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AN INEQUALITY FOR THE MIXED DISCRIMINANT OF A MATRIX

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Abstract

In this paper, a new inequality for the mixed discriminant of matrix is established, which is the matrix analogue of inequality for a symmetric function and the mixed volume function, respectively.

1. Introduction

Let $x_1, ..., x_n$ be a set of nonnegative quantities and $E_i(x)$ be the ith

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elementary symmetric function of an n-tuple $x = x(x_1, \ldots, x_n)$ of positive reals defined by $E_0(x) = 1$ and

$$E_i(x) = \sum_{1 < j_1 < \cdots < j_i \leq n} x_{j_1} x_{j_2} \cdots x_{j_i}, \quad 1 \leq i \leq n.$$

An interesting inequality for the symmetric function was established ([1], also see [2, p. 33]) as follows:

$$\frac{E_i(x+y)}{E_{i-1}(x+y)} \geq \frac{E_i(x)}{E_{i-1}(x)} + \frac{E_i(y)}{E_{i-1}(y)}. \tag{1.1}$$

A matrix analogue of (1.1) is the following result of Bergstrom [3].

Let K and L be positive definite matrices, and let K_i and L_i denote the sub-matrices obtained by deleting the ith row and column. Then

$$\frac{\det(K+L)}{\det(K_i + L_i)} \geq \frac{\det(K)}{\det(K_i)} + \frac{\det(L)}{\det(L_i)}. \tag{1.2}$$

An interesting proof is due to Bellman [4] (also see [2, p. 67]). A generalization of (1.2) was established by Fan [5] (also see [6, 7]).

There is a remarkable similarity between inequalities about symmetric functions (or determinants of symmetric matrices) and inequalities about the mixed volumes of convex bodies. In 1991, Milman asked if there is a version of (1.1) or (1.2) in the theory of mixed volumes and it was stated as the following open question (see [8]):

Question 1.1. For which values of i and every pair of convex bodies K and L in $\mathbb{R}^n$, is it true that

$$\frac{W_i(K+L)}{W_{i+1}(K+L)} \geq \frac{W_i(K)}{W_{i+1}(K)} + \frac{W_i(L)}{W_{i+1}(L)}? \tag{1.3}$$

The convex body is the compact and convex subset with non-empty interiors in $\mathbb{R}^n$. $W_i(K)$ denotes the quermassintegral of convex body K and $W_{i+1}(K)$

denotes the mixed volumes $V(\underbrace{K, ..., K}_{n-i-1}, \underbrace{B, ..., B}_{i+1})$. The sum + is the usual Minkowski vector sum.

A partial answer (L must be a ball) of (1.3) was established by Giannopoulos et al. (for details, see [9]). It can be proved that (1.3) is true in full generality only when $i = n - 1$ or $i = n - 2$ (see [10]).

If K and L are convex bodies in $\mathbb{R}^n$, and i is equal to $n - 1$ or $n - 2$, then

$$\frac{W_i(K + L)}{W_{i+1}(K + L)} \geq \frac{W_i(K)}{W_{i+1}(K)} + \frac{W_i(L)}{W_{i+1}(L)}. \tag{1.4}$$

In the paper, we establish a new matrix analogue of inequality (1.1) or (1.4).

Theorem 1.2. *Let K and L be symmetric positive define matrices. If i is equal to $n - 1$ or $n - 2$, then*

$$\frac{D_i(K + L)}{D_{i+1}(K + L)} \geq \frac{D_i(K)}{D_{i+1}(K)} + \frac{D_i(L)}{D_{i+1}(L)}. \tag{1.5}$$

Here $D_i(K) = D(\underbrace{K, ..., K}_{n-i}, \underbrace{I, ..., I}_{i})$ is the mixed discriminant (see Section 2).

2. Mixed Discriminants and Aleksandrov's Inequality

Recall that for positive definite $n \times n$ matrices $K_1, ..., K_N$ and $\lambda_1, ..., \lambda_N$, the determinant of the linear combination $\lambda_1 K_1 + \cdots + \lambda_N K_N$ is a homogeneous polynomial of degree n in the λ_i (see e.g., [11]),

$$\det(\lambda_1 K_1 + \cdots + \lambda_N K_N) = \sum_{1 \leq i_1, ..., i_n \leq N} \lambda_{i_1} \cdots \lambda_{i_n} D(K_{i_1}, ..., K_{i_n}), \tag{2.1}$$

where the coefficient $D(K_{i_1}, ..., K_{i_n})$ depends only on $K_{i_1}, ..., K_{i_n}$ (and not on other K_j's) and thus may be chosen to be symmetric in its arguments. The coefficient $D(K_{i_1}, \ldots, K_{i_n})$ is called the *mixed discriminant* of $K_{i_1}, \ldots, K_{i_n}$.

