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ON THE HAUSDORFF DIMENSION OF THE GRAPH OF THE WEIERSTRASS FUNCTION

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Abstract

It is still an open problem whether or not for λ an integer greater than 1 and $0 < \alpha < 1$, the Hausdorff dimension of the graph of the Weierstrass function $W(x) = \sum_{i=1}^{\infty} \lambda^{-i\alpha} \sin \lambda^i \pi x$ equals to $2 - \alpha$. This paper provides a partial solution of the open problem, i.e., it is shown that the Hausdorff dimension of the graph of Weierstrass function equals to $2 - \alpha$ for large integers λ. Moreover, our proof is based on the method, it is called power law combining $\theta - \tau$ technique. This method may be used to treated some non-linear problem.

A. Introduction

It is an open question that for $\lambda > 1$, $0 < \alpha < 1$, whether or not the Hausdorff dimension of the graph of Weierstrass function

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$$W(x) = \sum_{i=1}^{\infty} \lambda^{-i\alpha} \sin \lambda^{i} \pi x, \quad x \in R \tag{1.1}$$

equals to $2 - \alpha$ [3, p. 649]. There are many attempts to solve the problem stated in (1.1). We list some of them, Berry and Lewls [1], Kaplan et al. [5], Mauldin and Williams [7], Mandelbrot [6], Hu and Lau [4], Hua [2], Sun and Wen [8]. But all of these works do not exactly solve this problem. The goal of present paper is to show that the Hausdorff dimension of the graph of Weierstrass function equals to $2 - \alpha$ for large integer λ.

B. Preliminary

B.1. Notation and terminology

In the following, we will state the notation and terminology.

Unless explicitly stated otherwise, throughout present paper, let λ, i, j, k, l, m, n be positive integers, θ and τ be real numbers with $-1 \leq \theta \leq 1$ and $0 \leq \tau \leq 1$. Value θ and τ may change at each step. Different fixed values of θ and τ will be denoted as $\theta, \theta', \theta'', \ldots$ and $\tau, \tau', \tau'', \ldots$. Special fixed values of θ and τ will be denoted as θ_i, τ_i.

Notation can be adjusted by stating, fox example,

$$\theta' \sin(ax^n \pm b) + \theta'' \cos(cx^m \pm d) = 2\theta.$$

Of cause, we do not know the exact value 2θ, but we do know the exact bound of 2θ. By using this technique, we can track the magnitude exactly and simplified expression. Thus, two conflicting goads, accuracy and simplicity, can be achieved by wisely using this technique. We call this method $\theta - \tau$ *technique*.

B.2. $\theta - \tau$ technique combining power law

We will make the following assumption on the power law.

Let $0 < \alpha < \beta$ and let a, b be real numbers, note that

$$a\lambda^{\alpha} + b\lambda^{\beta} = (1 + o(\lambda))a\lambda^{\alpha} \text{ as } \lambda \to \infty.$$

This means the sum $a\lambda^{\alpha} + b\lambda^{\beta}$ merges into $a\lambda^{\alpha}$. For simplified expression, we call this *property power law*. In the following, we will explain the method of $\theta - \tau$ technique combining power law.

We want to solve following complicated equation:

$$\left(\cos\frac{2k'\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha} - 1} + \frac{(k + k')\pi}{\lambda}\right)\frac{2(k' - k)\pi}{\lambda}$$

$$+ \frac{\sin\frac{2k'\pi}{\lambda}}{2}\left(\frac{2(k' - k)}{\lambda}\right)^2 - \frac{\tau}{3(\lambda^{3-\alpha} - 1)}\left(\frac{2(k' - k)\pi}{\lambda}\right)^3 = \lambda^{-\alpha},$$

where $0 \le k < k' \le \lambda$. By $\theta - \tau$ technique,

$$\frac{(k' + k)\pi}{\lambda^2(\lambda^{2-\alpha} - 1)} = \frac{2\theta\pi}{\lambda(\lambda^{2-\alpha} - 1)},$$

$$\frac{\tau}{3(\lambda^{3-\alpha} - 1)}\left(\frac{2(k' - k)\pi}{\lambda}\right)^3 = \frac{(2\pi)^2\tau'}{3(\lambda^{3-\alpha} - 1)}\frac{2(k' - k)\pi}{\lambda}.$$

We label the left hand side of above equation by I:

$$I = \left(\cos\frac{2k'\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha} - 1} + \frac{2\theta\pi}{\lambda(\lambda^{2-\alpha} - 1)}\right)\frac{2(k' - k)\pi}{\lambda}$$

$$+ \frac{(2\pi)^2\tau'}{3(\lambda^{3-\alpha} - 1)}\frac{2(k' - k)\pi}{\lambda} + \frac{\sin\frac{2k'\pi}{\lambda}}{2}\left(\frac{2(k' - k)\pi}{\lambda}\right)^2.$$

By power law, we have

$$I = \left(\cos\frac{2k'\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)\frac{2(k' - k)\pi}{\lambda} + \frac{\sin\frac{2k'\pi}{\lambda}}{2}\left(\frac{2(k' - k)\pi}{\lambda}\right)^2.$$

Thus, original complicated equation almost equivalent to following simpler equation for large integer λ:

$$\left(\cos\frac{2k'\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)\frac{2(k' - k)\pi}{\lambda} + \frac{\sin\frac{2k'\pi}{\lambda}}{2}\left(\frac{2(k' - k)\pi}{\lambda}\right)^2 = \lambda^{-\alpha}.$$

B.3. A variant of the net measure

We now state the variant of net measure. The net measure is equate to the Hausdorff measure and behaves much more convenient for our purpose. It is worth mentioning that Theorem 5.1 of [3] continuous to hold, even though Falconer has

$$\lambda = 2.$$

In the following, we now give the definition of measure $\tilde{\mu}(E)$. Let N be the collection of all 2-dimensional half-open λ squares S, that is,

$$S = [\lambda^{-k}m_1, \lambda^{-k}(m_1 + 1)] \times [\lambda^{-k}m_2, \lambda^{-k}(m_2 + 1)],$$

where k is a non-negative integer and m_1, m_2 are integers. A square S is called a kth *square* if its length equals to λ^{-k}. Let N_k be the collection of all 2-dimensional half-open kth squares S. A collection A of squares is called an $(n, n+m)$ *set* if $A \subset \bigcup_{j=n}^{n+m} N_j$. Let $E \subset R^2$ and $E \subset \bigcup_{S_i \in A} S_i$, we will call A an $(n, n+m)$ *cover* of E. We now define

$$\tilde{\mu}^s_{n, n+m}(E) = \inf\left\{\sum |S_i|^s : S_i \in A \text{ and } A \text{ is an } (n, n+m) \text{ cover of } E\right\},$$

where $|S_i|$ denote the length of square S_i. Furthermore, we define

$$\tilde{\mu}^s(E) = \lim_{n\to\infty} \lim_{m\to\infty} \tilde{\mu}^s_{n, n+m}(E). \tag{2.1}$$

In the following, we will prove a proposition, that is,

$$\mu^s(C) = \tilde{\mu}^s(C), \; C \text{ is a compact set.}$$

In order to prove the proposition, we need the following definition. The cover A of C is called a λ^{-n} -*cover* of C, if A is a collection of k-squares, where $k \geq n$:

$$\tilde{\mu}^{2-\alpha}_{\lambda^{-n}}(C) = \inf\left\{\sum |S_i|^{2-\alpha} : S_i \in A, A \text{ is a collection of } k\text{-squares, where } k \geq n\right\}.$$

Proposition 3.1. *Let C be a compact set. Then*

$$\mu^s(C) = \tilde{\mu}^s(C).$$

Proof. We easily see that any $(n,\ n+m)$ cover of C is a λ^{-n} -cover of C. This implies

$$\tilde{\mu}^{2-\alpha}_{n,\,n+m}(C) \geq \tilde{\mu}^{2-\alpha}_{\lambda^{-n}}(C) \text{ for any } m \geq 0.$$

Therefore, we obtain

$$\lim_{m\to\infty} \tilde{\mu}^{2-\alpha}_{n,\,n+m}(C) \geq \tilde{\mu}^{2-\alpha}_{\lambda^{-n}}(C). \tag{2.2}$$

We will show $\lim\limits_{m\to\infty} \tilde{\mu}^{2-\alpha}_{n,\,n+m}(C) \leq \tilde{\mu}^{2-\alpha}_{\lambda^{-n}}(C)$. Given any $\varepsilon > 0$, by definition of $\tilde{\mu}^{2-\alpha}_{\lambda^{-n}}(C)$, there is a λ^{-n} -cover of C such that

$$\sum_{S_i \in A} |\, S_i \,|^{2-\alpha} < \mu^{2-\alpha}_{\lambda^{-n}}(C) + \varepsilon.$$

Note that S-dimension net measure of the boundary of a square equals to 0 for $S > 1$. Hence, without loss of generality, we may suppose A is an open cover. Then, by Heine-borel theorem, there is a finite subset B of A which covers C, that is,

$$\sum_{S_i \in B} |\, S_i \,|^{2-\alpha} \leq \sum_{S_i \in A} |\, S_i \,|^{2-\alpha}.$$

Note that B is a $(n_\varepsilon,\ n_\varepsilon + m_\varepsilon)$ cover of C, we have

$$\tilde{\mu}^{2-\alpha}_{n_\varepsilon,\ n_\varepsilon + m_\varepsilon}(C) \leq \mu^{2-\alpha}_{\lambda^{-n}}(C) + \varepsilon.$$

Clearly, we have

$$\lim_{m\to\infty} \tilde{\mu}^{2-\alpha}_{n_\varepsilon}(C) \leq \mu^{2-\alpha}_{\lambda^{-n}}(C) + \varepsilon.$$

Since this is true for each ε, so that

$$\lim_{m\to\infty} \tilde{\mu}^{2-\alpha}_{n_\varepsilon,\, m}(C) \leq \mu^{2-\alpha}_{\lambda^{-n}}(C). \tag{2.3}$$

Combining (2.2) and (2.3), we have

$$\lim_{m\to\infty} \tilde{\mu}^{2-\alpha}_{n_\varepsilon,\, m}(C) = \mu^{2-\alpha}_{\lambda^{-n}}(C).$$

For each δ in interval $\left(0, \frac{1}{\lambda}\right]$, there is an uniquely n_δ such that $\frac{1}{\lambda^{n_\delta}} \le \delta < \frac{1}{\lambda^{(n_\delta - 1)}}$,

so

$$\lim_{\delta \to 0} \mu_\delta^{2-\alpha}(C) = \lim_{n\to\infty} \lim_{m\to\infty} \tilde{\mu}_{n,m}^{2-\alpha}(C).$$

This proves our proposition.

B.4. Three geometry properties

In the following, we specify three geometry properties:

B.4.a. Skelton of the graph of function $W(x)$

We note that $W_n\left(\frac{j}{\lambda^n}\right) = W_{n+1}\left(\frac{j}{\lambda^n}\right) = \cdots$. Consequently, we deduce

$$W\left(\frac{j}{\lambda^n}\right) = W_n\left(\frac{j}{\lambda^n}\right).$$

In other words, the partial sum $W_n\left(\frac{j}{\lambda^n}\right)$ is exact value of $W(x)$. Let us call the point $\left(\frac{j}{\lambda^n}, W_n\left(\frac{j}{\lambda^n}\right)\right)$ a *fixed point*. The collection of all fixed points $\left(\frac{j}{\lambda^n}, W_n\left(\frac{j}{\lambda^n}\right)\right)_{n=1,2,\ldots;\ j=0,1,2,\ldots,\lambda^n-1}$ forms a skeleton of the graph of function $W(x)$, since the function $W(x)$ is continuous and the length of period $\frac{2}{\lambda^n} \to 0$ as $n \to \infty$. Moreover, the collection of fixed points has regular configuration.

The collection of abscissa of fixed points forms a net in interval $[0, 2]$. First, we divide the interval $[0, 2]$ into λ equal subintervals

$$\left[\frac{j}{\lambda} + \frac{2k}{\lambda^2}, \frac{j}{\lambda} + \frac{2(k+1)}{\lambda^2}\right]_{k=0,1,2,\ldots,\lambda-1}.$$

By repeating above partition, we get a net $\mu\left\{\left[\frac{2j}{\lambda^n}, \frac{2(j+1)}{\lambda^n}\right]_{n=1,2,\ldots;\ j=0,1,2,\ldots,\lambda^n-1}\right\}$.

B.4.b. The sine-like curve

Below we present the shape of the graph of function $W_n(x)$ based on $\left[\frac{2j}{\lambda^n}, \frac{2(j+1)}{\lambda^n}\right]$. We easily shift the graph of $W_n(x)$ based on $\left[\frac{2j}{\lambda^n}, \frac{2(j+1)}{\lambda^n}\right]$ onto the graph of $W_n(x)$ based on $\left[0, \frac{2}{\lambda^n}\right]$. Consequently, we get recent function $g_n(x)$:

$$g_n(x) = \left(W_{n-1}\left(\frac{2j}{\lambda^n} + x\right) - W_{n-1}\left(\frac{2j}{\lambda^n}\right)\right) + \lambda^{-n\alpha} \sin \lambda^n \pi x.$$

We label first term of $g_n(x)$ by I:

$$\begin{aligned} I &= \sum_{i=1}^{n-1} \lambda^{-i\alpha}\left(\sin \lambda^i \pi\left(\frac{2j}{\lambda^n} + x\right) - \sin \lambda^i \pi \frac{2j}{\lambda^n}\right) \\ &= 2\sum_{i=1}^{n-1} \lambda^{-i\alpha} \sin \frac{\lambda^i \pi x}{2} \cos \lambda^i \pi\left(\frac{2j}{\lambda^n} + \frac{x}{2}\right). \end{aligned}$$

Next, we note $0 \le x \le \frac{2}{\lambda^n}$, we see

$$|\, I \,| \le \sum_{i=1}^{n-1} \pi x \lambda^{i(1-\alpha)} \le \frac{\pi x \lambda^n \lambda^{-n\alpha}}{\lambda^{1-\alpha} - 1}.$$

Therefore, we can write

$$g_n(x) = \frac{\theta \pi x \lambda^n}{\lambda^{1-\alpha} - 1} \lambda^{-n\alpha} + \sin \pi x \lambda^n \lambda^{-n\alpha}.$$

Since $0 \le x \le \frac{2}{\lambda^n}$, we obtain

$$g_n(x) = \frac{\theta x}{\lambda^{1-\alpha} - 1} \lambda^{-n\alpha} + \sin x \lambda^{-n\alpha}, \text{ where } 0 \le x \le 2\pi.$$

Thus, the second term is much larger than first term for large λ. Since the second term dominates function $g_n(x)$ and shift does not change the shape of graph of function $W_n(x)$, so all graphs of $W_n(x)$ based on $\left[\frac{2j}{\lambda^n}, \frac{2(j+1)}{\lambda^n}\right]$ are more or less slightly deviation from the position of graph of $\lambda^{-n\alpha} \sin \lambda^n \pi x + W_{n-1}\left(\frac{2j}{\lambda^n}\right)$ on each interval $\left[\frac{2j}{\lambda^n}, \frac{2(j+1)}{\lambda^n}\right]$. In this sense, we call the graph of $W_n(x)$ based on $\left[\frac{2j}{\lambda^n}, \frac{2(j+1)}{\lambda^n}\right]$ *lever n sine-like curve*.

