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UNIFIED TREATMENT OF p-VALENTLY ANALYTIC FUNCTIONS

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Abstract

In this paper we introduce and study the unified class $\mathcal{U}[p, n, \alpha, \beta]$,

$$p, n \in \mathbf{N} := \{1, 2, 3, ...\}; \quad 0 \leq \alpha < \frac{1}{p}; \quad \beta \geq 0;$$

of p-valently starlike and p-valently convex functions of order α in the open unit disk E. Some properties of functions belonging to $\mathcal{U}[p, n, \alpha, \beta]$ are obtained, which include radii of starlikeness and radii of convexity involving Hadamard products (or convolution).

1. Introduction and Definition

Let $\mathcal{T}(p, n)$ denote the class of functions f of the form:

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k; \quad a_k \geq 0, \quad p, n \in \mathbf{N} = \{1, 2, 3, ...\}, \tag{1.1}$$

which are analytic and p-valent in the unit disk $E = \{z : z \in \mathbf{C}; |z| < 1\}$.

Let $ST_\alpha(p, n)$ and $CT_\alpha(p, n)$ be the subclasses of $T(p, n)$ consisting

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of p-valently starlike functions of order α and p-valently convex functions of order α, respectively, that is,

$$ST_\alpha(p, n) = \left\{f \in T(p, n) : \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in E)\right\}$$

and

$$CT_\alpha(p, n) = \left\{f \in T(p, n) : \mathrm{Re}\left\{1 + \frac{zf''(z)}{f(z)}\right\} > \alpha \quad (z \in E)\right\},$$

where $0 \leq \alpha < p$. Yamakawa [2] easily derived the following:

$$f \in ST_\alpha(p, n) \Leftrightarrow \sum_{k=p+n}^{\infty} (k - \alpha)a_k \leq p - \alpha; \quad 0 \leq \alpha \leq p, \tag{1.2}$$

and

$$f \in CT_\alpha(p, n) \Leftrightarrow \sum_{k=p+n}^{\infty} k(k - \alpha)a_k \leq p(p - \alpha); \quad 0 \leq \alpha \leq p. \tag{1.3}$$

In this paper we consider new class $\mathcal{T}(p, n, \alpha)$ given by Yamakawa [2]. A function $f \in \mathcal{T}(p, n)$ is said to be a *member* of the class $\mathcal{T}(p, n, \alpha)$ if it satisfies the inequality:

$$\mathrm{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > \alpha \quad (z \in E),$$

where $0 \leq \alpha < \frac{1}{p}$.

We note that $\mathcal{T}(p, n, \alpha)$ is a subclass of $\mathcal{T}_0(p, n)$, since

$$\mathrm{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > \alpha \Rightarrow \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0; \quad 0 \leq \alpha < \frac{1}{p}.$$

For the class $\mathcal{T}(p, n, \alpha)$, Yamakawa [2] has given the following lemma.

Lemma 1.1. *Let* $f \in \mathcal{T}(p, n)$ *satisfy the inequality*

$$\sum_{k=p+n}^{\infty} (2k - pk\alpha - p)a_k \leq p(1 - \alpha p); \quad 0 \leq \alpha \leq \frac{1}{p}. \tag{1.4}$$

Then $f \in \mathcal{T}(p, n, \alpha)$.

However, the converse of the lemma is not true and Yamakawa [2] defined the subclass $\mathcal{A}(p, n, \alpha)$ of $\mathcal{T}(p, n, \alpha)$ consisting of functions f which satisfy (1.4). And let $\mathcal{B}(p, n, \alpha)$ denote the subclass of $\mathcal{T}(p, n)$ consisting of functions f such that $zf'(z) \in \mathcal{A}(p, n, \alpha)$.

Thus Yamakawa [2] gave the following:

Lemma 1.2. *A function f defined by* (1.1) *is in the class* $\mathcal{B}(p, n, \alpha)$ *if and only if*

$$\sum_{k=p+n}^{\infty} k(2k - pk\alpha - p)a_k \leq p^2(1 - \alpha p); \quad 0 \leq \alpha \leq \frac{1}{p}. \tag{1.5}$$

In view of (1.4) and (1.5), we introduce and study some properties and characteristics of the following general class $\mathcal{U}[p, n, \alpha, \beta]$ of function $f \in \mathcal{T}(p, n)$ which also satisfy the inequality:

$$\sum_{k=p+n}^{\infty} (2k - pk\alpha - p)\left(1 - \beta + \beta\left(\frac{k}{p}\right)\right)a_k \leq p(1 - \alpha p); \quad 0 \leq \beta \leq 1. \tag{1.6}$$

