X, and hence $\lim_{n\to\infty} x_n = a \in X$ exists. As in the proof of Theorem 2.1, a is a unique common fixed point of S and T. □

Remark 2.1. Let a mapping $\varphi : P \to P$ be non-vanishing and subadditive cone integrable on each $[a, b] \subset P$ such that for each $c \in int(P)$, $\int_0^c \varphi d_P \in int(P)$.

Let $F(t) = \int_0^t \varphi d_P$. Then F satisfies (F1), (F3), (F4), and conditions (1) and (2) in Theorem 2.3.

Corollary 2.4. *Let (X, d) be a complete cone metric space with normal cone P. Suppose that a mapping $\varphi : P \to P$ is non-vanishing and subadditive cone integrable on each $[a, b] \subset P$ such that for each $c \in int(P)$, $\int_0^c \varphi d_P \in int(P)$. Suppose that mappings $S, T : X \to X$ are satisfying*

$$C\left(\int_0^{d(Tx, Sy)} \varphi d_P\right) \cap k\left\{\int_0^u \varphi d_P : u \in m(x, y)\right\} \neq \varnothing$$

for all $x, y \in X$, where $k \in [0, 1)$.

Then S and T have a unique common fixed point in X. Moreover, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ is convergent to the common fixed point of S and T.

Corollary 2.5 [2]**.** *Let (X, d) be a complete cone metric space with normal cone P. Suppose that a mapping $\varphi : P \to P$ is non-vanishing and subadditive cone integrable on each $[a, b] \subset P$ such that for each $c \in$*

$int(P)$, $\int_0^c \varphi d_P \in int(P)$. Suppose that mapping $T : X \to X$ is satisfying

$$\int_0^{d(Tx, Ty)} \varphi d_P \leq k \int_0^{d(x, y)} \varphi d_P$$

for all $x, y \in X$, where $k \in [0, 1)$.

Then T has a unique fixed point in X. Moreover, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{n+1} = Tx_n$ is convergent to the fixed point of T.

Theorem 2.6. *Let (X, d) be a complete cone metric space with normal cone P. Suppose that mappings $S, T : X \to X$ are satisfying*

$$C(F(d(Tx, Sy))) \cap kF(M(x, y)) \neq \varnothing \tag{2.12}$$

for all $x, y \in X$, where $k \in \left[0, \frac{1}{2}\right)$ and

$$M(x, y) = \{d(x, y), d(Tx, x), d(Sy, y), d(Tx, y), d(Sy, x)\},$$

and $F : P \to P$ is satisfying (F1), (F3) *and* (F4) *such that*

(1) *F is subadditive*;

(2) *if, for $\{c_n\} \subset P$, $\lim_{n\to\infty} F(c_n) = 0$, then $\lim_{n\to\infty} c_n = 0$.*

Then S and T have a unique common fixed point in X. Moreover, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ is convergent to the common fixed point of S and T.

Proof. As in the proof of Theorem 2.1, we have that any fixed point of T is also a fixed point of S, and conversely. Also, we have that if S and T have a common fixed point, then the common fixed point is unique.

Let $x_0 \in X$ be fixed, and let $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Sx_{2n+1}$ for all $n \geq 0$.

Then we may assume that, for any $n \geq 0$, $x_{n+1} \neq x_n$.

In fact, if there exists an $n \in \mathbb{N}$ such that $x_{n+1} = x_n$, then S or T has a fixed point, and so S and T have a common fixed point. Hence the proof is completed.

From (2.12) with $x = x_{2n}$ and $y = x_{2n-1}$, we have

$$C(F(d(Tx_{2n}, Sx_{2n-1}))) \cap kF(M(x_{2n}, x_{2n-1})) \neq \emptyset, \tag{2.13}$$

where

$$M(x_{2n}, x_{2n-1})$$

$$= \{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n-1}), d(x_{2n}, x_{2n})\}$$

$$= \{d(x_{2n}, x_{2n-1}), d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+1})\}.$$

Thus we have

$$C(F(d(Tx_{2n+1}, x_{2n}))) \cap kF(\{d(x_{2n}, x_{2n-1}), d(x_{2n+1}, x_{2n-1})\}) \neq \emptyset$$

which implies

$$C(F(d(x_{2n+1}, x_{2n})))$$

$$\cap\, kF(\{d(x_{2n}, x_{2n-1}), d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1})\}) \neq \emptyset. \tag{2.14}$$

If

$$F(d(x_{2n+1}, x_{2n})) \leq kF(d(x_{2n}, x_{2n-1})),$$

then $F(d(x_{2n+1}, x_{2n})) \leq \dfrac{k}{1-k} F(d(x_{2n}, x_{2n-1}))$ because $k \leq \dfrac{k}{1-k}$.