Next, we want to show that different lever sine-like curves are quasi-similar. We consider the sequence of function $\lambda^{-n\alpha} \sin \lambda^n \pi x$, $n = 1, 2, \ldots$. The ratio $r_n = \frac{h_n}{l_n}$ decides the shape of function $\lambda^{-n\alpha} \sin \lambda^n \pi x$, where $h_n(l_n)$ is the height (reps; length) of function $\lambda^{-n\alpha} \sin \lambda^n \pi x$. We note that $\frac{r_n}{r_{n+1}} = \frac{\frac{\lambda^{-n\alpha}}{\lambda^{-n}}}{\frac{\lambda^{-(n+1)\alpha}}{\lambda^{-(n+1)}}} = \lambda^{-(1-\alpha)}$, $n = 1, 2, \ldots$. Thus, we conclude that different lever sine-like curves are quasi-similar.

B.4.c. Quasi-self similar property

Below we present the construction of the graph of $W_{n+1}(x)$ based on $\left[\frac{2j}{\lambda^n}, \frac{2(j+1)}{\lambda^n}\right]$ from the graph of $W_n(x)$ based on $\left[\frac{2j}{\lambda^n}, \frac{2(j+1)}{\lambda^n}\right]$.

The graph of $W_n(x)$ based on $\left[\frac{2j}{\lambda^n}, \frac{2(j+1)}{\lambda^n}\right]$ is a lever n sine-like curve (see, e.g., B.4b) and the graph of $W_{n+1}(x)$ on each subinterval $\left[\frac{2j}{\lambda^n} + \frac{2k}{\lambda^{n+1}}, \frac{2j}{\lambda^n} + \frac{2(k+1)}{\lambda^{n+1}}\right]_{k=0,1,2,\ldots,\frac{\lambda}{2}}$ is a lever $n+1$ sine-like curve. Thus, the graph of $W_{n+1}(x)$ based on $\left[\frac{2j}{\lambda^n}, \frac{2(j+1)}{\lambda^n}\right]$ is to replace lever $(n+1)$ sine-like curve on the, respectively, arc of a lever n sine-like curve. This yields the construction of the graph of $W(x)$ is to repeat above proceed again and again. Therefore, the graph of $W(x)$ builds up by pieces quasi-similar to the entire set but on a smaller scale. Here, we note that same lever sine-like curves have almost same shape and differential lever sine-like curves are quasi-similar but in different scales.

In this way, we describe the shape of $W(x)$ not only in macro-scale, but also in micro-scale, we call this property *quasi-self similar*.

The construction of lever $(n+1)$ sine-like curve based on $\left[\frac{2j}{\lambda^n}, \frac{2(j+1)}{\lambda^n}\right]$ may view as a movement of lever $(n+1)$ sine-like curve; that is, the lever $(n+1)$ sine-like curve moves on the fixed points $\left\{\left(\frac{2j}{\lambda^n} + \frac{2k}{\lambda^{n+1}}, W_n\left(\frac{2j}{\lambda^n} + \frac{2k}{\lambda^{n+1}}\right)\right)\right\}_{k=0,1,\ldots,\lambda}$. A vertical change of $d_{j,k} = W_n\left(\frac{2j}{\lambda^n} + \frac{2(k+1)}{\lambda^{n+1}}\right) - W_n\left(\frac{2j}{\lambda^n} + \frac{2k}{\lambda^{n+1}}\right)$ units corresponds to a horizontal change of $dx = \frac{2}{\lambda^{n+1}}$ units as lever $(n+1)$ sine-like curve moving from left to right along *x*-axis. We call $d_{j,k}$ *k*th pace in *j*th *lever n* sin*e-like curve*. The *k*th pace $d_{j,k}$ measures the vertical increment in *k*th moving. Note that $d_{j,k} = \left(\frac{\theta_0}{\lambda^{1-\alpha}-1} + \cos\frac{2k'\pi}{\lambda}\right)\frac{2\pi}{\lambda}\lambda^{-n\alpha} \ll \lambda^{-(n+1)\alpha}$, which will be shown in Lemma 3.2. An immediate consequence is that $y = h$ (horizontal line) intersects in nearly mid-point of $S_{n+1,k}$ and height $S_{n+1,k}$ almost equals to $2\lambda^{-(n+1)\alpha}$. This yields the lever $(n+1)$ sine-like curves which move up and down not too far from $S_{n+1,k}$ position will intersect with $y = h$ either. In the following, we introduce the concept of intersection number for quantitative description of distribution of lever $(n+1)$ sine-like curves in a lever *n* sine-like curve.

Remark. In some scene, the shape of the graph of $W_{n+1}(x)$ based on $\left[\frac{2j}{\lambda^n}, \frac{2(j+1)}{\lambda^n}\right]$ looks like a saw-toothed sine curve.

B.4.d. Intersection number

We now give the definition of intersection number. Let $y = h$ be a horizontal line which intersects with the lever *n* sine-like curve, the intersection number $I_{n+1}(h)$ is defined as the number of lever $(n+1)$ sine-like curves which intersect with the line $y = h$. In addition, let $y = h$ be a horizontal line with

$$W_n\left(\frac{2j}{\lambda^n}+\frac{2k'\pi}{\lambda^{n+1}}\right)\le h < W_n\left(\frac{2j}{\lambda^n}+\frac{2k'\pi}{\lambda^{n+1}}\right).$$

We call 2k′ *integer phase of horizontal line* $y = h$. In above case, the intersection number $I_{n+1}(h)$ will be denoted by $I_{n+1}\left(\frac{2k'}{\lambda}\right)$. In the following, we will characterize the intersection number $I_{n+1}\left(\frac{2k'\pi}{\lambda}\right)$ in two cases.

The graph of $W_n(x)$ based on $\left[\frac{2j}{\lambda^n}, \frac{2j+1}{\lambda^n}\right]$ is approximately symmetric with respect to the vertical line $x = \frac{2j+\frac{1}{2}}{\lambda^n}$. In the right hand side, the situation is similar by symmetry of the graph. Therefore, in general case, collection of lever $(n+1)$ sine-like curves which intersect horizontal line $y = h$ forms two groups of four parts. Two graphs of lever $(n+1)$ sine-like curves can be visualized as two stairs. First, we roughly estimate the intersection number. We consider left hand side stair first. By projecting the fixed points

$$\left(\frac{2j}{\lambda^n}+\frac{2k}{\lambda^{n+1}}, W_n\left(\frac{2j}{\lambda^n}+\frac{2k}{\lambda^{n+1}}\right)\right)_{k=\dots,k'-1,k',k'+1,\dots}$$

on a vertical line, it is not hard to verify that the sequence $\{d_{j,k}\}_{k=\dots,k'-1,k',k'+1,\dots}$ is an approximately arithmetic sequence and $d_{j,k}$ is average value of this sequence. The situation in right hand side is similar. Thus, we conclude that estimate of $I_{n+1}(h)$ is two time of ratio $\frac{2\lambda^{-(n+1)\alpha}}{d_{j,k}}$, i.e., $\frac{4\lambda^{-(n+1)\alpha}}{d_{j,k}}$. By Lemma 3.2, we have

$$\frac{4\lambda^{-(n+1)\alpha}}{d_{j,k}} = \frac{2\lambda^{1-\alpha}}{\pi\left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)}.$$

In bottom, $y = h$ may not intersect with lever $(n+1)$ sine-like curves fully, the intersection number is less than in general case, so the estimate keeps true in weakly mean. The situation will roughly change as the line $y = h$ moves up to top. We see that two stairs become closer as h increases. When line $y = h_0$ in position below the top in a distance, then two stairs meet together. We claim that $I_{n+1}(h_0)$ is the maximum intersection number. First, we consider the case $h \le h_0$, by monotonously, we have $I_{n+1}(h) \le I_{n+1}(h_0)$. Then for the case $h_0 < h$ and $y = h$ below the top, in the case lever $(n+1)$ sine-like curves which interest with $y = h$ will intersect $y = h_0$ so that $I_{n+1}(h) \le I_{n+1}(h_0)$. This observation is a key for solving the open problem.

We now explain how to precisely estimate the intersection number. In above estimate, we use symmetry of the graph of $W_n(x)$ based on $\left[\frac{2j}{\lambda^n}, \frac{2j+1}{\lambda^n}\right]$. In fact, $W_n(x) = W_{n-1}(x) + \lambda^{-n\alpha} \sin \lambda^n \pi x$ is not symmetry with line $x = \frac{2j + \frac{1}{2}}{\lambda^n}$. We see

$$
\begin{aligned}
& w_n\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right) - w_n\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2k'}{\lambda^{n+1}}\right) \\
&= w_{n-1}\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2k'}{\lambda^{n+1}}\right) - w_{n-1}\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2k'}{\lambda^{n+1}}\right).
\end{aligned}
$$

In general, we have $w_{n-1}\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right) - w_{n-1}\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2k'}{\lambda^{n+1}}\right) \ne 0$. By property two, $\lambda^{-n\alpha} \sin \lambda^n \pi x$ is main term and $w_{n-1}(x)$ is small term which may view as an impact to a symmetric function. The impact makes a deviation from normal position. Now we suppose that $w_n\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right) = w_n\left(\frac{2j}{\lambda^n} + \frac{\lambda - k_1'}{\lambda^{n+1}}\right)$. Then $\frac{2(k' - k_1')}{\lambda}$ measures the deviation. For precisely estimating intersection number, we have to compute the deviation number $\frac{2(k' - k_1')}{\lambda}$ precisely.

C. Main Results

C.1. The statement of the main results

We now present Lemma 3.2 which provides base formula for following results. In the following, we will also characterize of the shape of $W_n(x)$ based on

$\left[\frac{2j}{\lambda^n}, \frac{2j+1}{\lambda^n}\right]$. Let us present a theorem about deviation number which may be sharp remainder estimates. The concept of intersection number is the key to solve this open problem. We also obtain following theorem which describes the distribution of intersection number. Let us state our main results about intersection number.

We now give some lemmas which give to compute the minimum cover of the graph $\Gamma_w \cap R_n$. Above formulas on form and content are alike. Since each n-square S in some sine-like curve, we may view first form formula as a global property and second formulas as a local property. This means that the global property and local property are comparable. This result may be extended, but first we have to show a proposition.

We begin by setting $r_1 = \frac{2}{\pi}\arccos\frac{2}{\pi} + 1 - \sqrt{1-\left(\frac{2}{\pi}\right)^2}$. Assume that r_{n-1} is constructed, we define $r_n = \frac{2r_{n-1}}{\pi}\arccos\frac{2r_{n-1}}{\pi} + 1 - \sqrt{1-\left(\frac{2r_{n-1}}{\pi}\right)^2}$, $n = 2, 3, \ldots$, we call r_n *contraction coefficient*. It is easy to show $\lim\limits_{n\to\infty} r_n = 0$.

We now can state the extension in the general setting. Finely, Theorem 3.13 solves the open problem.

C.2. Proofs of main results

C.2.a. Lemma 3.2. *Let* $W(x) = \sum_{i=1}^{\infty} \lambda^{-i\alpha} \sin \lambda^i \pi x$ *be the Weierstrass function and let* $0 \le k < k' < k'' \le \lambda$. *Then for large* λ, *we have*

(a)

$$w\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right) - w\left(\frac{2j}{\lambda^n} + \frac{2k}{\lambda^{n+1}}\right)$$

$$= \left[\left(\cos\frac{2k'\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha}-1}\right)\frac{2(k'-k)\pi}{\lambda} - \frac{\sin\frac{2k'}{\lambda}\pi}{2}\left(\frac{2(k'-k)\pi}{\lambda}\right)^2\right]\lambda^{-n\alpha}, \quad (3.1)$$

(b)

$$W\left(\frac{2j}{\lambda^n}+\frac{2k'}{\lambda^{n+1}}\right)-W\left(\frac{2j}{\lambda^n}+\frac{2k}{\lambda^{n+1}}\right)$$

$$=\sum_{i=1}^{\infty}\lambda^{-i\alpha}\left[\sin\lambda^i\pi\left(\frac{2j}{\lambda^n}+\frac{2k'\pi}{\lambda^{n+1}}\right)-\sin\lambda^i\pi\left(\frac{2j}{\lambda^n}+\frac{2k\pi}{\lambda^{n+1}}\right)\right]$$

$$+\lambda^{-n\alpha}\left(\sin\frac{2k'\pi}{\lambda}-\sin\frac{2k\pi}{\lambda}\right). \tag{3.2}$$

Proof. We may write

$$W\left(\frac{2j}{\lambda^n}+\frac{2k'}{\lambda^{n+1}}\right)-W\left(\frac{2j}{\lambda^n}+\frac{2k}{\lambda^{n+1}}\right)$$

$$=\sum_{i=1}^{\infty}\lambda^{-i\alpha}\left[\sin\lambda^i\pi\left(\frac{2j}{\lambda^n}+\frac{2k'\pi}{\lambda^{n+1}}\right)-\sin\lambda^i\pi\left(\frac{2j}{\lambda^n}+\frac{2k\pi}{\lambda^{n+1}}\right)\right]$$

$$+\lambda^{-n\alpha}\left(\sin\frac{2k'\pi}{\lambda}-\sin\frac{2k\pi}{\lambda}\right)$$

$$=A_{k',k}+B_{k',k}. \tag{3.3}$$

First, we estimate $A_{k',k}$,

$$A_{k',k}=\sum_{i=1}^{\infty}\lambda^{-i\alpha}\left[\sin\lambda^i\pi\left(\frac{2j}{\lambda^n}+\frac{2k'\pi}{\lambda^{n+1}}\right)-\sin\lambda^i\pi\left(\frac{2j}{\lambda^n}+\frac{2k\pi}{\lambda^{n+1}}\right)\right]$$

$$=\sum_{i=1}^{n-1}2\lambda^{-i\alpha}\sin\frac{(k'-k)\pi}{\lambda^{n+1-i}}\cos\lambda^i\pi\left(\frac{2j}{\lambda^n}+\frac{k'+k}{\lambda^{n+1}}\right)$$

$$=2\sum_{i=1}^{n-1}\lambda^{-i\alpha}\sin\frac{(k'-k)\pi}{\lambda^{n+1-i}}\cos\frac{2j\pi}{\lambda^{n+1-i}}$$

$$-4\sum_{i=1}^{n-1}\lambda^{-i\alpha}\sin\frac{(k'-k)\pi}{\lambda^{n+1-i}}\sin\frac{(k'+k)\pi}{\lambda^{n+1-i}}\sin\lambda^i\pi\left(\frac{2j}{\lambda^n}+\frac{k'+k}{2\lambda^{n+1}}\right). \tag{3.4}$$

Note that

$$\sum_{i=1}^{n-1} \lambda^{-i\alpha} \sin \frac{(k'-k)\pi}{\lambda^{n+1-i}} \cos \frac{2j\pi}{\lambda^{n-i}}$$

$$= \frac{(k'-k)\pi}{\lambda^{n+1}} \sum_{i=1}^{n-1} \lambda^{i(1-\alpha)} \cos \frac{2j\pi}{\lambda^{n-i}} + \frac{\theta}{6}\left(\frac{(k'-k)\pi}{\lambda}\right)^3 \frac{\lambda^{-n\alpha}}{\lambda^{3-1}-1}, \tag{3.5}$$

$$\sum_{i=1}^{n-1} \lambda^{-i\alpha} \sin \frac{(k'-k)\pi}{\lambda^{n+1-i}} \sin \frac{(k'+k)\pi}{2\lambda^{n+1-i}} \sin \lambda^i \pi\left(\frac{2j}{\lambda^n} + \frac{k'-k}{2\lambda^{n+1}}\right)$$

$$= \theta \frac{(k'-k)(k'+k)\pi^2}{\lambda^2(\lambda^{2-\alpha}-1)} \lambda^{-n\alpha}. \tag{3.6}$$

Set

$$\sum_{i=1}^{n-1} \lambda^{i(1-\alpha)} \cos \frac{2j\pi}{\lambda^{n-i}} = \frac{\theta_0 \lambda^{n(1-\alpha)}}{\lambda^{i-\alpha}-1}. \tag{3.7}$$