We can see easily

$$\mathcal{U}[p, n, \alpha, \beta] = (1 - \beta)\mathcal{A}(p, n, \alpha) + \beta\mathcal{B}(p, n, \alpha),$$

so that

$$\mathcal{U}[p, n, \alpha, 0] = \mathcal{A}(p, n, \alpha) \text{ and } \mathcal{U}[p, n, \alpha, 1] = \mathcal{B}(p, n, \alpha).$$

The main objective here is to give some properties involving the Hadamard products to the unified classes of $\mathcal{A}(p, n, \alpha)$ and $\mathcal{B}(p, n, \alpha)$ in a more general form $\mathcal{U}(p, n, \alpha, \beta)$. The idea is motivated from the work done by Srivastava et al. [1]. In [1], the authors gave results on distortion theorem and some characteristics on modified Hadamard products. In fact, the properties mentioned for unification of the classes $ST_\alpha(p, n)$ and $CT_\alpha(p, n)$ satisfying (1.2) and (1.3) respectively can be easily derived. We note that when $p = 1$ in the unification of classes $ST_\alpha(p, n)$ and $CT_\alpha(p, n)$, all the properties mentioned above reduce to [1].

2. Convolution Properties

Let the function f_m defined by

$$f_m(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,m} z^k, \quad m = (1, 2), \tag{2.7}$$

be in the class $\mathcal{T}(p, n)$, we denote by $(f_1 * f_2)(z)$ the convolution (or Hadamard product) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 * f_2)(z) := z^p - \sum_{k=p+n}^{\infty} a_{k,1} a_{k,2} z^k. \tag{2.8}$$

Theorem 2.1. *Let the functions f_m, for $m = (1, 2)$ defined by* (2.7) *be in class $\mathcal{U}[p, n, \alpha, \beta]$. Then*

$$(f_1 * f_2)(z) \in \mathcal{U}[p, n, \gamma, \beta],$$

where

$$\gamma = \frac{1}{p} - \frac{n(1 - p\alpha)^2}{[(1 - p\alpha)(p + n) + n]^2 [1 - \beta + \beta(p + n)/n] - p(p + n)(1 - p\alpha)^2}. \tag{2.9}$$

The result is sharp for functions f given by

$$f_m(z) = z^p - \frac{p(1 - p\alpha)}{[(1 - p\alpha)(p + n) + n][1 - \beta + \beta(p + n)/n]} z^{p+n}, \; m = (1, 2). \tag{2.10}$$

Proof. To prove Theorem 2.1, we must find the largest γ such that

$$\sum_{k=p+n}^{\infty} \frac{\left[(2k - \gamma pk - p)\left(1 - \beta + \beta\left(\frac{k}{p}\right)\right)\right]}{p(1 - \gamma p)} a_{k,1} a_{k,2} \leq 1,$$

for $f_m \in \mathcal{U}[p, n, \gamma, \beta]$, $(m = 1, 2)$. Since $f_m \in \mathcal{U}[p, n, \alpha, \beta]$, for $m = (1, 2)$, we have

$$\sum_{k=p+n}^{\infty} \frac{\left[(2k - \alpha pk - p)\left(1 - \beta + \beta\left(\frac{k}{p}\right)\right)\right]}{p(1 - \alpha p)} a_{k,i} \leq 1.$$

Therefore, by the Cauchy-Schwarz inequality, we get

$$\sum_{j=i+1}^{\infty} \frac{(2k-\alpha pk-p)\left(1-\beta+\beta\left(\frac{k}{p}\right)\right)}{p(1-p\alpha)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{2.11}$$

This implies that we need only show that

$$\frac{(2k-\gamma pk-p)\left(1-\beta+\beta\left(\frac{k}{p}\right)\right)}{(1-p\gamma)} \sqrt{a_{k,1}a_{k,2}}$$

$$\leq \frac{(2k-\alpha pk-p)\left(1-\beta+\beta\left(\frac{k}{p}\right)\right)}{(1-p\alpha)} a_{k,1}a_{k,2}, \quad (k \geq (p+n)),$$

or equivalently, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(1-p\gamma)(2k-\alpha pk-p)}{(1-p\alpha)(2k-\gamma pk-p)}, \quad (k \geq (p+n)).$$

Hence, by the inequality (2.11), it suffices to prove that

$$\frac{p(1-\alpha p)}{(2k-\alpha pk-p)\left(1-\beta+\beta\left(\frac{k}{p}\right)\right)} \leq \frac{(1-p\gamma)(2k-\alpha pk-p)}{(1-p\alpha)(2k-\gamma pk-p)}, (k \geq (p+n)). \tag{2.12}$$