The mixed discriminant $D(K, \ldots, K, I, \ldots, I)$, with $n - k$ copies of K and k copies of the identity matrix I, will be abbreviated by $D_k(K)$. From (2.1), we have

$$D_i(K + \lambda I) = \sum_{j=0}^{n-i} \binom{n-i}{j} \lambda^j D_{i+j}(K). \tag{2.2}$$

Note that the elementary mixed discriminants $D_0(K), \ldots, D_n(K)$ are thus defined as the coefficients of the polynomial

$$\det(K + \lambda I) = \sum_{i=0}^{n} \binom{n}{i} \lambda^i D_i(K). \tag{2.3}$$

Obviously, $D_0(K) = \det(K)$ while $nD_{n-1}(K)$ is the trace of K.

The well-known Aleksandrov's inequality for mixed discriminants can be stated as follows (see [12], also see [13, p. 383] or [14, p. 35]):

Let $K_1, \ldots, K_n$ be real symmetric $n \times n$ matrices, where $K_1, \ldots, K_n$ are positive definite. Then

$$D(K_1, K_2, K_3, \ldots, K_n)^2$$
$$\geq D(K_1, K_1, K_3, \ldots, K_n) D(K_2, K_2, K_3, \ldots, K_n), \tag{2.4}$$

with equality if and only if $K_1 = \lambda K_2$ with a real number λ.

3. A Matrix Analogue of Inequality (1.1) or (1.4)

Lemma 3.1. *Let* K_j, $j = 3, \ldots, n$ *be symmetric positive define matrices and* $\mathcal{M} = (K_3, \ldots, K_n)$ *and denote* $D(K, L, \mathcal{M})$ *by* $D(K, L)$. *If* K, L, M *are symmetric positive define matrices, then we have either*

$$D(L, M)D(K, K) \geq D(L, K)D(M, K) \tag{3.1}$$

or

$$[D(K, L)D(K, M) - D(L, M)D(K, K)]^2$$

$$\leq [D(K, L)^2 - D(K, K)D(L, L)][D(K, M)^2 - D(K, K)D(M, M)]. \tag{3.2}$$

Proof. From (2.4), we obtain for $t, s \geq 0$,

$$D(L + tK, M + sK)^2 - D(L + tK, L + tK)D(M + sK, M + sK) \geq 0.$$

From the linearity of mixed discriminant, we have

$$f(t, s) + t^2[D(M, K)^2 - D(K, K)D(M, M)]$$

$$+ s^2[D(L, K)^2 - D(K, K)D(L, L)]$$

$$+ 2st[D(L, M)D(K, K) - D(L, K)D(M, K)] \geq 0,$$

where $f(t, s)$ is a linear function of *t* and *s*. It follows that the quadratic term is non-positive and in view of the following fact:

$$D(M, K)^2 \geq D(K, K)D(M, M),$$

and

$$D(L, K)^2 \geq D(K, K)D(L, L),$$

and hence, either

$$D(L, M)D(K, K) \geq D(L, K)D(M, K)$$

or its discriminant is non-positive. This proves Lemma 3.1.

Lemma 3.2. *Let* K_j, $j = 3, \ldots, n$ *be symmetric positive define matrices and* $\mathcal{M} = (K_3, \ldots, K_n)$ *and denote* $D(K, L, \mathcal{M})$ *by* $D(K, L)$. *If K*, *L*, *M are symmetric positive define matrices*, *then we have*

$$\frac{D(L + M, L + M)}{D(L + M, K)} \geq \frac{D(L, L)}{D(L, K)} + \frac{D(M, M)}{D(M, K)}. \tag{3.3}$$

Proof. From Lemma 3.1 and the Arithmetic-Geometric means inequality, we obtain

$$D(L, K)D(M, K) - D(L, M)D(K, K)$$

$$\leq (D(L, K)^2 - D(K, K)D(L, L))^{1/2}(D(M, K)^2 - D(K, K)D(M, M))^{1/2}$$

$$\leq \frac{D(M, K)}{2D(L, K)}(D(L, K)^2 - D(K, K)D(L, L))$$

$$+ \frac{D(L, K)}{2D(M, K)}(D(M, K)^2 - D(K, K)D(M, M)).$$

Hence

$$2D(L, M) \geq \frac{D(M, K)}{D(L, K)} \times D(L, L) + \frac{D(L, K)}{D(M, K)} \times D(M, M). \tag{3.4}$$

From (3.4) and the linearity of mixed discriminant, we get

$$D(L + M, L + M)$$

$$= D(L, L) + 2D(L, M) + D(M, M)$$

$$\geq D(L, L)\left(1 + \frac{D(M, K)}{D(L, K)}\right) + D(M, M)\left(1 + \frac{D(L, K)}{D(M, K)}\right),$$

which is the inequality (3.3).

Theorem 3.3. *Let K and L be symmetric positive define matrices. If i is equal to* $n - 1$ *or* $n - 2$, *then*

$$\frac{D_i(K + L)}{D_{i+1}(K + L)} \geq \frac{D_i(K)}{D_{i+1}(K)} + \frac{D_i(L)}{D_{i+1}(L)}. \tag{3.5}$$

The case $i = n - 2$ is an immediate consequence of Lemma 3.2.

Observe that when $i = n - 1$, Theorem 3.3 reduces to the inequality

$$D_{n-1}(K + L) \geq D_{n-1}(K) + D_{n-1}(L),$$

which holds as an equality, a special case of (2.4).

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