Substituting to (3.5), we have

$$\sum_{i=1}^{n-1} \lambda^{-n\alpha} \sin \frac{(k'-k)\pi}{\lambda^{n+1-i}} \cos \frac{2j\pi}{\lambda^{n-i}}$$

$$= \frac{\theta_0}{\lambda^{1-\alpha}-1} \frac{(k'-k)\pi}{\lambda} + \frac{\theta}{3(\lambda^{3-\alpha}-1)}\left(\frac{(k'-k)\pi}{\lambda}\right)\lambda^{-n\alpha}. \tag{3.8}$$

Combining (3.4), (3.6) and (3.8), we get

$$A_{k',k} = \left(\frac{\theta_0}{\lambda^{1-\alpha}-1} \frac{2(k'-k)\pi}{\lambda} + \frac{\theta}{12(\lambda^{3-\alpha}-1)}\left(\frac{2(k'-k)}{\lambda}\right)^3 + \frac{\theta'(k'-k)\pi}{\lambda^2(\lambda^{2-\alpha}-1)}\right)\lambda^{-n\alpha}. \tag{3.9}$$

By elementary calculus, we have

$$B_{k',k} = \left(\sin \frac{2k'\pi}{\lambda} - \sin \frac{2k\pi}{\lambda}\right)\lambda^{-n\alpha}$$

$$= \left[\cos \frac{2k'\pi}{\lambda} \frac{2(k'-k)\pi}{\lambda} - \frac{\sin \frac{2k'\pi}{\lambda}}{2}\left(\frac{2(k'-k)\pi}{\lambda}\right)^2 + \frac{\tau}{6(\lambda^{3-\alpha}-1)}\left(\frac{2(k'-k)\pi}{\lambda}\right)^3\right]\lambda^{-n\alpha}. \tag{3.10}$$

By (3.9) and (3.10), we have

$$W\left(\frac{2j}{\lambda^n}+\frac{2k'}{\lambda^{n+1}}\right)-W\left(\frac{2j}{\lambda^n}+\frac{2k}{\lambda^{n+1}}\right)$$

$$=\left[\left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}+\frac{\theta(k'+k)\pi}{\lambda^2(\lambda^{2-\alpha}-1)}\right)\frac{2(k'-k)\pi}{\lambda}\right.$$

$$\left.-\frac{\sin\frac{2k'\pi}{\lambda}}{2}\left(\frac{2(k'-k)\pi}{\lambda}\right)^2+\frac{\tau}{6(\lambda^{3-\alpha}-1)}\left(\frac{2(k'-k)\pi}{\lambda}\right)^3\right]\lambda^{-n\alpha}.$$

Without loss of generality, we may suppose $\frac{2(k'-k)}{\lambda}<\frac{1}{2}$, thus we have

$$W\left(\frac{2j}{\lambda^n}+\frac{2k'}{\lambda^{n+1}}\right)-W\left(\frac{2j}{\lambda^n}+\frac{2k}{\lambda^{n+1}}\right)$$

$$=\left[\left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}+\frac{\theta}{\lambda^{2-\alpha}-1}\right)\frac{2(k'-k)\pi}{\lambda}\right.$$

$$\left.-\left(\frac{\sin\frac{2k'\pi}{\lambda}}{2}+\frac{\tau\pi}{12(\lambda^{3-\alpha}-1)}\right)\left(\frac{2(k'-k)\pi}{\lambda}\right)^2\right]\lambda^{-n\alpha}.$$

Clearly, $\frac{\theta\pi}{\lambda^{2-\alpha}-1}\ll\frac{\theta_0}{\lambda^{1-\alpha}-1}$ and $\frac{\tau\pi}{12(\lambda^{3-\alpha}-1)}\ll\frac{\sin\frac{2k'\pi}{\lambda}}{2}$ as $\lambda\to\infty$, by power law, we have

$$W\left(\frac{2j}{\lambda^n}+\frac{2k'}{\lambda^{n+1}}\right)-W\left(\frac{2j}{\lambda^n}+\frac{2k}{\lambda^{n+1}}\right)$$

$$=\left[\left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)\frac{2(k'-k)\pi}{\lambda}-\frac{\sin\frac{2k'\pi}{\lambda}}{2}\left(\frac{2(k'-k)\pi}{\lambda}\right)^2\right]\lambda^{-n\alpha}.$$

This proves (3.1). Note that for $k' < k''$, we have

$$\sin\frac{2k''\pi}{\lambda} - \sin\frac{2k'\pi}{\lambda}$$

$$= \cos\frac{2k'\pi}{\lambda}\frac{2(k''-k')}{\lambda} + \frac{\sin\frac{2k'\pi}{\lambda}}{2}\left(\frac{2(k''-k')}{\lambda}\right)^2 + \frac{\theta}{6}\left(\frac{2(k''-k')\pi}{\lambda}\right)^3.$$

Then formula (3.2) comes immediately.

3.c.b. Theorem 3.3. *Let* $W(x) = \sum_{i=1}^{\infty}\lambda^{-i\alpha}\sin\lambda^i\pi x$ *be the Weierstrass function. Then we have*

1. *There is a positive integer* k_u *such that*

$$w\left(\frac{2j}{\lambda^n}\right) < w\left(\frac{2j}{\lambda^n} + \frac{2}{\lambda^{n+1}}\right) < \cdots < w\left(\frac{2j}{\lambda^n} + \frac{2(k_u-1)}{\lambda^{n+1}}\right) < w\left(\frac{2j}{\lambda^n} + \frac{2k_u}{\lambda^{n+1}}\right)$$

$$> w\left(\frac{2j}{\lambda^n} + \frac{2(k_u+1)}{\lambda^{n+1}}\right) > \cdots > w\left(\frac{2j+1}{\lambda^n}\right). \tag{3.11}$$

2. *There is a positive integer* k_d *such that*

$$w\left(\frac{2j+1}{\lambda^n}\right) > w\left(\frac{2j+1}{\lambda^n} + \frac{2}{\lambda^{n+1}}\right) > \cdots > w\left(\frac{2j+1}{\lambda^n} + \frac{2(k_d-1)}{\lambda^{n+1}}\right)$$

$$> w\left(\frac{2j+1}{\lambda^n} + \frac{2k_d}{\lambda^{n+1}}\right) < w\left(\frac{2j+1}{\lambda^n} + \frac{2(k_d+1)}{\lambda^{n+1}}\right)$$

$$< \cdots < w\left(\frac{2(j+1)}{\lambda^n}\right). \tag{3.12}$$

3. *Let*

$$h_u = \max_{0\le k_1, k_2\le\frac{\lambda}{2}}\left\{W\left(\frac{2j}{\lambda^n} + \frac{2k_1}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{2k_2}{\lambda^{n+1}}\right)\right\}$$

and

$$h_d = \max_{0\le k_1, k_2\le\frac{\lambda}{2}}\left\{W\left(\frac{2j+1}{\lambda^n} + \frac{2k_1}{\lambda^{n+1}}\right) - W\left(\frac{2j+1}{\lambda^n} + \frac{2k_2}{\lambda^{n+1}}\right)\right\}.$$

Then

$$h_u = \left\{\left(1 + \frac{\pi\theta_0}{2(\lambda^{1-\alpha} - 1)} + \frac{\theta^2}{2(\lambda^{1-\alpha} - 1)^2}\right)\lambda^{-n\alpha},\ \theta_0 \geq 0\right\} \textit{ and}$$

$$h_u = \left\{\left(1 - \frac{\pi\theta_0}{2(\lambda^{1-\alpha} - 1)} - \frac{\theta^2}{2(\lambda^{1-\alpha} - 1)^2}\right)\lambda^{-n\alpha},\ \theta_0 < 0\right\}, \tag{3.13}$$

$$h_d = \left\{\left(1 + \frac{\pi\theta_0}{2(\lambda^{1-\alpha} - 1)} - \frac{\theta^2}{2(\lambda^{1-\alpha} - 1)^2}\right)\lambda^{-n\alpha},\ \theta_0 \geq 0\right\} \textit{ and}$$

$$h_d = \left\{\left(1 - \frac{\pi\theta_0}{2(\lambda^{1-\alpha} - 1)} + \frac{\theta^2}{2(\lambda^{1-\alpha} - 1)^2}\right)\lambda^{-n\alpha},\ \theta_0 < 0\right\}. \tag{3.14}$$

Proof. We consider pace $d(k) = W\left(\frac{2j}{\lambda^n} + \frac{2(k+1)}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{2k}{\lambda^{n+1}}\right)$. By Lemma 3.2 (3.1), we have $d(k) = \left(\cos\frac{2k\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha} - 1} + \frac{\pi}{\lambda}\sin\frac{2k\pi}{\lambda}\right)\frac{2\pi}{\lambda}\lambda^{-n\alpha}$.

Since $\frac{\pi\sin\frac{2k\pi}{\lambda}}{\lambda} \ll \frac{\theta_0}{\lambda^{1-\alpha} - 1}$ as $\lambda \to \infty$, by power law, we have $d(k) = \left(\cos\frac{2k\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)\frac{2\pi}{\lambda}\lambda^{-n\alpha}$. Obviously, function $f(x) = \cos\pi x + \frac{\theta_0}{\lambda^{1-\alpha} - 1}$ has two points $x_1 = \frac{1}{2} + \frac{\theta_0}{\pi(\lambda^{1-\alpha} - 1)}$ and $x_2 = \frac{3}{2} - \frac{\theta_0}{\pi(\lambda^{1-\alpha} - 1)}$ such that $f'(x) > 0$ for $x \in [0, x_1) \cup (x_2, 1]$ and $f'(x) < 0$ for (x_1, x_2). Let $\frac{2k_u\pi}{\lambda} \leq x_1\pi$ and $\frac{2(k_u + 1)\pi}{\lambda} > x_1\pi$. Then $\frac{2k_u\pi}{\lambda} = \frac{\pi}{2} + \frac{\theta_o}{\lambda^{1-\alpha} - 1} + \frac{2\pi\tau}{\lambda}$. By power law, we have equality $\frac{2k_u\pi}{\lambda} = \frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}$. This shows (3.11), formula (3.12) may be shown in a similar way. The proof of statement 3 is simple. Clearly, if $\theta_0 \geq 0$, then we have

$$\left(\sin\frac{2k_u}{\lambda} + \frac{2k_u\pi\theta_0}{\lambda(\lambda^{1-\alpha} - 1)}\right)\lambda^{-n\alpha}$$

$$= \left(1 + \frac{\pi\theta_0}{2(\lambda^{1-\alpha} - 1)} + \frac{\theta^2}{2(\lambda^{1-\alpha} - 1)^2}\right)\lambda^{-n\alpha}.$$

If $\theta_0 < 0$, then

$$h_u = W\left(\frac{2j}{\lambda^n} + \frac{2k_u}{\lambda^{n+1}}\right) - W\left(\frac{2j+1}{\lambda^n}\right)$$

$$= \left(1 - \frac{\pi\theta_0}{2(\lambda^{1-\alpha} - 1)} - \frac{\theta^2}{2(\lambda^{1-\alpha} - 1)^2}\right)\lambda^{-n\alpha}.$$

Precisely, we have

$$h_u = \left(1 + \frac{\pi|\theta_0|}{2(\lambda^{1-\alpha} - 1)} + \frac{\theta^2}{2(\lambda^{1-\alpha} - 1)^2}\right)\lambda^{-n\alpha}.$$

It follows (3.13) and (3.14) may be treated in a similar way.

We can now state the following results that will be used in the sharp estimate.

Theorem 3.4. *Let* $W(x) = \sum_{i=1}^{\infty} \lambda^{-i\alpha} \sin \lambda^i \pi x$ *be the Weierstrass function and let* Γ_{w_n} *be lever n sine-like curve and* $\theta_0 \le 2k' < 2k_u$. *Then we have for large* λ,

(a)

$$\frac{2(k' - k_1')\pi}{\lambda} = \left(\frac{\tau\pi}{2}\right)\frac{2\theta_0}{\lambda^{1-\alpha} - 1}, \tag{3.15}$$

where $\frac{2}{\pi} \le \tau \le 1$ *and*

$$w\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2(k_1 + 1)}{\lambda^{n+1}}\right) > w\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right) \ge w\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2k_1'}{\lambda^{n+1}}\right) \tag{3.16}$$

with $\frac{2k'}{\lambda} \in \left[\frac{\theta_0}{\lambda^{1-\alpha} - 1}, \frac{1}{2}\right]$

b. *If*

$$W\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2k_1'}{\lambda^{n+1}}\right) \ge W\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right) > W\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2(k_1' + 1)}{\lambda^{n+1}}\right),$$

$$\textit{here } \frac{2k'}{\lambda} \in \left[\frac{\theta_0}{\lambda^{1-\alpha}-1}, \frac{1}{2}\right]. \tag{3.17}$$

Then we have for large λ,

$$\frac{2(k'-k_1')\pi}{\lambda} = \frac{2\tau\theta_0}{\lambda^{1-\alpha}-1}. \tag{3.18}$$

Proof. a. Inequality (3.16) does not have solution for small $2k'$. First we will find a lower bound for $2k'$ such that inequality (3.16) has a solution. Suppose

$$W\left(\frac{2j}{\lambda^n} + \frac{2(k_0-1)}{\lambda^{n+1}}\right) \le w\left\{\frac{2j+1}{\lambda^n}\right\} < w\left(\frac{2j}{\lambda^n} + \frac{2k_0}{\lambda^{n+1}}\right).$$

Then we have

$$W\left(\frac{2j}{\lambda^n} + \frac{2k_0}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n}\right)$$

$$= W\left(\frac{2j+1}{\lambda^n}\right) - W\left(\frac{2j}{\lambda^n}\right) - \tau\left[w\left(\frac{2j}{\lambda^n} + \frac{2k_0}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{2(k_0-1)}{\lambda^{n+1}}\right)\right].$$

Thus, by Lemma 3.2, we have

$$\sin\frac{2k_0\pi}{\lambda} + \frac{2k_0\pi\theta_0}{\lambda(\lambda^{1-\alpha}-1)} + \frac{2k_0\pi\theta}{\lambda(\lambda^{2-\alpha}-1)} = \frac{\pi\theta_0}{\lambda^{1-\alpha}-1} - \tau\left(\frac{2\pi}{\lambda} - \frac{\pi\theta}{\lambda(\lambda^{2-\alpha}-1)}\right).$$

Solving above equation, we get

$$\frac{2k_0\pi}{\lambda}\left(1 + \frac{\theta_0}{\lambda^{1-\alpha}-1}\right) = \frac{\pi\theta_0}{\lambda^{1-\alpha}-1}.$$

That is for large enough λ,

$$2k_0 = \theta_0\lambda^a.$$

Since function $W\left(\frac{2j}{\lambda^n} + \frac{\lambda-2k_1'}{\lambda^{n+1}}\right)$ monotone increases for $2k_1' \le \lambda - 2k_u$, inequality (3.16) has unique solution $\lambda - 2k_1'$. Consequently, we have

$$W\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{2k_1'}{\lambda^{n+1}}\right)$$

$$= W\left(\frac{2j}{\lambda^n} + \frac{\lambda - k_1'}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{2k_1'}{\lambda^{n+1}}\right)$$

$$- \tau\left[W\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2(k_1'+1)}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2k_1'}{\lambda^{n+1}}\right)\right].$$

Recall (3.16), we get

$$\left(\cos\frac{2k'\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha}-1} + \frac{2\pi\theta}{\lambda}\right)\frac{2(k'-k_1')\pi}{\lambda} + \frac{\sin\frac{2k'\pi}{\lambda}}{2}\left(\frac{2(k'-k_1')\pi}{\lambda}\right)^2$$

$$= \theta_0\,\frac{2\pi\left(\frac{1}{2} - \frac{2k_1'}{\lambda}\right)}{\lambda^{1-\alpha}-1} + \tau\left(\frac{2\pi}{\lambda} + \frac{2\pi}{\lambda(\lambda^{1-\alpha}-1)}\right).$$