If

$$F(d(x_{2n+1}, x_{2n})) \leq kF(d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n})),$$

then $F(d(x_{2n+1}, x_{2n})) \leq kF(d(x_{2n}, x_{2n-1})) + kF(d(x_{2n+1}, x_{2n}))$, and so $F(d(x_{2n+1}, x_{2n})) \leq \dfrac{k}{1-k} F(d(x_{2n}, x_{2n-1}))$.

Thus, (2.14) implies $F(d(x_{2n+1}, x_{2n})) \leq \frac{k}{1-k} F(d(x_{2n}, x_{2n-1}))$.

Similarly, we have $F(d(x_{2n+2}, x_{2n+1})) \leq \frac{k}{1-k} F(d(x_{2n+1}, x_{2n}))$.

Thus, we have

$$F(d(x_{n+1}, x_n)) \leq hF(d(x_n, x_{n-1})) \text{ for all } n \in \mathbb{N}, \text{ where } h = \frac{k}{1-k}.$$

Hence

$$F(d(x_{n+1}, x_n)) \leq h^n F(d(x_1, x_0)).$$

As in the proof of Theorem 2.3, we have that $\{x_n\}$ is a Cauchy sequence in *X*, and so the limit $\lim_{n\to\infty} x_n = a \in X$ exists.

From (2.12) with $x = x_{2n}$ and $y = a$, we obtain

$$C(F(d(x_{2n+1}, Sa))) \cap kF(M(x_{2n}, a)) \neq \emptyset, \tag{2.15}$$

where

$$M(x_{2n}, a) = \{d(x_{2n}, a), d(x_{2n+1}, x_{2n}), d(Sa, a), d(x_{2n+1}, a), d(Sa, x_{2n})\}.$$

Letting $n \to \infty$ in (2.15), we obtain

$$C(F(d(a, Sa))) \cap kF(\{0, d(Sa, a)\}) \neq \emptyset.$$

By Lemma 1.3, $d(a, Sa) = 0$. Thus, $Sa = a$.

Therefore, *a* is a unique common fixed point of *S* and *T*. □

Corollary 2.7. *Let (X, d) be a complete cone metric space with normal cone P. Suppose that a mapping $\varphi : P \to P$ is non-vanishing and subadditive cone integrable on each $[a, b] \subset P$ such that for each $c \in int(P)$, $\int_0^c \varphi d_P \in int(P)$. Suppose that mappings $S, T : X \to X$ are satisfying*

$$C\left(\int_0^{d(Tx, Sy)} \varphi d_P\right) \cap k\left\{\int_0^u \varphi d_P : u \in M(x, y)\right\} \neq \varnothing$$

for all $x, y \in X$, *where* $k \in \left[0, \frac{1}{2}\right)$.

Then S and T have a unique common fixed point in X. Moreover, for each $x_0 \in X$, *the iterated sequence* $\{x_n\}$ *with* $x_{2n+1} = Tx_{2n}$ *and* $x_{2n+2} = Sx_{2n+1}$ *is convergent to the common fixed point of S and T.*

The following example illustrates our main theorem.

Example 2.1. Let $E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$.

Let $X = \left\{\frac{1}{n} : n = 1, 2, \ldots\right\} \cup \{0\}$, and let $d(x, y) = (|x - y|, |x - y|)$. Then (X, d) is a complete cone metric space and P is regular.

Let $F(s, t) = \left(s^{\frac{1}{s}}, t^{\frac{1}{t}}\right)$ for all $(s, t) \in P$ and $\phi(a, b) = \frac{1}{2}(a, b)$ for all $(a, b) \in int(P) \cup \{0\}$.

Suppose that mappings S and T are defined by

$$Tx = Sx = \begin{cases} 0, & x = 0, \\ \frac{1}{n+1}, & x = \frac{1}{n}, \; n \geq 1. \end{cases}$$

We show that (2.1) is satisfied.

We consider the following three cases:

Case 1. $x = y$.

Then $d(x, y) = 0$ and $F(d(x, y)) = 0$. Thus $F(d(Tx, Sy)) = F(0) = \phi(F(0)) = \phi(F(d(x, y)))$, and so (2.1) is satisfied.