It follows from (2.12), that

$$\gamma \leq \frac{1}{p} + \frac{(p-k)(1-p\alpha)^2}{(2k-\alpha pk-p)^2\left[1-\beta+\beta\left(\frac{k}{p}\right)\right]-pk(1-p\alpha)^2}, (k \geq (p+n)). \tag{2.13}$$

Now, defining the function $\tau(k)$ by

$$\tau(k) := \frac{1}{p} + \frac{(p-k)(1-p\alpha)^2}{(2k-\alpha pk-p)^2\left[1-\beta+\beta\left(\frac{k}{p}\right)\right]-pk(1-p\alpha)^2}, (k \geq (p+n)). \tag{2.14}$$

We see that $\tau(k)$ is an increasing function of $k = p+n$. Therefore, we conclude that

$$\gamma \leq \tau(p+n) := \frac{1}{p} - \frac{n(1-p\alpha)^2}{[(1-\alpha p)(p+n)+n]^2\left[1-\beta+\beta\left(\frac{p+n}{p}\right)\right]-p(p+n)(1-p\alpha)^2}. \tag{2.15}$$

The proof is complete.

Letting $\beta = 0$ and $\beta = 1$, we will find Corollary 2.2 and Corollary 2.3, respectively.

Corollary 2.2. *Let the functions* f_m, *for* $m = (1, 2)$ *defined by* (2.7) *be in class* $\mathcal{A}(p, n, \alpha)$. *Then*

$$(f_1 * f_2)(z) \in \mathcal{A}(p, n, \gamma),$$

where

$$\gamma = \frac{1}{p} - \frac{n(1-p\alpha)^2}{[(1-p\alpha)(p+n)+n]^2 - p(p+n)(1-p\alpha)^2}. \tag{2.16}$$

The result is sharp for functions f given by

$$f_m(z) = z^p - \frac{p(1-p\alpha)}{(1-p\alpha)(p+n)+n} z^{p+n}, \quad m = (1, 2). \tag{2.17}$$

Corollary 2.3. *Let the functions* f_m, *for* $m = (1, 2)$ *defined by* (2.7) *be in class* $\mathcal{B}(p, n, \alpha)$. *Then*

$$(f_1 * f_2)(z) \in \mathcal{B}(p, n, \gamma),$$

where

$$\gamma = \frac{1}{p} - \frac{n(1-p\alpha)^2}{[(1-p\alpha)(p+n)+n]^2\left(\frac{p+n}{p}\right) - p(p+n)(1-p\alpha)^2}. \tag{2.18}$$

The result is sharp for functions f given by

$$f_m(z) = z^p - \frac{p(1-p\alpha)}{((1-p\alpha)(p+n)+n)\left(\frac{p+n}{2}\right)} z^{p+n}, \quad m = (1, 2). \tag{2.19}$$

Theorem 2.4. *Let the functions* f_m, *for* $m = (1, 2)$ *defined by* (2.7) *be in class* $\mathcal{U}[p, n, \alpha, \beta]$. *Then the function* $h(z)$ *defined by*

$$h(z) := z^p - \sum_{k=p+n}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k, \tag{2.20}$$

belongs to the class $\mathcal{U}[p, n, \gamma, \beta]$, *where*

$$\gamma = \frac{1}{p} - \frac{2n(1-p\alpha)^2}{[(1-p\alpha)(p+n)+n]^2[1-\beta+\beta(p+n)/n] - 2p(p+n)(1-p\alpha)^2}. \tag{2.21}$$

The result is sharp for the functions

$$f_m(z) = z^p - \frac{p(1-p\alpha)}{[(1-p\alpha)(p+n)+n][1-\beta+\beta(p+n)/n]} z^{p+n}, \ m = (1, 2). \tag{2.22}$$

Proof. Noting that

$$\sum_{k=p+n}^{\infty} \frac{\{2k-\alpha pk-p\}^2\left(1-\beta+\beta\left(\frac{k}{p}\right)\right)^2}{p^2(1-\alpha p)^2} a_{k,m}^2$$

$$\leq \left[\sum_{k=p+n}^{\infty} \frac{\{2k-\alpha pk-p\}\left(1-\beta+\beta\left(\frac{k}{p}\right)\right)}{p(1-\alpha p)} a_{k,m}\right]^2 \leq 1,$$

for $f_m \in \mathcal{U}[p, n, \alpha, \beta]$, for $m = (1, 2)$, we have

$$\sum_{k=p+n}^{\infty} \frac{\{2k-\alpha pk-p\}^2\left(1-\beta+\beta\left(\frac{k}{p}\right)\right)^2}{2p^2(1-\alpha p)^2} [a_{k,1}^2 + a_{k,2}^2] \leq 1.$$