By power law, for large λ, we may solve the following simplified equation:

$$\left(\cos\frac{2k'\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha}-1}\right)\frac{2(k'-k_1')\pi}{\lambda} + \frac{\sin\frac{2k'\pi}{\lambda}}{2}\left(\frac{2(k'-k_1')\pi}{\lambda}\right)^2 = \theta_0\,\frac{2\pi\left(\frac{1}{2} - \frac{2k_1'}{\lambda}\right)}{\lambda^{1-\alpha}-1}.$$

Note that

$$\theta_0\,\frac{2\pi\left(\frac{1}{2} - \frac{2k_1'}{\lambda}\right)}{\lambda^{1-\alpha}-1} = \theta_0\,\frac{2\pi\left(\frac{1}{2} - \frac{2k'}{\lambda}\right)}{\lambda^{1-\alpha}-1} + \frac{2\theta_0}{\lambda^{1-\alpha}-1}\,\frac{2(k'-k_1')\pi}{\lambda}.$$

We have

$$\frac{\sin\frac{2k'\pi}{\lambda}}{2}\left(\frac{2(k'-k_1')\pi}{\lambda}\right)^2 + \left(\cos\frac{2k'\pi}{\lambda} - \frac{\theta_0}{\lambda^{1-\alpha}-1}\right)\frac{2(k'-k_1')\pi}{\lambda} - \theta_0\,\frac{2\pi\left(\frac{1}{2} - \frac{2k'}{\lambda}\right)}{\lambda^{1-\alpha}-1} = 0.$$

Without loss of generality, we may suppose $\frac{2(k'-k_1')\pi}{\lambda} > 0$. Thus, we obtain

$$\frac{2(k'-k_1')\pi}{\lambda}$$

$$= \frac{1}{\sin\frac{2k'\pi}{\lambda}}\left[\sqrt{\left(\cos\frac{2k'\pi}{\lambda} - \frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2 + \frac{4\pi\sin\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}-1}} - \left(\cos\frac{2k'\pi}{\lambda} - \frac{\theta_0}{\lambda^{1-\alpha}-1}\right)\right]. \quad (3.19)$$

From (3.19), we have

$$\frac{2(k'-k_1')\pi}{\lambda}$$

$$=\frac{1}{\sin\frac{2k'\pi}{\lambda}}\left[\left(\cos\frac{2k'\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)\sqrt{1+\frac{4\pi\sin\frac{2k'\pi}{\lambda}\left(\frac{1}{2}-\frac{2k'\pi}{\lambda}\right)}{(\lambda^{1-\alpha}-1)\left(\cos\frac{2k'\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2}}-\left(\cos\frac{2k'\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)\right]$$

$$=\frac{2\pi\left(\frac{1}{2}-\frac{2k'}{\lambda}\right)}{\left(\cos\frac{2k'\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)}-\frac{2\sin\frac{2k'\pi}{\lambda}\left(\frac{1}{2}-\frac{2k'}{\lambda}\right)^2\theta_0^2}{\left(\cos\frac{2k'\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^3(\lambda^{1-\alpha}-1)^2}+\cdots.$$

By power law, we have

$$\frac{2(k'-k_1')\pi}{\lambda}=\frac{2\pi\left(\frac{1}{2}-\frac{2k'}{\lambda}\right)}{\cos\frac{2k'\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}}\frac{2\theta_0}{\lambda^{1-\alpha}-1}$$

$$=\frac{\frac{\pi}{2}-\frac{2k'\pi}{\lambda}}{\sin\left(\frac{\pi}{2}-\frac{2k'\pi}{\lambda}\right)-\frac{\theta_0}{\lambda^{1-\alpha}-1}}\frac{2\theta_0}{\lambda^{1-\alpha}-1}.$$

By calculus, the term $\dfrac{\frac{\pi}{2}-\frac{2k'\pi}{\lambda}}{\sin\left(\frac{\pi}{2}-\frac{2k'\pi}{\lambda}\right)-\frac{\theta_0}{\lambda^{1-\alpha}-1}}$ monotone decreases from $\frac{\pi}{2}$ to 1 on interval $\left[0, \frac{2k_0\pi}{\lambda}\right]$. This proves (3.15) and (3.16).

Now we consider the case $\frac{2k'}{\lambda}\in\left[\frac{1}{2}, \frac{2k_u}{\lambda}\right]$. By (3.17), without loss of generality, we may suppose $W\left(\frac{2j}{\lambda^n}+\frac{\lambda-2k_1'}{\lambda^{n+1}}\right)=W\left(\frac{2j}{\lambda^n}+\frac{2k'}{\lambda^{n+1}}\right)$. Thus, we have

$$W\left(\frac{2j}{\lambda^n}+\frac{\lambda-2k_1'}{\lambda^{n+1}}\right)-W\left(\frac{2j}{\lambda^n}+\frac{2k_u}{\lambda^{n+1}}\right)=W\left(\frac{2j}{\lambda^n}+\frac{2k'}{\lambda^{n+1}}\right)-W\left(\frac{2j}{\lambda^n}+\frac{\lambda+2k_u}{\lambda^{n+1}}\right).$$

By Lemma 3.2 and $\frac{2k_u}{\lambda} = \frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}$, we have

$$W\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{2k_u}{\lambda^{n+1}}\right)$$

$$= \left[\left(\cos\left(\frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right) + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)\frac{2(k' - k_u)\pi}{\lambda}\right.$$

$$\left. + \frac{\sin\left(\frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)}{2}\left(\frac{2(k' - k_u)\pi}{\lambda}\right)^2\right]\lambda^{-n\alpha}$$

$$= \frac{\sin\left(\frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)}{2}\left(\frac{2(k' - k_u)\pi}{\lambda}\right)^2\lambda^{-n\alpha},$$

$$W\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2k_1'}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{2k_u}{\lambda^{n+1}}\right)$$

$$= \left[\left(\cos\left(\frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right) + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)\frac{\lambda - 2k_1' - 2k_u}{\lambda}\pi\right.$$

$$\left. + \frac{\sin\left(\frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)}{2}\left(\frac{(\lambda - 2k_1' - 2k_u)\pi}{\lambda}\right)^2\right]\lambda^{-n\alpha}$$

$$= \frac{\sin\left(\frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)}{2}\left(\frac{(\lambda - 2k_1' - 2k_u)\pi}{\lambda}\right)^2\lambda^{-n\alpha}.$$

Thus,

$$\pi = \frac{2k_1'\pi}{\lambda} + \frac{2k'\pi}{\lambda}.$$

Note that $\frac{2k'\pi}{\lambda} \in \left[\frac{\pi}{2}, \frac{2k_u\pi}{\lambda}\right]$ and $\frac{2k_u\pi}{\lambda} = \frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}$, we may write $\frac{2k'\pi}{\lambda} = \frac{\pi}{2}$ $+ \frac{\tau\theta_0}{\lambda^{1-\alpha} - 1}$. Therefore, we have $\frac{2k_1'\pi}{\lambda} = \pi - \frac{2k'\pi}{\lambda} = \frac{\pi}{2} - \frac{\tau\theta_0}{\lambda^{1-\alpha} - 1}$. Equality (3.17) follows.

We are now ready to characterize the distribution of intersection number. Recall the intersection set A may divide into four subsets A_1, A_2, A_3 and A_4. In typical case, four subsets are approximately equal; however, the situation may change as horizontal line $y = h$ closes to the top or the bottom of the sine-like curve Γ_{w_n}.

Theorem 3.5. *Let* $W(x) = \sum_{i=1}^{\infty} \lambda^{-i\alpha} \sin \lambda^i \pi x$ *be the Weierstrass function and let* Γ_{w_n} *be lever* n *sine-like curve. Then the intersection set* A *is in typical case for horizontal line* $y = h$ *with integer phase* $2k'$, *where* $2k_0 \le 2k' \le 2\tilde{k}$. *Moreover, we have*

$$I_{n+1}\left(\frac{2\tilde{k}}{\lambda}\right) = \frac{2}{\pi}\lambda^{1-\frac{\alpha}{2}} \tag{3.20}$$

and $I_{n+1}\left(\frac{2\tilde{k}}{\lambda}\right)$ *is the maximum intersection number of lever n sine-like curve* Γ_{w_n}.

Proof. We want to show in some case, the horizontal line $y = h$ will intersect with lever $(n+1)$ sine-like curves $\Gamma_{w_{n+1}}$. Consider the following inequalities:

$$W\left(\frac{2j}{\lambda^n} + \frac{2(\tilde{k})}{\lambda^{n+1}}\right) > W\left(\frac{2j}{\lambda^n} + \frac{2k_u}{\lambda^{n+1}}\right) - h_{d, r_1, r_2, \ldots, r_n, 2k_u} \ge W\left(\frac{2j}{\lambda^n} + \frac{2\tilde{k}}{\lambda^{n+1}}\right). \tag{3.21}$$

Thus, by Lemma 3.2 and Theorem 3.4 (3.14), we have

$$\frac{\sin \frac{2k_u\pi}{\lambda}}{2}\left(\frac{2(k_u - \tilde{k})\pi}{\lambda}\right)^2 + \left(\cos\frac{2k_u\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha} - 1} + \frac{2\pi\theta}{\lambda^{2-\alpha} - 1}\right)\frac{2(k_u - \tilde{k})\pi}{\lambda}$$

$$= \left(1 + \frac{\pi\cos\frac{2k_u\pi}{\lambda}}{2(\lambda^{1-\alpha} - 1)} + \frac{\cos^2\frac{2k_u\pi}{\lambda}}{2(\lambda^{1-\alpha} - 1)^2}\right)\lambda^{-\alpha} + \tau\left(\frac{2\pi}{\lambda} + \frac{2\pi}{\lambda(\lambda^{1-\alpha} - 1)}\right).$$

Note that $\cos\frac{2k_u\pi}{\lambda} = -\frac{\theta_0}{\lambda^{1-\alpha}-1}$, we have

$$\frac{1}{2}\left(\frac{2(k_u-\tilde{k})\pi}{\lambda}\right)^2 = \lambda^{-\alpha} + \lambda^{-\alpha}\frac{\pi\theta}{2(\lambda^{1-\alpha}-1)} + \lambda^{-\alpha}\frac{\theta'}{2(\lambda^{1-\alpha}-1)^4}$$
$$+\frac{2\pi\tau}{\lambda} + \frac{2\pi\tau'}{\lambda(\lambda^{1-\alpha}-1)} + \frac{2\pi\theta''}{\lambda^{2-\alpha}-1}.$$

By power law, we have

$$\frac{1}{2}\left(\frac{2(k_u-\tilde{k})\pi}{\lambda}\right)^2 = \lambda^{-\alpha}.$$

Thus, we have

$$\frac{2(k_u-\tilde{k})\pi}{\lambda} = \sqrt{2}\lambda^{-\frac{\alpha}{2}}$$

so that

$$\frac{2\tilde{k}\pi}{\lambda} = \frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha}-1} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}. \tag{3.22}$$

Let $2k' \le 2\tilde{k}$. Then distance between the horizontal line $y = W\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right)$ and the top of the lever n sine-like curve Γ_{w_n} is large or equals to the height of lever $(n+1)$ sine-like curve. Therefore, we can find a lever $(n+1)$ sine-like curve with integer phase $2k'$ such that

$$W\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{2k}{\lambda^{n+1}}\right) = \lambda^{-(n+1)\alpha}.$$

Now we show in some case, the horizontal line $y = h$ will intersect lever $(n+1)$ sine-like curve. It is easy to estimate that if $\frac{2k'\pi}{\lambda} \ge \frac{1}{\lambda^\alpha}$, then the distance between line $y = W\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right)$ and the bottom of lever n sine-like curve Γ_{w_n} is large or equals to the height of lever $(n+1)$ sine-like curve, so that we can find a lever $(n+1)$ sine-like curve with integer phase $2k$ such that

$$W\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{2k}{\lambda^{n+1}}\right) = \lambda^{-(n+1)\alpha}.$$

By Theorem 3.4, if $\frac{2k'\pi}{\lambda} \geq \frac{1}{\lambda^{\alpha}}$, then the line $y = W\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right)$ intersects with right hand side graph of $W(x)$. This fact combines with above two equalities, we conclude that there is a number $\frac{2k_0\pi}{\lambda}$, it is about $\max\left\{\frac{\theta_0}{\lambda^{1-\alpha} - 1}, \frac{1}{\lambda^{\alpha}}\right\}$ such that the intersection set A is in typical case for the horizontal line $y = h$ with integer phase $2k'$, $2k_0 \leq 2k' \leq 2\tilde{k}$.

We claim that $I_{n+1}\left(\frac{2\tilde{k}}{\lambda}\right)$ is the maximum intersection number of lever n sine-like curve Γ_{w_n}. Since the value of function $W\left(\frac{2j}{\lambda^n} + \frac{2k}{\lambda^{n+1}}\right)$ monotonously increases for $k = 0, 1, 2, \ldots, k_u$; thus we may view the pace $d_{j,k}$ as speed, $\frac{2k'\pi}{\lambda}$ as time variable and $2\lambda^{-(n+1)\alpha}$ as distance. Since $d_{j,k} \downarrow$ as $2k' \uparrow$, $2k' \leq 2\tilde{k}$, so that intersection number $I_{n+1}\left(\frac{2k'}{\lambda}\right)$ (the time needs to go through the distance $2\lambda^{-(n+1)\alpha}$) increases. Thus,

$$I_{n+1}\left(\frac{2k'}{\lambda}\right) \leq I_{n+1}\left(\frac{2\tilde{k}}{\lambda}\right), \quad 2k' \leq 2\tilde{k}.$$

For the case $2\tilde{k} \leq 2k' \leq 2k_u$, the lever $(n+1)$ sine-like curves intersect with horizontal line $y = W\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right)$, which will intersect with horizontal line $y = W\left(\frac{2j}{\lambda^n} + \frac{2\tilde{k}}{\lambda^{n+1}}\right)$. Therefore, we have

$$I_{n+1}\left(\frac{2k'}{\lambda}\right) \leq I_{n+1}\left(\frac{2\tilde{k}}{\lambda}\right), \quad 2\tilde{k} \leq 2k' \leq 2k_u.$$

$$W\left(\frac{2j}{\lambda^n}+\frac{2\tilde{k}}{\lambda^{n+1}}\right)-W\left(\frac{2j}{\lambda^n}+\frac{2\bar{k}}{\lambda^{n+1}}\right)$$

$$=W\left(\frac{2j}{\lambda^n}+\frac{2\bar{k}}{\lambda^{n+1}}+\frac{2k_u(\bar{k})}{\lambda^{n+2}}\right)-W\left(\frac{2j}{\lambda^n}+\frac{2\bar{k}}{\lambda^{n+1}}\right)$$

$$-\tau\left[W\left(\frac{2j}{\lambda^n}+\frac{2\bar{k}}{\lambda^{n+1}}+\frac{2k_u(\bar{k})}{\lambda^{n+2}}\right)-W\left(\frac{2j}{\lambda^n}+\frac{2(\bar{k}-1)}{\lambda^{n+1}}+\frac{2k_u(\bar{k}-1)}{\lambda^{n+2}}\right)\right].$$

This yields that $I_{n+1}\left(\frac{2\tilde{k}}{\lambda}\right)$ is the maximum value of the $I_{n+1}\left(\frac{2k'}{\lambda}\right)$, $0 \le 2k' \le 2k_u$.

