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PRISONER'S DILEMMA POPULATION GAME WITH COOPERATIVE EQUILIBRIA

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Abstract

We propose an imitation protocol for 2-person symmetric population games, which leads to multiple stable equilibria in the aspiration based Prisoner's Dilemma (PD) game. The imitation protocol describes populations in which the aspiration levels of players depend on the current population state. Bifurcation of a unique stable cooperative equilibrium is discussed.

I. Introduction

Evolution of cooperation is one of the most important social phenomena studied extensively in social, economic, biological and other contexts. The qualitative measures of cooperation are important characteristics of social groups.

The PD game, one of the most popular games in social sciences, is studied as a paradigm for the evolution of cooperation, cf. for example [3, 8]. In the models of continuous systems of agents playing at each instant of time

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a one-stage PD, for example in the replicator dynamics, the unique asymptotic equilibrium state is defection. There are various ways out of the dilemma, leading to cooperation in the long run. In particular, various approaches to overcome the "dilemma of the PD" are provided by the evolutionary game theory. The dynamics of evolutionary games can be formally derived from the revision protocols, cf. for example [9, 11]. The revision protocol ρ_{ij} determines the probability of change from strategy *i* to *j* by an agent who plays strategy *i*. In particular, the revision protocol $\rho_{ij}(x, \Pi) = x_j(K - \Pi_i)$, where x is the state of the population, Π_i is the mean payoff of strategy *i*, *K* is a constant, referred to as the aspiration level, leads to the replicator dynamics [2, 9]. The protocol describes imitation driven by dissatisfaction: an *i* strategy (*i*-type) player who revises his strategy, compares its current payoff Π_i with the aspiration level *K* (assumed to be larger than the highest feasible payoff), and switches to a randomly chosen strategy *j* with probability proportional to $K - \Pi_i$.

In general, the aspiration levels may depend on the state of the system and the type of the player. Such an idea has been considered, in a different setting, for example in [7], where it has been shown that strategy and state dependent aspiration levels may lead to the increase of cooperation in the PD game. There can be a threshold of the state variable (frequency threshold), above which the players of a given type change their aspiration level. For example, the players who are in majority may have different aspiration levels from those in minority. The question arises whether a dynamics based on such dissatisfaction driven imitation with state dependent aspirations can lead to qualitative changes in the behavior of the population, comparing with behavior described by the standard replicator dynamics. For particular types of imitation protocols this question can be answered analytically. We prove, using elementary mathematical tools, that if players of different types have a preference towards being in majority, then the population which play the PD game can have two stable stationary states, in which both types of players are present.

We consider an infinite population of individuals, whose interactions are described by a PD game, with the revision protocol $\rho_{ij}(x, \Pi) = x_j[K_i(x) - \Pi_i]$, where the aspiration levels K_i depend on the state x of the population, $i = 1$ corresponds to cooperation, $i = 2$ to defection. We demonstrate that the resulting evolutionary dynamics admits multiple stable stationary states if, for a given player's type, the aspirations of majority significantly differ from that in minority. The difference will be modeled using the step functions and their smoothed approximations. In the case of smooth dependence of the aspiration levels on the state variable we discuss the bifurcation of a unique stable interior stationary state into two stable (and one unstable) stationary states.

In the next section, we introduce the model. Sections III and IV describe the populations with linear and nonlinear aspirations, respectively. In Section V, we discuss the results and indicate some extensions.

II. Model

We consider an infinite homogeneous population of agents who at each instant of time are randomly matched in pairs to play a 2-person symmetric PD game with the payoff matrix

$$\begin{array}{c|cc} & C & D \\ \hline C & R & S \\ D & T & P \end{array} \tag{1}$$

where $T > R > P > S$. We assume that the players who play strategy i, $i = 1, 2$, review their strategy according to the Poisson process with the arrival rate r_i. We model the corresponding stochastic processes as a deterministic flow ([11], [9]): $\dot{x}_i(t) = \sum_{j=1}^{2}[x_j r_j \rho_{ji} - x_i r_i \rho_{ij}]$, $i = 1, 2$, where x_j denotes the frequency of strategy j in the population. The function ρ_{ij}, referred to as the revision protocol, is proportional to the probability that an agent playing strategy i will switch to strategy j. Assuming r_i =

$const.$, $i = 1, 2,$ with $x_2 = 1 - x_1$ the rescaled dynamics reads

$$\dot{x}_1 = (1 - x_1)\rho_{21} - x_1\rho_{12}. \tag{2}$$

The state of the population is identified with the proportion x_1 of the agents who play the first (C) strategy. The initial state of the population is denoted $x_1(0) = x_{10}$.