Therefore, we have to find the largest γ such that

$$\frac{\{2k-\gamma pk-p\}}{1-p\gamma} \leq \frac{\{2k-\alpha pk-p\}^2\left(1-\beta+\beta\left(\frac{k}{p}\right)\right)}{2p(1-p\alpha)^2}, \tag{2.23}$$

that is,

$$\gamma \leq \frac{1}{p} + \frac{2(p-k)(1-p\alpha)^2}{(2k-\alpha pk-p)^2\left[1-\beta+\beta\left(\frac{k}{p}\right)\right] - 2pk(1-p\alpha)^2}, \ (k \geq (p+n)). \tag{2.24}$$

Now, defining the function $\tau(k)$ by

$$\tau(k) := \frac{1}{p} + \frac{2(p-k)(1-p\alpha)^2}{(2k-\alpha pk-p)^2\left[1-\beta+\beta\left(\frac{k}{p}\right)\right] - 2pk(1-p\alpha)^2}, \ (k \geq (p+n)). \tag{2.25}$$

We see that $\tau(k)$ is an increasing function of k. Thus, we conclude that

$$\gamma \le \tau(p+n) := \frac{1}{p} - \frac{2n(1-p\alpha)^2}{[(1-\alpha p)(p+n)+n]^2\left[1-\beta+\beta\left(\frac{p+n}{p}\right)\right]-2p(p+n)(1-p\alpha)^2}. \tag{2.26}$$

The proof is completed.

By setting $\beta = 0$, we will arrive at the following corollary.

Corollary 2.5. *Let the functions* f_m, *for* $m = (1, 2)$ *defined by* (2.7) *be in class* $\mathcal{A}(p, n, \alpha)$. *Then the function* $h(z)$ *defined by* (2.20) *belongs to the class* $\mathcal{A}(p, n, \gamma)$, *where*

$$\gamma = \frac{1}{p} - \frac{2n(1-p\alpha)^2}{[(1-p\alpha)(p+n)+n]^2 - 2p(p+n)(1-p\alpha)^2}. \tag{2.27}$$

The result is sharp for the functions given by (2.17).

Letting $\beta = 1$, we will find a corollary as follows:

Corollary 2.5′. *Let the functions* f_m, *for* $m = (1, 2)$ *defined by* (2.7) *be in class* $\mathcal{B}(p, n, \alpha)$. *Then the function* $h(z)$ *defined by* (2.20) *belongs to the class* $\mathcal{B}(p, n, \gamma)$, *where*

$$\gamma = \frac{1}{p} - \frac{2n(1-p\alpha)^2}{[(1-p\alpha)(p+n)+n]^2\left(\frac{p+n}{p}\right) - 2p(p+n)(1-p\alpha)^2}. \tag{2.28}$$

The result is sharp for the functions given by (2.19).

3. Radii Convexity and Starlikeness

The radii of convexity for class $\mathcal{U}[p, n, \alpha, \beta]$ is given by the following theorem.

Theorem 3.6. *Let the functions f be in the class* $\mathcal{U}[p, n, \alpha, \beta]$. *Then the function f is p-valently convex in the disk* $|z| < r_1(p, n, \alpha, \delta)$, *where*

$$r_1(p, n, \alpha, \delta) = \inf_k \left\{\frac{(p-\delta)(2k-pk\alpha-p)\left(1-\beta+\beta\left(\frac{k}{p}\right)\right)}{k(1-p\alpha)(k-\delta)}\right\}^{\frac{1}{k-p}}. \tag{3.29}$$

Proof. It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)} - (p-1)\right| = \left|\frac{\sum_{k=p+n}^{\infty} k(k-\alpha)a_k z^{k-p}}{p - \sum_{k=p+n}^{\infty} ka_k z^{k-p}}\right|$$

$$\leq \frac{\sum_{k=p+n}^{\infty} k(k-\alpha)a_k |z|^{k-p}}{p - \sum_{k=p+n}^{\infty} ka_k |z|^{k-p}} \tag{3.30}$$

which implies that

$$(p-\delta) - \left|\frac{zf''(z)}{f'(z)} - (p-1)\right|$$

$$\geq (p-\delta) - \frac{\sum_{k=p+n}^{\infty} k(k-\alpha)a_k z^{k-p}}{p - \sum_{k=p+n}^{\infty} ka_k z^{k-p}}$$

$$\geq \frac{p(p-\delta) - \sum_{k=p+n}^{\infty} [k(p-\delta) + k(k-p)]a_k |z|^{k-p}}{p - \sum_{k=p+n}^{\infty} ka_k |z|^{k-p}}$$

$$\geq \frac{p(p-\delta) - \sum_{k=p+n}^{\infty} k(k-\delta)a_k |z|^{k-p}}{p - \sum_{k=p+n}^{\infty} ka_k |z|^{k-p}}. \tag{3.31}$$