For computing $I_{n+1}\left(\frac{2\tilde{k}}{\lambda}\right)$, we consider the following inequalities:

$$W\left(\frac{2j}{\lambda^n}+\frac{2(\bar{k}-1)}{\lambda^{n+1}}+\frac{2k_u(\bar{k}-1)}{\lambda^{n+2}}\right)<W\left(\frac{2j}{\lambda^n}+\frac{2\tilde{k}}{\lambda^{n+1}}\right)\le W\left(\frac{2j}{\lambda^n}+\frac{2\bar{k}}{\lambda^{n+1}}+\frac{2k_u(\bar{k})}{\lambda^{n+2}}\right).$$

By Lemma 3.2 and Theorem 3.3, we have

$$\left[\left(\cos\frac{2\tilde{k}\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)\frac{2(\tilde{k}-\ddot{\bar{k}})\pi}{\lambda}+\frac{\sin\frac{2\tilde{k}\pi}{\lambda}}{2}\left(\frac{2(\tilde{k}-\bar{k})\pi}{\lambda}\right)^2\right]\lambda^{-n\alpha}$$

$$=\left(1+\frac{\pi\theta_1}{2(\lambda^{1-\alpha}-1)}+\frac{\theta^2}{2(\lambda^{1-\alpha}-1)^2}\right)\lambda^{-(n+1)\alpha}-\tau\left(\frac{\theta}{\lambda^{\alpha}(\lambda^{\alpha}-1)}+\frac{2\pi\theta}{\lambda}\right)\lambda^{-n\alpha}.$$

By power law, we have

$$\frac{\sin\frac{2\tilde{k}\pi}{\lambda}}{2}\left(\frac{2(\tilde{k}-\bar{k})\pi}{\lambda}\right)^2+\left(\cos\frac{2\tilde{k}\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)\frac{2(\tilde{k}-\bar{k})\pi}{\lambda}-\lambda^{-\alpha}=0.$$

Therefore, we have

$$\frac{2(\tilde{k}-\bar{k})\pi}{\lambda}=\frac{\sqrt{\left(\cos\frac{2\tilde{k}\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)+2\sin\frac{2\tilde{k}\pi}{\lambda}\lambda^{-\alpha}}-\left(\cos\frac{2\tilde{k}\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)}{\sin\frac{2\tilde{k}\pi}{\lambda}}.$$

Note that $\frac{2\tilde{k}\pi}{\lambda} = \frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}$, we get

$$\frac{2(\tilde{k} - \bar{k})\pi}{\lambda} = \frac{\sqrt{\left(\frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right)^2 + \left\{1 - \left[\frac{1}{2}\left(\frac{\theta_0}{\lambda^{1-\alpha} - 1} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right)\right]^2\right\}} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}}{1 - \left[\frac{1}{2}\left(\frac{\theta_0}{\lambda^{1-\alpha} - 1} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right)\right]^2}$$

$$= \frac{2 - \sqrt{2}}{\lambda^{\frac{\alpha}{2}}} - \left[\frac{1}{2}\left(\frac{\theta_0}{\lambda^{1-\alpha} - 1} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right)\right]^2 \frac{2 - \sqrt{2}}{\lambda^{\frac{\alpha}{2}}}.$$

By power law, we have

$$\frac{2(\tilde{k} - k)\pi}{\lambda} = (2 - \sqrt{2})\lambda^{-\frac{\alpha}{2}}.$$

Combining the fact $\frac{2k_u\pi}{\lambda} = \frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}$ and $\frac{2\tilde{k}\pi}{\lambda} = \frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}$, it follows that the left hand side intersection number equals to $\frac{1}{\pi}\lambda^{1-\frac{\alpha}{2}}$.

We are now ready to estimate right hand side intersection number. Similarly, we have

$$W\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2(\tilde{k}_0 - 1)}{\lambda^{n+1}} + \frac{2k_u(\lambda - 2(\tilde{k} - 1))}{\lambda^{n+2}}\right)$$

$$< W\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2\tilde{k}_0}{\lambda^{n+1}}\right) \le W\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2\tilde{k}_0}{\lambda^{n+1}} + \frac{2k_u(\lambda - 2\tilde{k}_0)}{\lambda^{n+2}}\right).$$

Thus, same arguments as above, we see that

$$W\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2\tilde{k}_1}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2\tilde{k}_0}{\lambda^{n+1}}\right)$$

$$= W\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2\tilde{k}_0}{\lambda^{n+1}} + \frac{2k_u(\tilde{k}_0)}{\lambda^{n+2}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2\tilde{k}_0}{\lambda^{n+1}}\right)$$

$$- \tau\left[W\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2\tilde{k}_0}{\lambda^{n+1}} + \frac{2k_u(\lambda - 2\tilde{k}_0)}{\lambda^{n+2}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{\lambda - 2(\tilde{k}_0)}{\lambda^{n+1}} + \frac{\lambda - 2(\tilde{k}_0 - 1)}{\lambda^{n+2}}\right)\right].$$

That is,

$$\left[\left(\cos\frac{(\lambda - 2\tilde{k}_1)\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)\frac{2(\tilde{k}_0 - \tilde{k}_1)\pi}{\lambda} + \frac{\sin\frac{(\lambda - \tilde{k}_1)}{\lambda}}{2}\left(\frac{2(\tilde{k}_0 - \tilde{k}_1)\pi}{\lambda}\right)^2\right]\lambda^{-n\alpha}$$

$$= \left(1 + \frac{\pi\theta_1}{2(\lambda^{1-\alpha})} + \frac{\theta_2^2}{2(\lambda^{1-\alpha} - 1)^2}\right)\lambda^{-(n+1)} - \tau\left(\frac{\theta}{\lambda^{\alpha}(\lambda^{1-\alpha} - 1)} + \frac{2\pi}{\lambda}\right).$$

Thus,

$$\left(\cos\frac{2\tilde{k}\pi}{\lambda} - \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)\frac{2(\tilde{k}_1 - \tilde{k}_0)\pi}{\lambda} + \frac{\sin\frac{2\tilde{k}_1\pi}{\lambda}}{2}\left(\frac{2(\tilde{k}_1 - \tilde{k}_0)\pi}{\lambda}\right)^2 - \lambda^{-\alpha} = 0.$$

Solving above equality, this yields

$$\frac{2(\tilde{k}_1 - \tilde{k}_0)\pi}{\lambda} = \frac{\sqrt{\left(\cos\frac{2\tilde{k}_1\pi}{\lambda} - \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)^2 + 2\sin\frac{2\tilde{k}_1\pi}{\lambda}\lambda^{-\alpha}} - \left(\cos\frac{2\tilde{k}_1\pi}{\lambda} - \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)}{\sin\frac{2\tilde{k}_1\pi}{\lambda}}.$$

From Theorem 3.4 and (3.21), we obtain that

$$\frac{2\tilde{k}_1\pi}{\lambda} = \frac{\pi}{2} - \frac{\theta_0}{\lambda^{1-\alpha} - 1} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}.$$

Same arguments as above yield

$$\frac{2(\tilde{k}_1 - \tilde{k}_0)\pi}{\lambda} = (2 - \sqrt{2})\lambda^{-\frac{\alpha}{2}}.$$

Thus, the right hand side intersection number also equals to $\frac{2}{\pi}\lambda^{1-\frac{\alpha}{2}}$, which proves (3.20).

Theorem 3.6. *Let* $W(x) = \sum_{i=1}^{\infty} \lambda^{-i\alpha} \sin \lambda^i \pi x$ *be the Weierstrass function and let* Γ_{w_n} *be a lever n sine-like curve. Then*

$$\frac{2(k'' - k')}{\lambda} + \frac{2(k' - k)}{\lambda}$$

$$= \frac{2\lambda^{-\alpha}}{\pi\left(\cos \frac{2k'\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)} + \frac{2 \cdot 1 \cdot 1 \cdot 3 \cdot 2^3 \lambda^{-3\alpha}}{\pi 2 \cdot 4 \cdot 6\left(\cos \frac{2k'\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)^5}$$

$$+ \frac{2 \cdot 1 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 2^5 \lambda^{-5\alpha}}{\pi 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10\left(\cos \frac{2k'\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)^9} + \cdots, \tag{3.23}$$

$$\frac{2(k_2' - k_1')}{\lambda} + \frac{2(k_1' - k_0')}{\lambda}$$

$$= \frac{2\lambda^{-\alpha}}{\pi\left(\cos \frac{2k_1'\pi}{\lambda} - \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)} + \frac{2 \cdot 1 \cdot 1 \cdot 3 \cdot 2^3 \lambda^{-3\alpha}}{\pi 2 \cdot 4 \cdot 6\left(\cos \frac{2k_1'\pi}{\lambda} - \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)^5}$$

$$+ \frac{2 \cdot 1 \cdot 1 \cdot 3 \cdot 5 \cdot 7\left(\sin \frac{2k'\pi}{\lambda}\right)^4 2^5 \lambda^{-5\alpha}}{\pi 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10\left(\cos \frac{2k_1'\pi}{\lambda} - \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)^9} + \cdots. \tag{3.24}$$

Here $y = h$ *is a horizontal line with integer phase* $2k'$, $2k_0 \le 2k' \le 2\tilde{k}$.

Moreover,

$$\frac{2(\tilde{k}' - \tilde{k})}{\lambda} + \frac{2(\tilde{k} - k)}{\lambda} = \frac{2(\tilde{k}_2 - \tilde{k}_1)}{\lambda} + \frac{2(\tilde{k}_1 - \tilde{k}_0)}{\lambda} = \frac{2}{\pi}\lambda^{-\frac{\alpha}{2}}.$$

Proof. We compute the intersection number in first quarter, by solving the following inequalities:

$$W\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right) \le W\left(\frac{2j}{\lambda^n} + \frac{2k}{\lambda^{n+1}} + \frac{2k_u(k)}{\lambda^{n+2}}\right) < W\left(\frac{2j}{\lambda^n} + \frac{2(k'+1)}{\lambda^{n+1}}\right).$$

Same arguments as in proof of Theorem 3.5 yield

$$\frac{2(k'-k)\pi}{\lambda}$$

$$= \left[\frac{1}{\sin\frac{2k'\pi}{\lambda}}\sqrt{\left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2+2\sin\frac{2k'\pi}{\lambda}\lambda^{-\alpha}}-\left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)\right]. \tag{3.25}$$

A slight modification of above proof shows that

$$\frac{2(k''-k')\pi}{\lambda}$$

$$= \frac{1}{\sin\frac{2k'\pi}{\lambda}}\left[\left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)-\sqrt{\left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2-2\sin\frac{2k'\pi}{\lambda}\lambda^{-\alpha}}\right], \tag{3.26}$$

$$\frac{2(k_1'-k_0')\pi}{\lambda}$$

$$= \frac{1}{\sin\frac{2k_1'\pi}{\lambda}}\left[\sqrt{\left(\cos\frac{2k_1'\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2+2\sin\frac{2k_1'\pi}{\lambda}\lambda^{-\alpha}}-\left(\cos\frac{2k_1'\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)\right], \tag{3.27}$$

$$\frac{2(k_2'-k_1')\pi}{\lambda}$$

$$= \frac{1}{\sin\frac{2k_1'\pi}{\lambda}}\left[\left(\cos\frac{2k_1'\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)-\sqrt{\left(\cos\frac{2k_1'\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2-2\sin\frac{2k_1'\pi}{\lambda}\lambda^{-\alpha}}\right]. \tag{3.28}$$

Thus, in view of (3.25), (3.26), we have

$$\frac{2(k''-k')\pi}{\lambda}+\frac{2(k'-k)\pi}{\lambda}$$

$$= \frac{1}{\sin\frac{2k'\pi}{\lambda}}\left[\sqrt{\left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2 - 2\sin\frac{2k'\pi}{\lambda}\lambda^{-\alpha}} - \left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)\right.$$

$$\left. + \left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right) + \sqrt{\left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2 - 2\sin\frac{2k'\pi}{\lambda}\lambda^{-\alpha}}\right]$$

$$= \frac{\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}}{\sin\frac{2k'\pi}{\lambda}}\left[\sqrt{1+\frac{2\sin\frac{2k'\pi}{\lambda}\lambda^{-\alpha}}{\left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2}} - \sqrt{1-\frac{2\sin\frac{2k'\pi}{\lambda}\lambda^{-\alpha}}{\left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2}}\right]$$

$$= \frac{2\lambda^{-\alpha}}{\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}} + \frac{2\cdot 1\cdot 1\cdot 3\cdot 2^3\left(\sin\frac{2k'\pi}{\lambda}\right)^2\lambda^{-3\alpha}}{2\cdot 4\cdot 6\left(\cos\frac{2k'\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^5} + \cdots. \tag{3.29}$$

Note (3.27) and (3.28), copying above arguments, we obtain

$$\frac{2(k_2'-k_1')\pi}{\lambda} + \frac{2(k_1'-k_0')\pi}{\lambda}$$

$$= \frac{2\lambda^{-\alpha}}{\cos\frac{2k_1'\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}} + \frac{2\cdot 1\cdot 1\cdot 3\cdot 2^3\left(\sin\frac{2k_1'\pi}{\lambda}\right)^2}{2\cdot 4\cdot 6\left(\cos\frac{2k_1'\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^5}\lambda^{-3\alpha} + \cdots. \tag{3.30}$$

Before we are going to prove (3.23), we need some formulas. Using the power series of functions $\sqrt{1\pm x}$, we have

$$\sqrt{2} = 1+\frac{1}{2}-\frac{1\cdot 1}{2\cdot 4}+\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}+\frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}-\cdots,$$

$$\sqrt{0} = 1-\frac{1}{2}+\frac{1\cdot 1}{2\cdot 4}-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}+\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}-\frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}+\cdots,$$

where $x = \pm 1$. Clearly, this yields

$$\sqrt{2}-\sqrt{0} = \frac{2}{2}+\frac{2\cdot 1\cdot 1\cdot 3}{2\cdot 4\cdot 6}+\frac{2\cdot 1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}+\cdots.$$

Putting $\frac{2\tilde{\tilde{k}}\pi}{\lambda} = \frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}$ in (3.28), we have

$$\frac{2(\tilde{\tilde{k}} - \tilde{k}_0)\pi}{\lambda} + \frac{2(\tilde{k}' - \tilde{\tilde{k}})\pi}{\lambda}$$

$$= \frac{2\lambda^{-\alpha}}{\cos\left(\frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right) + \frac{\theta_0}{\lambda^{1-\alpha} - 1}}$$

$$+ \frac{2 \cdot 1 \cdot 1 \cdot 3 \cdot 2^3 \sin^2\left(\frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right)\lambda^{-3\alpha}}{2 \cdot 4 \cdot 6\left(\cos\left(\frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right) + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)^5} + \cdots$$

$$= \frac{2\lambda^{-\alpha}}{\sqrt{2}\lambda^{\frac{\alpha}{2}}} + \frac{2 \cdot 1 \cdot 1 \cdot 3\lambda^{-3\alpha}}{2 \cdot 4 \cdot 6\left(\sqrt{2\lambda^{\frac{\alpha}{2}}}\right)^3} + \frac{2 \cdot 1 \cdot 1 \cdot 3\left(1 - \sin^2\left(\frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right)\right)}{2 \cdot 4 \cdot 6\left(\sqrt{2}\lambda^{\frac{\alpha}{2}}\right)^3} + \cdots$$

$$= \sqrt{2}\lambda^{-\frac{\alpha}{2}}\left[\frac{2}{2} + \frac{2 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} + \cdots + \left(\frac{2 \cdot 1 \cdot 1 \cdot 3\left(1 - \sin^2\left(\frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right)\right)}{2 \cdot 4 \cdot 6} + \cdots\right)\right].$$