Case 2. $x = 0,\ y = \frac{1}{n}\left(\text{or } x = \frac{1}{n},\ y = 0\right).$

Then

$$\begin{aligned}
F(d(Tx, Sy)) &= F\left(d\left(0, \frac{1}{n+1}\right)\right)\\
&= F\left(\frac{1}{n+1}, \frac{1}{n+1}\right)\\
&= \left(\left(\frac{1}{n+1}\right)^{n+1}, \left(\frac{1}{n+1}\right)^{n+1}\right)\\
&= \left(\frac{1}{n+1}\left(\frac{1}{n+1}\right)^{n}, \frac{1}{n+1}\left(\frac{1}{n+1}\right)^{n}\right)\\
&\leq \left(\frac{1}{2}\left(\frac{1}{n}\right)^{n}, \frac{1}{2}\left(\frac{1}{n}\right)^{n}\right)\\
&= \phi(F(d(x, y))).
\end{aligned}$$

Case 3. $x = \frac{1}{n},\ y = \frac{1}{m}.$

Then

$$\begin{aligned}
&F(d(Tx, Sy))\\
&= F\left(d\left(\frac{1}{n+1}, \frac{1}{m+1}\right)\right)\\
&= F\left(\left|\frac{1}{n+1} - \frac{1}{m+1}\right|, \left|\frac{1}{n+1} - \frac{1}{m+1}\right|\right)\\
&= \left(\left(\frac{|n-m|}{(n+1)(m+1)}\right)^{\frac{(n+1)(m+1)}{|n-m|}}, \left(\frac{|n-m|}{(n+1)(m+1)}\right)^{\frac{(n+1)(m+1)}{|n-m|}}\right)
\end{aligned}$$

$$= \left(\left(\frac{|n-m|}{(n+1)(m+1)} \right)^{\frac{n+m+1}{|n-m|}}, \left(\frac{nm}{(n+1)(m+1)} \right)^{\frac{nm}{|n-m|}} \left(\frac{|n-m|}{nm} \right)^{\frac{nm}{|n-m|}}, \right.$$

$$\left. \left(\frac{|n-m|}{(n+1)(m+1)} \right)^{\frac{n+m+1}{|n-m|}}, \left(\frac{nm}{(n+1)(m+1)} \right)^{\frac{nm}{|n-m|}} \left(\frac{|n-m|}{nm} \right)^{\frac{nm}{|n-m|}} \right)$$

$$\leq \left(\frac{1}{2} \cdot 1 \cdot \left(\frac{|n-m|}{nm} \right)^{\frac{nm}{|n-m|}}, \frac{1}{2} \cdot 1 \cdot \left(\frac{|n-m|}{nm} \right)^{\frac{nm}{|n-m|}} \right)$$

$$= \frac{1}{2} \left(\left(\frac{|n-m|}{nm} \right)^{\frac{nm}{|n-m|}}, \left(\frac{|n-m|}{nm} \right)^{\frac{nm}{|n-m|}} \right)$$

$$= \phi(F(d(x, y))).$$

Thus S and T satisfy all conditions of Theorem 2.1, and so they have a unique common fixed point.

But we cannot obtain

$$C(d(Tx, Sy)) \cap km(x, y) \neq \emptyset$$

for all $x, y \in X$, where $k \in [0, 1)$.

That is, the following general contractive inequality is not satisfied:

There exists a $k \in [0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Sy) \leq ku \tag{2.16}$$

for some $u \in m(x, y)$.

Suppose that (2.16) is satisfied.

Let $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$.

Then we have

$$d(Tx, Sy) = \left(\frac{1}{(n+1)(n+2)}, \frac{1}{(n+1)(n+2)} \right)$$

and

$$m(x,\ y) = \left\{\left(\frac{1}{n(n+1)},\ \frac{1}{n(n+1)}\right),\ \left(\frac{1}{(n+1)(n+2)},\ \frac{1}{(n+1)(n+2)}\right)\right\}.$$

If

$$u = \left(\frac{1}{(n+1)(n+2)},\ \frac{1}{(n+1)(n+2)}\right),$$

then

$$\left(\frac{1}{(n+1)(n+2)},\ \frac{1}{(n+1)(n+2)}\right) = d(Tx,\ Sy)$$
$$\leq k\left(\frac{1}{(n+1)(n+2)},\ \frac{1}{(n+1)(n+2)}\right)$$

which is a contradiction.

If

$$u = \left(\frac{1}{n(n+1)},\ \frac{1}{n(n+1)}\right),$$

then

$$\left(\frac{1}{(n+1)(n+2)},\ \frac{1}{(n+1)(n+2)}\right) = d(Tx,\ Sy) \leq k\left(\frac{1}{n(n+1)},\ \frac{1}{n(n+1)}\right).$$

Thus we obtain

$$\left(\frac{k}{n(n+1)} - \frac{1}{(n+1)(n+2)},\ \frac{k}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right) \in P,$$

and so

$$0 \leq \frac{k}{n(n+1)} - \frac{1}{(n+1)(n+2)},$$

and hence $k \geq 1$, which is a contradiction.

Hence (2.16) is not satisfied.

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