We introduce the general revision protocol

$$\rho_{ij}(x, \Pi) = x_j r_{ij}(x_1, \Pi) \equiv x_j[K_i(x_i) - \Pi_i], \tag{3}$$

where $\Pi_1 = S + x_1(R - S), \Pi_2 = P + x_1(T - P),$ and the aspiration levels K_i, $i = 1, 2,$ depend on the current state of the population. The functions r_{ij} are referred to as the *conditional transition rates*, cf. [9]. The inequalities

$$r_{ij}(x_1, \Pi) > 0 \quad \forall x_1 \in (0, 1) \tag{4}$$

have to be satisfied to guarantee positivity of the revision protocols. The multiplier x_j in (3) indicates that the player chosen to revise his strategy picks up the strategy *j* of a randomly chosen partner. Inserting (3) into the dynamics (2), we obtain the population dynamics

$$\dot{x}_1 = x_1(1 - x_1)[K_2(1 - x_1) - K_1(x_1) + \Pi_1 - \Pi_2]. \tag{5}$$

In the next section, we discuss the stationary states of (5) for aspiration levels which depend linearly on the population state, and find sufficient conditions for the existence of a unique polymorphic equilibrium in populations playing the PD game. In the subsequent section, we show that if the players of each type have preferences for majority, described by the relevant aspiration levels, then two stable polymorphic equilibria are possible. For brevity, we introduce the notation:

the cases $\lim_{t\to\infty} x_1(t) = 1\ (0)$ are referred to as the full cooperation (full defection),

the case $\lim_{t\to\infty} x_1(t) = x_1^* \in (0, 1)$ corresponds to the partial cooperation.

III. Linear Aspirations

Let us assume that the aspirations K_i of i-strategists are linear functions of x_i:

$$K_i(x_i) = c_i + b_i x_i, \quad b_i, c_i \in R, \quad x_i \in (0, 1), \quad i = 1, 2. \tag{6}$$

With the notation $d := R - S + P - T - b_1 - b_2$, $e := c_2 - c_1 + b_2 + S - P$, the dynamics (5) reads: $\dot{x}_1 = x_1(1 - x_1)(dx_1 + e)$ which straightforwardly implies

Proposition 1.

(1) *If* $e > 0$ *and* $d + e > 0$, *then* $x_1^* = 1$ *is stable in* $(0, 1]$.

(2) *If* $e > 0$ *and* $d + e < 0$ $(e < 0$ *and* $d + e > 0)$, *then* $x_1^* = -\dfrac{e}{d} \in (0, 1)$ *is stable* (*unstable*) *in* (0, 1).

(3) *If* $e < 0$ *and* $d + e < 0$, *then* $x_1^* = 0$ *is stable in* $[0, 1)$.

In particular, in agreement with intuition, if $c_2 \gg c_1$, then full cooperation is the asymptotic limit for all initial data $x_{10} \neq 1$, cf. p.1. For intermediate values of $c_2 - c_1$ (obtained from p.2), the population tends to partial stable cooperation. The particular case of constant aspirations: $b_1 = b_2 = 0$ is reported in the Appendix, p.1. The linear aspirations do not admit multiple cooperation levels.