We estimate the following term:

$$\frac{2 \cdot 1 \cdot 1 \cdot 3\left(1 - \sin^2\left(\frac{\pi}{2} + \frac{\theta_0}{\lambda^{1-\alpha} - 1} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right)\right)}{2 \cdot 4 \cdot 6}$$

$$+\frac{2\cdot 1\cdot 1\cdot 3\cdot 5\cdot 7\left(1-\sin^4\left(\frac{\pi}{2}+\frac{\theta_0}{\lambda^{1-\alpha}-1}-\frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right)\right)}{2\cdot 4\cdot 6\cdot 8\cdot 10}+\cdots$$

$$=\frac{2\cdot 1\cdot 1\cdot 3\left(1-\sin^2\left(\frac{\pi}{2}+\frac{\theta_0}{\lambda^{1-\alpha}-1}-\frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right)\right)}{2\cdot 4\cdot 6}$$

$$+\cdots+\frac{2\cdot 1\cdot 1\cdot 3\cdots(4n-1)\left(1-\sin^{2n}\left(\frac{\pi}{2}+\frac{\theta_0}{\lambda^{1-\alpha}-1}-\frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right)\right)}{2\cdot 4\cdot 6\cdots(4n+2)}$$

$$+\frac{2\cdot 1\cdot 1\cdots(4n+3)\left(1-\sin^{2(n+1)}\left(\frac{\pi}{2}+\frac{\theta_0}{\lambda^{1-\alpha}-1}-\frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right)\right)}{2\cdot 4\cdots(4n+6)}+\cdots$$

$$=\left(1-\sin\left(\frac{\pi}{2}+\frac{\theta_0}{\lambda^{1-\alpha}-1}-\frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right)\right)\tau(2+4+\cdots+2n)+\frac{2\cdot 1\cdots(4n+3)2\tau'}{2\cdots(4n+6)}+\cdots$$

$$=\frac{1}{2}\left(\frac{\theta_0}{\lambda^{1-\alpha}-1}-\frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right)^2 n(n-1)\tau+\frac{2\cdot 1\cdots(4n+3)2\tau'}{2\cdots(4n+6)}+\cdots.$$

Taking $n=\left[\dfrac{1}{\sqrt{\left|\dfrac{\theta_0}{\lambda^{1-\alpha}}-\dfrac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right|}}\right]$, using the known expansion of $\sqrt{2}$, we have

$$\frac{2(\tilde{k}''-\tilde{k}')\pi}{\lambda}+\frac{2(\tilde{k}'-\tilde{k})\pi}{\lambda}$$

$$=\sqrt{2}\lambda^{-\frac{\alpha}{2}}\left[\left(\frac{2}{2}+\frac{2\cdot 1\cdot 1\cdot 3}{2\cdot 4\cdot 6}+\cdots\right)-\left(\sqrt{\left|\frac{\theta_0}{\lambda^{1-\alpha}-1}-\frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}\right|}+\varepsilon(n)\right)\right]=2\lambda^{-\frac{\alpha}{2}}.$$

Putting $\frac{2\tilde{k}_1\pi}{\lambda} = \frac{\pi}{2} - \frac{\theta_0}{\lambda^{1-\alpha} - 1} - \frac{\sqrt{2}}{\lambda^{\frac{\alpha}{2}}}$ in (3.30) repeating above arguments, we have

$$\frac{2(\tilde{k}_2 - \tilde{k}_1)\pi}{\lambda} + \frac{2(\tilde{k}_1 - \tilde{k}_0)\pi}{\lambda} = 2\lambda^{-\frac{\alpha}{2}}.$$

Ultimately, we obtain the value of intersection number

$$I\left(\frac{2\tilde{k}}{\lambda}\right) = [(\tilde{k}'' - \tilde{k}') + (\tilde{k}' - \tilde{k}) + (\tilde{k}_2 - \tilde{k}_1) + (\tilde{k}_1 - \tilde{k}_0)] = \frac{2}{\pi}\lambda^{1-\frac{\alpha}{2}}.$$

This result coincides with Theorem 3.5 (3.21), but two proofs come from total different points of view.

Lemma 3.7. *Let* $W(x) = \sum_{i=1}^{\infty} \lambda^{-i\alpha} \sin \lambda^i \pi x$ *be the Weierstrass function and let* Γ_{w_n} *be any lever n sine-like curve. Then we have*

$$\tilde{\mu}^{2-\alpha}_{n,\,n+1}(\Gamma_{w_n}) = \left(\frac{2}{\pi}\arccos\frac{2}{\pi} + 1 - \sqrt{1 - \left(\frac{2}{\pi}\right)^2}\right)\lambda^{-\alpha}. \tag{3.31}$$

Proof. Let S be a n-square with integer phase $2k'$ in the lever n sine-like curve Γ_{w_n}. Note that the pace $\frac{2\pi}{\lambda}\left(\cos\frac{2k'\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)\lambda^{-n\alpha}$ is much larger than the length λ^{-n} for $2k' \le 2\tilde{k}$. Therefore, a lever $(n+1)$ sine-like curve intersects with the bottom (top) going through top(bottom) except at most four. By Theorem 3.6, the number of lever $(n+1)$ sine-like curves which intersects with S is $I_{n+1}\left(\frac{2k'}{\lambda}\right)$ $= \frac{2\lambda^{1-\alpha}}{\pi\left(\cos\frac{2k'\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)}$ for $\frac{2k'}{\lambda} \in \left[\frac{2k_0}{\lambda}, \frac{2\tilde{k}}{\lambda}\right]$. Thus for large enough λ, we have

$$\tilde{\mu}^{2-\alpha}_{n+1}(\Gamma_{w_n} \cap S) = \frac{2\lambda^{1-\alpha} \bullet \lambda}{\pi\left(\cos\frac{2k'\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha} - 1}\right)}\lambda^{-(n+1)(2-\alpha)}$$

$$= \frac{2}{\pi\left(\cos\frac{2k'\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha}-1}\right)}\lambda^{-n(2-\alpha)}.$$

Let l be an integer such that $\cos\frac{2l\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha}-1} \geq \frac{2}{\pi} > \cos\frac{2(l+1)\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha}-1}$, $0 < l < \frac{\lambda}{2}$. Without loss of generality, we may suppose $\cos\frac{2l\pi}{\lambda} + \frac{\theta_0}{\lambda^{1-\alpha}-1} = \frac{2}{\pi}$. So we obtain that

$$\tilde{\mu}_{n+1}^{2-\alpha}(\Gamma_{w_n} \cap S) \geq \tilde{\mu}_n^{2-\alpha}(\Gamma_{w_n} \cap S) \text{ if } 2k' \geq 2l,$$

$$\tilde{\mu}_{n+1}^{2-\alpha}(\Gamma_{w_n} \cap S) < \tilde{\mu}_n^{2-\alpha}(\Gamma_{w_n} \cap S) \text{ if } 2k' < 2l.$$

Therefore,

$$\tilde{\mu}_{n,n+1}^{2-\alpha}(\Gamma_{w_n}) = {}^{\#}A\lambda^{-n(2-\alpha)} + {}^{\#}B\lambda^{-(n+1)(2-\alpha)}, \tag{3.32}$$

where

$${}^{\#}A = \{S : S \text{ is } n\text{-square with integer phase } 2k' \text{ in } \Gamma_{w_n} \text{ and } 2k' \geq 2l\},$$

${}^{\#}B = \left\{S : S \text{ is a } (n+1)\text{-}\textit{square} \text{ below the horizontal line } y_0 = W\left(\frac{2j}{\lambda^n} + \frac{2l}{\lambda^{n+1}}\right)\right\}$ in Γ_{w_n} and $S \cap \Gamma_{w_n} \neq \varnothing$.

By Theorem 3.3, we have

$$W\left(\frac{2j}{\lambda^n} + \frac{2k_u}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{2l}{\lambda^{n+1}}\right)$$

$$= W\left(\frac{2j}{\lambda^n} + \frac{2k_u}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{\frac{1}{2}\lambda}{\lambda^{n+1}}\right) + W\left(\frac{2j}{\lambda^n} + \frac{\frac{1}{2}\lambda}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{2l}{\lambda^{n+1}}\right)$$

$$= \left\{1 - \sqrt{1 - \left(\frac{2}{\pi}\right)^2} + \frac{\left(\frac{\pi}{2} - \frac{2l\pi}{\lambda}\right)\theta_0}{\lambda^{1-\alpha}-1} + \frac{1}{2}\left(\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2\right\}\lambda^{-n\alpha}.$$

Thus,

$$^{\#}A = \left[\frac{\left\{1-\sqrt{1-\left(\frac{2}{\pi}\right)^2}+\frac{\left(\frac{\pi}{2}-\frac{2l\pi}{\lambda}\right)\theta_0}{\lambda^{1-\alpha}-1}+\frac{1}{2}\left(\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2\right\}\lambda^{-n\alpha}}{\lambda^n}\right]$$

$$= \left[\left\{1-\sqrt{1-\left(\frac{2}{\pi}\right)^2}+\frac{\left(\frac{\pi}{2}-\frac{2l\pi}{\lambda}\right)\theta_0}{\lambda^{1-\alpha}-1}+\frac{1}{2}\left(\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2\right\}\lambda^{n(1-\alpha)}\right].$$

By power law, we have

$$^{\#}A = \left(1-\sqrt{1-\left(\frac{2}{\pi}\right)^2}\right)\lambda^{n(1-\alpha)}. \tag{3.33}$$

We are now ready to compute $^{\#}B$. Let $B_1 = \{S : S \in B, \text{ and } \Gamma_{w_n} \cap \Gamma_{y_0} \neq \varnothing\}$ and $B_2 = B - B_1$. Clearly, we have $B = B_1 \cup B_2$. Intuitively, We see that

$$B_2 = \{S : S \text{ is a } (n+1)\textit{-square} \text{ and } S \subset \Gamma_{w_{n,j}}(j \le 2k_l) \text{ or } S \subset \Gamma_{w_{n,\lambda-1}}(j \ge 2k_l)\};$$

that is, B_2 includes entire lever $(n+1)$ sine-like curve. We can make the B_1 includes entire lever $(n+1)$ sine-like curves too. Since we can exchange the part of intercepts which are above line $y = y_0$ and the part of intercepts which are below the $y = y_0$. Because the sequences of intercepts of up(down) lever $(n+1)$ sine-like curve divided by the line $y = y_0$ are approximately arithmetic sequences and the number $k_l'' - k_{0,l}$ almost equal to $l - k_l$, the number $k_{2,l} - k_{1,l}$ almost equal to $k_{1,l} - k_{0,l}$. Indeed, according to Theorem 3.6, for large λ, we have

$$\frac{2(k_l''-l)\pi}{\lambda}+\frac{2(l-k_l)\pi}{\lambda}$$

$$= 2\left(\cos\frac{2l\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)-\left[\sqrt{\left(\cos\frac{2l\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2+2\sin\frac{2l\pi}{\lambda}\lambda^{-\alpha}}\right.$$

$$+\sqrt{\left(\cos\frac{2l\pi}{\lambda}+\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2-2\sin\frac{2l\pi}{\lambda}\lambda^{-\alpha}}\,\Bigg]=2\tau\lambda^{-2\alpha},$$

$$\frac{2(k_{2,l}-k_{1,l})\pi}{\lambda}-\frac{2(k_{1,l}-k_{0,l})\pi}{\lambda}$$

$$=2\left(\cos\frac{2k_{1,l}\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)-\Bigg[\sqrt{\left(\cos\frac{2k_{1,l}\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2+2\sin\frac{2k_{1,l}\pi}{\lambda}\lambda^{-\alpha}}$$

$$+\sqrt{\left(\cos\frac{2k_{1,l}\pi}{\lambda}-\frac{\theta_0}{\lambda^{1-\alpha}-1}\right)^2-2\sin\frac{2k_{1,l}\pi}{\lambda}\lambda^{-\alpha}}\,\Bigg]=2\tau\lambda^{-2\alpha}.$$

Thus, noting Theorem 3.4, we show that

$$^{\#}B=(2l+2k_{1,l})\lambda^{(n+1)(1-\alpha)}+2\tau\lambda^{1-2\alpha}\lambda^{(n+1)(1-\alpha)}$$

$$=\left(\frac{2}{\pi}\arccos\frac{2}{\pi}-\frac{\tau\theta_0}{\lambda^{1-\alpha}-1}+2\tau\lambda^{-2\alpha}\right)\lambda\lambda^{(n+1)(1-\alpha)}.$$

Again, by power law, we already show

$$^{\#}B=\frac{2}{\pi}\arccos\frac{2}{\pi}\lambda\lambda^{(n+1)(1-\alpha)}. \tag{3.34}$$

Finally, by replacing (3.33), (3.34) into (3.32), we obtain

$$\tilde{\mu}^{2-\alpha}_{n,n+1}(\Gamma_{w_n})=\left(1-\sqrt{1-\left(\frac{2}{\pi}\right)^2}\right)\lambda^{n(1-\alpha)}\bullet\lambda^{-n(2-\alpha)}$$

$$+\frac{2}{\pi}\arccos\frac{2}{\pi}\lambda\lambda^{(n+1)(1-\alpha)}\bullet\lambda^{-(n+1)(2-\alpha)}$$

$$=\left(\frac{2}{\pi}\arccos\frac{2}{\pi}+1-\sqrt{1-\left(\frac{2}{\pi}\right)^2}\right)\lambda^{-n}.$$

We obtain (3.32). We may call $2l$ *turning integer phase*.

We have solved the problem: find a minimum cover of a lever n sine-like curve Γ_{w_n} by n-squares and $(n+1)$-squares. It seems natural to consider the problem, find a minimum cover of the graph $S \cap \Gamma_{w_n}$ by $(n+1)$-squares and $(n+2)$-squares. Since the intercept sequences of lever $(n+1)$ sine-like curves divided by line $y = h$ are approximately arithmetic sequences; therefore Lemma 3.8 provides a similar result with Lemma 3.7 for *n*-square S_n in the lever *n* sine-like curve Γ_{w_n}.