Hence from (3.29), if

$$|z|^{k-p} \leq \frac{(p-\delta)}{k(k-\delta)} \frac{(2k - pk\alpha - p)\left(1 - \beta + \beta\left(\frac{k}{p}\right)\right)}{(1-p\alpha)}. \tag{3.32}$$

According to (1.6)

$$p(p-\delta) - \sum_{k=p+n}^{\infty} k(k-\delta)a_k |z|^{k-p} > p(p-\delta) - p(p-\delta) = 0. \tag{3.33}$$

Hence from (3.31), we obtain

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| < p - \delta.$$

Therefore

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0,$$

which shows that f is p-valently convex in the disk $|z| < r_1(p, n, \alpha, \delta)$.

By setting $\beta = 0$ and $\beta = 1$, we have Corollary 3.7 and Corollary 3.8, respectively.

Corollary 3.7. *Let the functions f be in the class* $\mathcal{A}(p, n, \alpha)$. *Then the function f is p-valently convex in the disk* $|z| < r_2(p, n, \alpha, \delta)$, *where*

$$r_2(p, n, \alpha, \delta) = \inf_k \left\{\frac{(p-\delta)(2k - pk\alpha - p)}{k(1 - p\alpha)(k - \delta)}\right\}^{\frac{1}{k-p}}. \tag{3.34}$$

Corollary 3.8. *Let the functions f be in the class* $\mathcal{B}(p, n, \alpha)$. *Then the function f is p-valently convex in the disk* $|z| < r_3(p, n, \alpha, \delta)$, *where*

$$r_3(p, n, \alpha, \delta) = \inf_k \left\{\frac{(p-\delta)(2k - pk\alpha - p)\left(\frac{k}{p}\right)}{k(1 - p\alpha)(k - \delta)}\right\}^{\frac{1}{k-p}}. \tag{3.35}$$

Theorem 3.9. *Let the functions f be in the class* $\mathcal{U}[p, n, \alpha, \beta]$. *Then the function f is p-valently starlike in the disk* $|z| < r_4(p, n, \alpha, \delta)$, *where*

$$r_4(p, n, \alpha, \delta) = \inf_k \left\{\frac{(p-\delta)(2k - pk\alpha - p)\left(1 - \beta + \beta\left(\frac{k}{p}\right)\right)}{p(1 - p\alpha)(k - \delta)}\right\}^{\frac{1}{k-p}}. \tag{3.36}$$

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \delta.$$

By using the similar method of Theorem 3.6 and (1.6), we will obtain (3.36).

Letting $\beta = 0$ and $\beta = 1$, we have Corollary 3.10 and Corollary 3.11, respectively.

Corollary 3.10. *Let the functions f be in the class* $\mathcal{A}(p, n, \alpha)$. *Then the function f is p-valently starlike in the disk* $|z| < r_5(p, n, \alpha, \delta)$, *where*

$$r_5(p, n, \alpha, \delta) = \inf_k \left\{ \frac{(p - \delta)(2k - pk\alpha - p)}{p(1 - p\alpha)(k - \delta)} \right\}^{\frac{1}{k-p}}. \tag{3.37}$$

Corollary 3.11. *Let the functions f be in the class* $\mathcal{B}(p, n, \alpha)$. *Then the function f is p-valently starlike in the disk* $|z| < r_6(p, n, \alpha, \delta)$, *where*

$$r_6(p, n, \alpha, \delta) = \inf_k \left\{ \frac{(p - \delta)(2k - pk\alpha - p)\left(\frac{k}{p}\right)}{p(1 - p\alpha)(k - \delta)} \right\}^{\frac{1}{k-p}}. \tag{3.38}$$

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References

[1] H. M. Srivastava, S. Owa, M. Obradovic and M. Nikic, A unified presentation of certain classes of starlike and convex functions with negative coefficients, Utilitas Math. 36 (1989), 107-113.

[2] R. Yamakawa, Certain subclasses of p-valently starlike functions with negative coefficients, Current Topics in Analytic Function Theory, H. M. Srivastava and S. Owa, eds., World Scientific Publishing Company, Singapore, New Jersey, London, Hong Kong, 1992, pp. 393-402.

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