IV. Nonlinear Aspirations

In this section, we study populations for which the aspiration level of a given type of players who are in minority differs significantly from their aspiration level when they are in majority. To model such preferences, we choose step functions and their smooth approximations. We prove that such aspirations admit two stable interior stationary states of (5). We define

$$K_1(x_1) = h_1 \text{ for } x_1 \leq 0.5, \quad h_2 \text{ for } x_1 > 0.5, \tag{7}$$

$$K_2(x_2) = h_3 \text{ for } x_2 \le 0.5, \quad h_4 \text{ for } x_2 > 0.5, \tag{8}$$

where $h_i \in R^+$ are fixed constants, $x_2 = 1 - x_1$. We consider populations in which both C- and D-players prefer to be in majority, i.e., to play more frequent strategy. In order to model such preferences, we assume $h_1 > h_2$ and $h_3 > h_4$. For example, the assumption $h_1 > h_2$ implies that the aspiration K_1 is bigger for $x_1 \le 0.5$ (C-players in minority) than for $x_1 > 0.5$ (C-players in majority). In consequence, the transition from strategy $i = 1$ to $i = 2$ is faster when C-players are in minority: $x_1^l < 0.5 < x_1^r \Rightarrow r_{12}(x_1^l, \Pi) > r_{12}(x_1^r, \Pi)$. To see this, we calculate from (3) and (7),

$$r_{12}(x_1^l, \Pi) - r_{12}(x_1^r, \Pi) = h_1 - h_2 + (x_1^r - x_1^l)(R - S) > 0. \tag{9}$$

Simple calculation shows that for the PD game the positivity requirements (4) for the conditional transition rates hold if the following four inequalities are satisfied:

$$h_1 > S + \frac{1}{2}(R - S), \quad h_2 > R, \quad h_3 > T, \quad h_4 > P + \frac{1}{2}(T - P). \tag{10}$$

The dynamics (5) has the form

$$\dot{x}_1 = x_1(1 - x_1)G(x_1), \tag{11}$$

with the piecewise continuous function $(U := R - S + P - T \neq 0)$,

$$G(x_1) := \begin{cases} h_4 - h_1 + S - P + Ux_1, & 0 < x_1 \le 0.5, \\ h_3 - h_2 + S - P + Ux_1, & 0.5 < x_1 < 1. \end{cases} \tag{12}$$

The number of stationary points of (11) and their stability are determined by the limits of $G(x_1)$ at $x_1 = 0$, $x_1 = 0.5$, $x_1 = 1$: $e_1 := h_4 - h_1 + S - P$, $e_2 := h_4 - h_1 + S - P + \frac{1}{2}U$, $e_3 := h_3 - h_2 + S - P + \frac{1}{2}U$, $e_4 := h_3 - h_2 + R - T$.

In particular, the existence of two stable stationary points is proved by

Proposition 2. *If* $e_1 > 0$, $e_2 < 0$, $e_3 > 0$, $e_4 < 0$, *then there exist two* *locally stable stationary points of the dynamics* (11):

$$x_1^* = \frac{h_1 - h_4 + P - S}{U} \in (0,\ 0.5), \quad x_2^* = \frac{h_3 - h_2 + P - S}{U} \in (0.5,\ 1). \quad (13)$$

For example for the payoffs $[R, S, T, P] = [2, 0, 7, 1]$ and $[h_1, h_2, h_3, h_4] = [5, 4, 8, 7]$ (note $h_1 > h_2$, $h_3 > h_4$), there exist two locally stable equilibria $x_1^* = \frac{1}{4}$, $x_2^* = \frac{3}{4}$. Other examples of populations with stepwise aspirations, which satisfy the positivity conditions (10) and have two interior stationary states are given in the Appendix, p.2.

Discontinuity of aspirations is not necessary to obtain dynamics with multiple internal states. Let us consider the aspirations in the form of a hyperbolic tangent. We demonstrate a bifurcation of the unique interior equilibrium when the bifurcation parameter (the slope of the hyperbolic tangent) increases. We choose $K_i = a_i + b_i \tanh\left[c\left(x_i - \frac{1}{2}\right)\right]$, $i = 1, 2$, the PD payoffs $[R, S, T, P] = [2, 0, 7, 1]$ and $a_1 = \frac{9}{2}$, $a_2 = \frac{15}{2}$, $b_1 = b_2 = -\frac{1}{2\tanh\left(\frac{c}{2}\right)}$, with c determining the slope of the aspirations. For "very steep" slopes (e.g., $c = 10$) there are three interior equilibria, $x_1^* = 0.25$, $x_2^* = 0.50$, $x_3^* = 0.75$, the middle one unstable, the other two stable. For $c = 3.87$, we still have three equilibria $x_1^* \approx 0.46$, $x_2^* = 0.48$, $x_3^* \approx 0.54$ with the same stability properties. For $c = 3.86$, two stable equilibria merged with the unstable one, and the unique equilibrium is $x_1^* \approx 0.54$. Similar bifurcations exist for the "intermediate" case of piecewise linear aspirations. We omit details.