Lemma 3.8. *Let* $W(x) = \sum_{i=1}^{\infty} \lambda^{-i\alpha} \sin \lambda^i \pi x$ *be the Weierstrass function. Let* S_n *be a n-square with integer phase* $2k'$ *in a lever n sine-like curve* Γ_{w_n}, $2k_0 \le 2k' \le 2\tilde{k}$. *Then*

$$\tilde{\mu}^{2-\alpha}_{n+1,\,n+2}(\Gamma_{w_n} \cap S_n) = \left(\frac{2}{\pi}\arccos\frac{2}{\pi} + 1 - \sqrt{1-\left(\frac{2}{\pi}\right)^2}\right)\frac{2\lambda^{-n(2-\alpha)}}{\pi\cos\frac{2k'\pi}{\lambda}}$$

$$= \left(\frac{2}{\pi}\arccos\frac{2}{\pi} + 1 - \sqrt{1-\left(\frac{2}{\pi}\right)^2}\right)\tilde{\mu}^{2-\alpha}_{n+1}(\Gamma_{w_n} \cap S_n). \quad (3.35)$$

Proof. By Theorem 3.5, the intersection set *A* is in typical case. We consider first quarter A_1, other cases can be treated in the same way. Without loss of generality, we may suppose that $h = W\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right)$, where *h* is the distance between the bottom of the square *S* and the *X*-axis. Since it now allows to cover the graph by $(n+2)$-squares, we have to know the integer phase of the horizontal line $y = h$ in the lever $(n+1)$ sine-like curve $\Gamma_{w_{n,\,2(k'-l)}}$. Let $2S_l$ be the integer phase of the horizontal line $y = W\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right)$ in the lever $(n+1)$ sine-like curve $\Gamma_{w_{n,\,2(k'-l)}}$. Then

$$W\left(\frac{2j}{\lambda^n} + \frac{2(k'-l)}{\lambda^{n+1}} + \frac{2S_l}{\lambda^{n+2}}\right) \le W\left(\frac{2j}{\lambda^n} + \frac{2k'}{\lambda^{n+1}}\right) < W\left(\frac{2j}{\lambda^n} + \frac{2(k'-l)}{\lambda^{n+1}} + \frac{2(S_l+1)}{\lambda^{n+2}}\right).$$

Thus, by Lemma 3.2 and Theorem 3.6, we have

$$\left(1 + \frac{\frac{1}{2}\pi\tau'\theta}{\lambda^{1-\alpha} - 1}\right)\sin\frac{2S_l\pi}{\lambda} = l\left(\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)}\right) + \frac{\tau}{2}\frac{(2\pi l)^2}{\lambda^{2-\alpha}}.$$

Consequently,

$$\sin\frac{2S_l\pi}{\lambda}=\left[l\left(\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}}+\frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)}\right)+\frac{\tau}{2}\frac{(2\pi l)^2}{\lambda^{2-\alpha}}\right]\left(1-\frac{\frac{\pi}{2}\theta'}{\lambda^{1-\alpha}}\right).$$

By power law, we reduce to

$$\sin\frac{2S_l\pi}{\lambda}=l\left(\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}}+\frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)}\right)+\frac{\tau}{2}\frac{(2\pi l)^2}{\lambda^{2-\alpha}}. \tag{3.36}$$

Next, we consider the measure of the graph of function $W(x)$ in the $\Gamma_{w_{n,2(k'-l)}}\cap S_n$, $l=0, 1, 2, \ldots, k'-k$. Let $\tilde{\mu}_1(\tilde{\mu}_2)$ be the measure of the graph $W(x)$ in $\Gamma_{w_{n,2(k'-l)}}$ using $(n+1)$-squares $((n+2)\text{-squares})$. Clearly, we have

$$\tilde{\mu}_1=\lambda\bullet\lambda^{-(n+1)(2-\alpha)}. \tag{3.37}$$

In view of Theorem 3.6 and (3.36), we have

$$I_{n+2}\left(\frac{2S_l}{\lambda}\right)=\frac{2\lambda^{1-\alpha}}{\pi\cos\arcsin l\left(\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}}+\frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)}+\frac{\tau}{2}\frac{(2\pi l)^2}{\lambda^{2-\alpha}}\right)}.$$

This implies

$$\tilde{\mu}_2=\frac{2\lambda^{1-\alpha}}{\pi\cos\arcsin l\left(\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}}+\frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)}+\frac{\tau}{2}\frac{(2\pi l)^2}{\lambda^{2-\alpha}}\right)}\bullet\lambda^2\bullet\lambda^{-(n+2)(2-\alpha)}. \tag{3.38}$$

By comparing (3.37) and (3.38), we conclude that

$$\text{if }\frac{2}{\pi\cos\arcsin l\left(\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}}+\frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)}+\frac{\tau}{2}\frac{(2\pi l)^2}{\lambda^{2-\alpha}}\right)}<1,\text{ then }\tilde{\mu}_2<\tilde{\mu}_1;$$

otherwise,

$$\text{if } \frac{2}{\pi\cos\arcsin l\left(\dfrac{2\pi\cos\dfrac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}}+\dfrac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)}+\dfrac{\tau}{2}\dfrac{(2\pi l)^2}{\lambda^{2-\alpha}}\right)}\geq 1, \text{ then } \tilde{\mu}_2\geq\tilde{\mu}_1.$$

Let l_1 be an integer such that

$$\cos\arcsin(l_1-1)\left(\frac{2\pi\cos\dfrac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}}+\frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)}+\frac{\tau(2\pi(l_1-1))^2}{2\lambda^{2-\alpha}}\right)$$

$$>\frac{2}{\pi}\geq\cos\arcsin l_1\left(\frac{2\pi\cos\dfrac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}}+\frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)}+\frac{\tau(2\pi l_1)^2}{2\lambda^{2-\alpha}}\right).$$

Without loss of generality, we may suppose

$$\cos\arcsin l_1\left(\frac{2\pi\cos\dfrac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}}+\frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)}+\frac{\tau(2\pi l_1)^2}{2\lambda^{2-\alpha}}\right)=\frac{2}{\pi}.$$

In other words, we may write

$$l_1\left(\frac{2\pi\cos\dfrac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}}+\frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)}+\frac{\tau(2\pi l_1)^2}{2\lambda^{2-\alpha}}\right)=\sqrt{1-\left(\frac{2}{\pi}\right)^2}. \tag{3.39}$$

Thus, we have

$$\tilde{\mu}^{2-\alpha}_{n+1,n+2}(\Gamma_W\cap S)={}^{\#}A_{1,1}\lambda^{-(n+1)(2-\alpha)}+{}^{\#}A_{1,2}\lambda^{-(n+1)(2-\alpha)}, \tag{3.40}$$

where

$$A_{1,1}=\{S_{n+2}:S_{n+2}\text{ is a }(n+2)\textit{-square } S_{n+2}\subset\Gamma_{W_{n,2(k'-l)}}\text{ and}$$

$$S_{n+2}\cap\Gamma_W\neq\varnothing,\ \ 1=1,2,\ldots,l_1\}$$

and

$$A_{1,2} = \{S_{n+1} : S_{n+1} \text{ is a } (n+1)\text{-}square\ S_{n+1} \subset \Gamma_{W_{n,2(k'-l)}}$$

$$\text{and } S_{n+1} \cap \Gamma_w \neq \varnothing,\ \ l = l_1 + 1, \ldots, k' - k\}.$$

We want to use the definite integral method to compute $^{\#}A_{1,1}$. From Theorem 3.6, we have

$$^{\#}A_{1,1} = \lambda^2 \sum \frac{2\lambda^{1-\alpha}}{\pi \cos \arcsin l \left(\dfrac{2\pi \cos \frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \dfrac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha} - 1)} + \dfrac{\tau(2\pi l)^2}{2\lambda^{2-\alpha}} \right)}. \tag{3.41}$$

We rewrite the right hand side of above formula as a special sum, which can be treated as Riemann sum

$$= \lambda^2 \sum \frac{2\lambda^{1-\alpha}\left\{ l\left[\dfrac{2\pi \cos \frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \dfrac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \dfrac{\tau(2\pi l)^2}{\lambda^{2-\alpha}} \right] - \left(l - 1\left[\dfrac{2\pi \cos \frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \dfrac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \dfrac{\tau(2\pi(l-1))^2}{\lambda^{2-\alpha}} \right]\right)\right\}}{\pi \cos \arcsin l \left[\dfrac{2\pi \cos \frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \dfrac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \dfrac{\tau(2\pi l)^2}{\lambda^{2-\alpha}} \right] \cdot \left\{ l\left[\dfrac{2\pi \cos \frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \dfrac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \dfrac{\tau(2\pi l)^2}{\lambda^{2-\alpha}} \right] - \left((l-1)\left[\dfrac{2\pi \cos \frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \dfrac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \dfrac{\tau[2\pi(l-1)]^2}{\lambda^{2-\alpha}} \right]\right)\right\}}$$

$$= \frac{2\lambda^{3-\alpha}}{\pi}\sum \frac{l\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l)^2}{\lambda^{2-\alpha}}\right] - \left((l-1)\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi(l-1))^2}{\lambda^{2-\alpha}}\right]\right)}{\cos\arcsin l\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l)^2}{2\lambda^{2-\alpha}}\right] \cdot \left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{(2\pi)^2[\tau' l^2 - \tau'(l-1)^2]}{\lambda^{2-\alpha}}\right]}.$$

Note that

$$\frac{1}{A+\varepsilon} = \frac{1}{A} - \frac{\varepsilon}{A(A+\varepsilon)},$$

$$^{\#}A_{1,1} = \frac{2\lambda^{3-\alpha}}{\pi}\sum_{l=1}^{l_1} \frac{l\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l)^2}{\lambda^{2-\alpha}}\right] - (l-1)\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi(l-1))^2}{\lambda^{2-\alpha}}\right]}{\left(\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)}\right) \cdot \cos\arcsin l\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l)^2}{2\lambda^{2-\alpha}}\right]}$$

$$= \sum_{l=1}^{l_1} \frac{\begin{array}{l}\left(l\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l)^2}{2\lambda^{2-\alpha}}\right] \right. \\ \left. \quad - (l-1)\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau'(2\pi(l-1))^2}{2\lambda^{2-\alpha}}\right]\right) \\ \quad \cdot \frac{(2\pi)^2[l^2-(l-1)^2]}{\lambda^{2-\alpha}}\end{array}}{\begin{array}{l}\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)}\right] \\ \quad \cos\arcsin l\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l)^2}{2\lambda^{2-\alpha}}\right] \\ \quad \cdot \left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l)^2}{2\lambda^{2-\alpha}}\right]\end{array}}.$$

For large enough λ, we can consider the following sum as Riemann sum. Then we have

$$\sum_{l=1}^{l_1} \frac{\begin{array}{l} l\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l)^2}{\lambda^{2-\alpha}}\right] \\ \quad - (l-1)\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l)^2}{\lambda^{2-\alpha}}\right]\end{array}}{\cos\arcsin l\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l)^2}{2\lambda^{2-\alpha}}\right]}$$

$$= \int_0^{l_1\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l)^2}{2\lambda^{2-\alpha}}\right]} \frac{dx}{\cos\arcsin}$$

$$= \int_0^{\arcsin l_1\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l_1)^2}{2\lambda^{2-\alpha}}\right]} dy \qquad (y = \arcsin x)$$

$$= \arcsin l_1\left[\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l_1)^2}{2\lambda^{2-\alpha}}\right]$$

$$= \arccos\frac{2}{\pi}. \tag{3.42}$$

It is routine to check

$$\frac{\frac{(2\pi)^2[\tau l^2 - (l-1)^2]}{\lambda^{2-\alpha}}}{\frac{2\pi\cos\frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l)^2}{\lambda^{2-\alpha}}} \to 0 \text{ as } \lambda \to \infty.$$

Combining (3.43) and (3.44), we have

$$^{\#}A_{1,1} = \frac{\lambda^2 \frac{2}{\pi}\lambda^{2(1-\alpha)} \arccos\frac{2}{\pi}}{2\pi\cos\frac{2k'\pi}{\lambda} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}-1}}. \tag{3.43}$$

Noting Theorem 3.6 and (3.39), we have

$$^{\#}A_{1,2} = \lambda\sum_{l=l_1}^{k'-k} 1$$

$$= \frac{\lambda \bullet \lambda^{1-\alpha}}{2\pi\cos\frac{2k'\pi}{\lambda} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}-1}}\left(1 - \sqrt{1-\left(\frac{2}{\pi}\right)^2}\right). \tag{3.44}$$

For large λ, we can express (3.43), (3.44) at least for most of $\left[\frac{2k_0}{\lambda}, \frac{2\tilde{k}}{\lambda}\right]$,

$$^{\#}A_{1,1} = \frac{\lambda^2 \frac{2}{\pi}\lambda^{2(1-\alpha)} \arccos\frac{2}{\pi}}{2\pi\cos\frac{2k'\pi}{\lambda}},$$

$$^{\#}A_{1,2} = \frac{\lambda^{2-\alpha}}{2\pi\cos\frac{2k'\pi}{\lambda}}\left(1-\sqrt{1-\left(\frac{2}{\pi}\right)^2}\right).$$

Same results hold for A_2, A_3, A_4. By replacing (3.43), (3.44) to (3.40), we obtain

$$\begin{aligned}\tilde{\mu}^{2-\alpha}_{n+1,n+2}(\Gamma_W \cap S) &= \frac{2}{\pi\cos\frac{2k'\pi}{\lambda}}\Bigg[\frac{2}{\pi}\arccos\frac{2}{\pi}\lambda^2 \bullet \lambda^{2(1-\alpha)} \bullet \lambda^{-(n+2)(2-\alpha)} \\ &\quad + \left(1-\sqrt{1-\left(\frac{2}{\pi}\right)^2}\right)\lambda^{2-\alpha} \bullet \lambda^{-(n+1)(2-\alpha)}\Bigg] \\ &= \frac{2}{\pi\cos\frac{2k'\pi}{\lambda}}\left(\frac{2}{\pi}\arccos\frac{2}{\pi}+1-\sqrt{1-\left(\frac{2}{\pi}\right)^2}\right)\lambda^{-n(2-\alpha)}.\end{aligned}$$

The desire follows.

Remark 3.9. Each n-square S has a minimum $(n+1, n+2)$ cover, all the minimum cover $(n+1, n+2)$ cover of n-squares S are similar. Rephrase the formulas of Lemma 3.7, Lemma 3.8,

$$\tilde{\mu}^{2-\alpha}_{n,n+1}(\Gamma_{w_n}) = \left(\frac{2}{\pi}\arccos\frac{2}{\pi}+1-\sqrt{1-\left(\frac{2}{\pi}\right)^2}\right)\tilde{\mu}^{2-\alpha}_{n}(\Gamma_{w_n}),$$

$$\tilde{\mu}^{2-\alpha}_{n+1,n+2}(\Gamma_w \cap S) = \left(\frac{2}{\pi}\arccos\frac{2}{\pi}+1-\sqrt{1-\left(\frac{2}{\pi}\right)^2}\right)\tilde{\mu}^{2-\alpha}_{n+1}(\Gamma_w \cap S).$$

Above formulas on form and content are alike. Since each n-square is included in some rectangle, we may view first formula as a global property and second formula as a local property. This means that the global property and local property are comparable.

From Lemma 3.8, we know that

$$\tilde{\mu}^{2-\alpha}_{n+1,n+2}(\Gamma_W \cap S) = \frac{2\left(\frac{2}{\pi\arccos\frac{2}{\pi}}+1-\sqrt{1-\left(\frac{2}{\pi}\right)^2}\right)}{\pi} \bullet \lambda^{-n(2-\alpha)}$$

for the n-square S with integer phase $2k'$. This gives us a way to find integer phase for the minimum cover of graph Γ_{W_n} by n-squares, $(n+1)$-squares and $(n+2)$-squares.

Lemma 3.10. *Let* $W(x) = \sum_{i=1}^{\infty} \lambda^{-i\alpha} \sin \lambda^i \pi x$ *be the Weierstrass function. Let* Γ_{W_n} *be any lever n sine-like curve. Then we have*

$$\tilde{\mu}^{2-\alpha}_{n,n+2}(\Gamma_{W_n}) = \left(\frac{2r_1}{\pi} \arccos \frac{2r_1}{\pi} + 1 - \sqrt{1 - \left(\frac{2r_1}{\pi}\right)^2} \right) \lambda^{-n}, \tag{3.45}$$

where $r_1 = \frac{2}{\pi} \arccos \frac{2}{\pi} + 1 - \sqrt{1 - \left(\frac{2}{\pi}\right)^2}$.

Proof. Let integer phase $2l_1$ be defined in Lemma 3.7, by Lemma 3.8, we have

$$\tilde{\mu}^{2-\alpha}_{n+1,n+2}(\Gamma_{W_n} \cap S_{2l_1}) = \frac{\left(\frac{2}{\pi} \arccos \frac{2}{\pi} + 1 - \sqrt{1 - \left(\frac{2}{\pi}\right)^2} \right) 2}{\pi \cos \frac{2l_1}{\lambda}} \lambda^{-n(2-\alpha)}$$

$$= \frac{r_1 2}{\pi \cos \frac{2l_1 \pi}{\lambda}} \lambda^{-n(2-\alpha)}$$

$$= r_1 \lambda^{-n(2-n)}.$$

It will be shown in the following proposition $0 < r_1 < 1$. This implies that the measure of $\Gamma_W \cap S_{2l_1}$ covered by $(n+1)$-squares and $(n+2)$-squares will be less than the measure of $\Gamma_W \cap S_{2l_1}$ covered by n-square.

Thus, we can move up integer phase to include more n-squares covered by $(n+1)$-squares and $(n+2)$-squares. Clearly, there is $2l_2$ such that $\cos \frac{2(l_2 - 1)\pi}{\lambda} > \frac{2r_1}{\pi} \geq \cos \frac{2l_2 \pi}{\lambda}$. Without loss of generality, we may suppose that $\cos \frac{2l_2 \pi}{\lambda} = \frac{2r_1}{\pi}$; that is $\frac{2l_2 \pi}{\lambda} = \arccos \frac{2r_1}{\pi}$ Let

$$A_1 = \{S : S \text{ are } n\text{-squares with integer phase } 2k',\ 2k' \geq 2l_2\},$$

$$A_2 = \{A(S) : A(S) \text{ is a minimum } (n+1,\ n+2) \text{ cover of a } n\text{-square } S$$

$$\text{with integer phase } 2k',\ 2k' \leq 2\tilde{k}\}$$

and $A = A_1 \cup A_2$, clearly, A is a $(n+1,\ n+2)$ cover of Γ_{W_n}. Note that in the proof of Lemma 3.7, the graph $\Gamma_W S's$ which is below integer phase $2l_2$ is covered by $(n+1)$-squares and $(n+2)$-squares. By Lemma 3.8, we may write

$$\tilde{\mu}_{n+1,\,n+2}^{2-\alpha}(\Gamma_W \cap S) = \frac{2\left(\frac{2}{\pi}\arccos\frac{2}{\pi} + 1 - \sqrt{1 - \left(\frac{2}{\pi}\right)^2}\right)}{\pi\cos\frac{2k'\pi}{\lambda}} \bullet \lambda^{-(2-\alpha)}$$

$$= \frac{2r_1}{\pi\cos\frac{2k'\pi}{\lambda}} \bullet \lambda^{-n(2-\alpha)}.$$

Using $\tilde{\mu}_{n+1,\,n+2}^{2-\alpha}(\Gamma_W \cap S) = \frac{2r_1}{\pi\cos\frac{2k'\pi}{\lambda}} \bullet \lambda^{-(2-\alpha)}$ instead of $\tilde{\mu}_{n+1}^{2-\alpha}(\Gamma_W \cap S) = \frac{2}{\pi\cos\frac{2k'\pi}{\lambda}} \bullet \lambda^{-(2-\alpha)}$ in Lemma 3.7, we may copy the proof of Lemma 3.7. Thus, we get

$$\sum_{S \in A_2} |S|^{2-\alpha} = (2l_2 + 2k_{l_2})r_1\lambda^{-(n+1)} = \frac{2r_1}{\pi}\arccos\frac{2r_1}{\pi}\lambda \bullet \lambda^{-(n+1)}$$

$$= \frac{2r_1}{\pi}\arccos\frac{2r_1}{\pi}\lambda^{-n}$$

and $\sum_{S \in A_2} |S|^{2-\alpha} = \left(1 - \sqrt{1 - \left(\frac{2r_1}{\pi}\right)^2}\right)\lambda^{-n}$. Let $\tilde{\mu}$ be the measure of cover A. Finally, we have

$$\tilde{\mu} = \left(\frac{2r_1}{\pi}\arccos\frac{2r_1}{\pi} + 1 - \sqrt{1 - \left(\frac{2r_1}{\pi}\right)^2}\right)\lambda^{-n}.$$

Clearly, the cover A is the minimum cover of the graph Γ_{W_n}. As above, we call the integer phase $2l_1,\ 2l_2$ *turning integer phase*. The proof completes.

Remark. Each lever n sine-like curve Γ_{w_n} has a minimum $(n,\ n+2)$ cover, all $(n,\ n+2)$ minimum covers A's are quasi-similar. This result may be extended, but first we show a proposition.

We begin by setting $r_1 = \frac{2}{\pi}\arccos\frac{2}{\pi} + 1 - \sqrt{1 - \left(\frac{2}{\pi}\right)^2}$. Assume that r_{n-1} is constructed. We define $r_n = \frac{2r_{n-1}}{\pi}\arccos\frac{2r_{n-1}}{\pi} + 1 - \sqrt{1 - \left(\frac{2}{\pi}\right)^2}$, $n = 2, 3, \ldots,$ r_n is called *contraction coefficient*.

Proposition 3.11. *Let* r_n *be a contraction coefficient. Then the sequence* $\{r_n\}_{n=1}^{\infty}$ *is a strictly decreasing sequence. Moreover, we have*

$$\lim_{n\to\infty} r_n = 0.$$

Proof. Let $f(x) = \frac{2}{\pi}x\arccos\frac{2}{\pi}x + 1 - \sqrt{1 - \left(\frac{2x}{\pi}\right)^2}$, it is routine to show

$$f'(x) = \frac{2}{\pi}\arccos\frac{2}{\pi}x.$$

By mean value theorem, we have

$$\frac{2}{\pi}x\arccos x + 1 - \sqrt{1 - \left(\frac{2x}{\pi}\right)^2} = \left(\frac{2}{\pi}\arccos\frac{2}{\pi}\xi\right)x, \tag{3.46}$$

where $0 < \xi < x$. Therefore, we have

$$\frac{\frac{2}{\pi}x\arccos\frac{2}{\pi} + 1 - \sqrt{1 - \left(\frac{2}{\pi}\right)^2}}{x} < 1. \tag{3.47}$$

Let $x = r_{n-1}$. Then

$$\frac{r_n}{r_{n-1}} = \frac{\frac{2}{\pi}r_{n-1}\arccos\frac{2r_{n-1}}{\pi} + 1 - \sqrt{1 - \left(\frac{2r_{n-1}}{\pi}\right)^2}}{r_{n-1}} < 1.$$

This shows the sequence $\{r_n\}_{n=1}^{\infty}$ is strictly decreasing. This implies that $\lim_{n\to\infty} r_n = r_0$, $r_0 \geq 0$. We claim that $r_0 = 0$, otherwise $r_0 > 0$. Taking limit in (3.46), we have

$$r_0 = \frac{2r_0}{\pi}\arccos\frac{2r_0}{\pi} + 1 - \sqrt{1-\left(\frac{2r_0}{\pi}\right)^2}.$$

That is

$$\frac{\frac{2r_0}{\pi}\arccos\frac{2r_0}{\pi} + 1 - \sqrt{1-\left(\frac{2r_0}{\pi}\right)^2}}{r_0} = 1.$$

This contradicts to (3.47). The desire follows.

Lemma 3.12. *Let* $W(x) = \sum_{i=1}^{\infty} \lambda^{-i\alpha} \sin \lambda^i \pi x$ *be Weierstrass function and let* S_n *be a n-square with integer phase* $2k'$ *in lever n sine-like curve* Γ_{W_n}, $2k_0 \leq 2k' \leq 2\tilde{k}$. *Then*

$$\tilde{\mu}^{2-\alpha}_{n+1,\, n+1+m}(\Gamma_W \cap S_n) = r_m \tilde{\mu}^{2-\alpha}_{n+1}(\Gamma_W \cap S_n), \tag{3.48}$$

where $2l_m \leq 2\tilde{k}$.

Proof. We prove statement (3.46) by induction. By Lemma 3.10, the statement holds for $m = 1$. Now suppose statement (3.46) holds for $m = p$. Using induction suppose to the graph $\Gamma_W \cap S_{n+1}$, we have

$$\tilde{\mu}^{2-\alpha}_{n+1,\,(n+1)+p}(\Gamma_W \cap S_{n+1}) = r_p \tilde{\mu}^{2-\alpha}_{n+1}(\Gamma_W \cap S_{n+1}).$$

Using $r_p \tilde{\mu}^{2-\alpha}_{n+1}(\Gamma_W \cap S_{n+1})$ instead of $\tilde{\mu}^{2-\alpha}_{n+1}(\Gamma_W \cap S_{n+1})$ in Lemma 3.8, the intersection number $I_{n+1}\left(\frac{2S_l}{\lambda}\right)$ becomes

$$I_{n+1}\left(\frac{2S_l}{\lambda}\right) = \frac{2r_p \lambda^{1-\alpha}}{\pi \cos \arcsin l \left(\frac{2\pi \cos \frac{2k'\pi}{\lambda}}{\lambda^{1-\alpha}} + \frac{2\pi\theta_0}{\lambda^{1-\alpha}(\lambda^{1-\alpha}-1)} + \frac{\tau(2\pi l)^2}{2\lambda^{2-\alpha}} \right)}.$$

By copying the minimum proceed in the proof of Lemma 3.8, we get

$$\tilde{\mu}_{n,\,n+m}^{2-\alpha}(\Gamma_W \cap S_n) = \frac{2}{\pi\cos\frac{2k'\pi}{\lambda}}\left(\frac{2r_p}{\pi}\arccos\frac{2r_p}{\pi} + 1 - \sqrt{1-\left(\frac{2r_p}{\pi}\right)^2}\right)\lambda^{-n(2-\alpha)}$$

$$= r_{p+1}\tilde{\mu}_{n+1}^{2-\alpha}(\Gamma_W \cap S_n).$$

This proves the lemma.

Theorem 3.13. *Let* $W(x) = \sum_{i=1}^{\infty}\lambda^{-i\alpha}\sin\lambda^i\pi x$ *be the Weierstrass function. Then we have*

$$D_H(\Gamma_W) = 2 - \alpha \tag{3.49}$$

with large λ.

Proof. By Theorem 3.6, the intersection number is defined in

$$\left[\frac{2k_0}{\lambda},\ \frac{1}{2} + \frac{\theta_0}{\pi(\lambda^{1-\alpha} - 1)} - \frac{1}{\pi\lambda^{\frac{\alpha}{2}}}\right].$$

First, we will show there is a minimum cover for each lever n sine-like curve Γ_{w_n}. Moreover, we have estimation $\tilde{\mu}_{n,\,n+m}^{2-\alpha}(\Gamma_{w_n}) = r_m\lambda^{-n}$ for $m \le n_0$. Second, we will extend above estimation for any integer m. Finally, we prove the graph Γ_w is a $(2-\alpha)$*-set*.

In view of Lemma 3.12, the sequence $\{2l_i\}$ of turning integer phases is a monotone increasing sequence. Therefore, there is an integer phase $2l_N$ such that $2l_{N-1} < 2\tilde{k} \le 2l_N$. In other words, after the procedure runs N step, the minimum cover of the graph of Γ_{W_n} decays to $(n+1,\ n+1+N)$ cover of the graph of Γ_{W_n}. We claim that

$$\tilde{\mu}_{n,\,n+m}^{2-\alpha}(\Gamma_{W_n}) = r_m\lambda^{-n}. \tag{3.50}$$

We prove the statement (3.50) by induction. By Lemma 3.7, the statement (3.50) holds for $m = 1$, assume statement (3.50) holds for $m \leq p$.

That is,

$$\tilde{\mu}^{2-\alpha}_{n+1,\, n+1+p}(\Gamma_{W_{n+1}}) \geq \frac{\pi}{2}\lambda^{-\frac{\alpha}{2}}\lambda^{-(n+1)}.$$

Clearly, never n sine-like curve can convert as disjoint union of λ never $n+1$ sine-like curves. Therefore, we have

$$\tilde{\mu}^{2-\alpha}_{n,\, n+p+1}(\Gamma_{W_n}) \geq \lambda \bullet \frac{\pi}{2}\lambda^{-\frac{\alpha}{2}}\lambda^{-(n+1)}.$$

This proves (3.50).

In the following, in order to make the contraction process more clear, we need the following arguments. Let $\cos\frac{2l_i\pi}{\lambda} = \frac{2r_i}{\pi}$, the point $\left(\frac{2j}{\lambda^n} + \frac{2l_i}{\lambda^{n+1}}, W\left(\frac{2j}{\lambda^n} + \frac{2l_i}{\lambda^{n+1}}\right)\right)$

is called a *dividing point* in Γ_{W_n}. We claim that there is a large integer M_0 such that n-squares S_n in a lever n sine-like curve Γ_{W_n} distinguish dividing points $\left(\frac{2j}{\lambda^n} + \frac{2l_i}{\lambda^{n+1}}, W\left(\frac{2j}{\lambda^n} + \frac{2l_i}{\lambda^{n+1}}\right)\right)$, $i = 1, 2, \ldots, N(\alpha, \lambda)$; for $n \geq M_0$. This means that each n-square S_n in the graph Γ_{W_n} contains at most one dividing point. Why is this possible! Note that contraction coefficients r_i are self-defined, it is independent of any parameter and the largest length of minimum bundle N_0 depends on parameters λ and α but it is independent of n. Keep this in mind, we compute the vertical distance of dividing points

$$\begin{aligned} D_i &= W\left(\frac{2j}{\lambda^n} + \frac{2l_{i+1}}{\lambda^{n+1}}\right) - W\left(\frac{2j}{\lambda^n} + \frac{2l_i}{\lambda^{n+1}}\right) \\ &= 2\sin\frac{l_{i+1} - l_i}{\lambda}\cos\frac{l_{i+1} + l_i}{\lambda}\lambda^{-n\alpha}. \end{aligned}$$

Now we consider the ratio $\frac{D_i}{\lambda^{-n}}$,

$$Min\left\{\frac{D_i}{\lambda^{-n}}\right\}_{i=1}^{N_0} = Min\left\{2\sin\frac{l_{i+1}-l_i}{\lambda}\pi\cos\frac{l_{i+1}+l_i}{\lambda}\pi\right\}_{i=1}^{N_0}\lambda^{n(1-\alpha)} \to \infty \text{ as } \lambda \to \infty.$$

Consequently, there is an integer M_0 such that for $n \geq M_0$, we have

$$Min_{i \leq N_0} D_i > \lambda^{-n}.$$

The desire follows.

Finally, let Υ be any $(n, n+m)$ cover of the graph of function Γ_W defined on $[0, 2]$. Let P be the vertical projection mapping defined by $P(x, y) = x$. Let $\Psi = \{I : P(S) = I, \text{ for } S \in \Upsilon\}$. Then Ψ is a cover of $[0, 2]$. An interval $I = \left[\frac{j(I)}{\lambda^{k(I)}}, \frac{j(I)+1}{\lambda^{k(i)}}\right]$ of Ψ is said to be a *structure interval* if $J \cap I \neq 0$, $I \in \Psi$, then $J \subseteq I$. Let Ψ_0 be the collection of all the structure intervals I. Then Ψ_0 is a net of $[0, 2]$ and $\sum_{I \in \Psi_0} |I| = 2$. Clearly, we have

$$\Upsilon = \bigcup_{I \in \Psi_0} \bigcup_{S \in \Upsilon, P(S) \subseteq I} S. \tag{3.51}$$

(3.51) means that Υ decomposes into a union of finite subsets, and each subset covers a graph Γ_{W_s}. Take a structure interval $I = \left[\frac{j(I)}{\lambda^{k(I)}}, \frac{j(I)+1}{\lambda^{k(I)}}\right]$, $\frac{j(I)}{\lambda^{k(I)}} = \frac{r_1}{\lambda} + \cdots + \frac{r_{k(I)}}{\lambda^{k(I)}}$, the set $\{S : S \in \Upsilon, P(S) \subseteq I\}$ covers $\Gamma_W \cap R_{r_1, r_2, \ldots, r_{k(I)}}$, since Υ covers the entire graph of function $W(x)$. By (3.50), we have

$$\sum_{S \in \Upsilon} |S|^{2-\alpha} = \sum_{I \in \Psi_0} \sum_{P(S) \subseteq I,\, S \in \Upsilon} |S|^{2-\alpha} \geq \sum_{I \in \Psi_0} \frac{\pi}{2}\lambda^{-\frac{\alpha}{2}}\lambda^{-k(I)}$$

$$= \frac{\pi}{2}\lambda^{-\frac{\alpha}{2}} \sum_{I \in \Psi_0} |I| = \pi\lambda^{-\frac{\alpha}{2}}.$$

This proves that Γ_W is a $(2-\alpha)$-set.

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