V. Discussion

In this note, we proposed an imitation protocol based on imitation driven by dissatisfaction, in which the aspiration levels of different types of the players depend on the state of the population. We proved, applying simple mathematical arguments, that if the aspiration of a given type of players when they are in minority differs significantly from their aspiration in majority, then the relevant evolutionary dynamics admits multiple interior stationary states. We focused on the PD game, however, the presented idea can be applied to any symmetric two-person game.

The method can be applied to other types of the revision protocols, for example $\rho_{ij}(x, \Pi) = x_j(K_i + \Pi_i)$, for which we obtain the dynamics (5). One could also consider state dependent conditional transition rates for asymmetric population games, in which the decision of changing strategy would depend on the states of both populations.

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VI. Appendix

1. For $b_1 = b_2 = 0,$ we obtain from (5), (6) the evolution equation

$$\dot{x}_1 = (T - R + S - P)x_1(1 - x_1)(x_1^* - x_1),$$

$$x_1^* := \frac{K + S - P}{T - R + S - P}, \quad K := c_2 - c_1. \tag{14}$$

For $T - R + S - P = 0,$ the evolution equation reads $\dot{x}_1 = x_1(1 - x_1)(K + S - P).$ $K > P - S$ $(K < P - S)$ implies full cooperation (full defection). In particular, for the benefit-cost formulation of the PD game: $T = b,$ $R = b - c,$ $P = 0,$ $S = -c,$ the dynamics (5) reads $\dot{x}_1 = x_1(1 - x_1)(K - c),$ with no interior equilibria.

For $K = 0,$ i.e., when the aspirations of both types of players are the same, (14) is the standard replicator dynamics, with full defection as the unique stationary state for all $x_{10} < 1.$ For $K \neq 0,$ the asymptotic behavior of the population depends on the relation between $T - R,$ $P - S$ and K:

A. For $K \geq \max\{T - R, P - S\},$ the population converges to full cooperation $\forall x_{10} \in (0, 1].$

B. For $\max\{T - R, P - S\} > K > \min\{T - R, P - S\},$ there exists the polymorphic equilibrium $x_1^* \in (0, 1),$ globally stable for $T - R > P - S$ and unstable for $T - R < P - S.$

C. For $K \leq \min\{T - R, P - S\},$ the population converges to full defection $\forall x_{10} \in [0, 1).$

2. Examples of two interior equilibria for the step aspirations [with discontinuity at $t \in (0, 1)$].

a. $h_1 > h_2, h_3 < h_4$: $t \approx 0.368$, $[R, S, T, P] = [3, 1, 7, 2]$, $[h_1, h_2, h_3, h_4] = [7, 5, 8, 9]$. The stationary states $x_1^* = \frac{1}{3}$, $x_2^* = \frac{2}{3}$ are stable.

b. $h_1 > h_2, h_3 < h_4$: $t \approx 0.605$, $[R, S, T, P] = [6, 1, 7, 5]$, $[h_1, h_2, h_3, h_4] = [5, 7, 9, 8]$. The stationary states $x_1^* = \frac{1}{3}$, $x_2^* = \frac{2}{3}$ are unstable. Note that due to the discontinuity of the rhs of (11), a numerical solution with initial state between the unstable stationary states would oscillate around the threshold t.

c. $h_1 > h_2, h_3 < h_4$: $t \approx 0.386$, $[R, S, T, P] = [3, 1, 7, 2]$, $[h_1, h_2, h_3, h_4] = [2, 5, 8, 4]$. The stationary states $x_1^* = \frac{1}{3}$, $x_2^* = \frac{2}{3}$ are unstable.