Far East Journal of Mathematical Sciences (FJMS)
Volume 62, Number 2, 2012, Pages 285-291
Published Online: February 2012
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

A NOTE ON PRODUCT OF CLOSURE SPACES

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Abstract

The purpose of the present paper is to investigate a characterization of closed sets and open sets in the product of closure spaces. Furthermore, it is proved that all projection maps are open.

1. Introduction

In [3], Čech has first introduced the concept of Čech closure spaces. A *Čech closure space* (X, u) is a set X equipped with a Čech closure operator u that is obtained from the Kuratowski ones by omitting the requirement of idempotency, i.e., u is grounded, extensive, additive. In [5], Šlapal introduced generalized Čech closure operators which are called *closure operators*. Such a closure operator is not even supposed to be additive but it retains the isotonicity. Later, some properties of continuous maps on closure spaces were studied by Boonpok in [1]. In addition, he provided a

2010 Mathematics Subject Classification: 54A05.
Keywords and phrases: closure space, closed set, open set, product of closure spaces.
The author would like to thank Asst. Prof. Chawalit Boonpok for suggestion.
Received October 8, 2011

characterization of closed sets of some kind in the product of closure spaces. In [2], Boonpok and Khampakdee gave characterizations of generalized closed sets of some kind in the product of closure spaces. In this paper, we give a characterization of closed sets in the product of closure spaces. Consequently, we obtain a characterization of open sets in the product of closure spaces. It follows that all projection maps are open.

2. Preliminaries

We recall some basic definitions and notations. A map $u : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a *closure operator* [5] on X and the pair (X, u) is called a *closure space* if the following axioms are satisfied:

C1: $u\varnothing = \varnothing$ (i.e., u is grounded),

C2: $A \subseteq uA$ for every $A \subseteq X$ (i.e., u is extensive),

C3: $A \subseteq B$ implies $uA \subseteq uB$ for each $A, B \subseteq X$ (i.e., u is isotonic).

A closure operator u on a set X is called *additive* (respectively, *idempotent*) if $u(A \cup B) = uA \cup uB$ for each $A, B \subseteq X$ (respectively, $uuA = uA$ for each $A \subseteq X$). A subset A of X is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement is closed. The empty set and the whole space are both open and closed. Let (X, u) and (X, w) be closure spaces. The closure u is said to be *finer* than the closure w, or w is said to be *coarser* than u, denoted by $u \le w$, if $uA \subseteq wA$ for every $A \subseteq X$. The relation $\le$ is a partial order on the set of all closures on X.

Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$. One can see that a map $f : (X, u) \to (Y, v)$ is continuous if and only if $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every $B \subseteq Y$. Clearly, if $f : (X, u) \to (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v). Similarly, if f is continuous, the $f^{-1}(G)$ is an open subset of (X, u) for every open subset G of (Y, v).

Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *closed* (respectively, *open*) [1, 2] if $f(F)$ is a closed map (respectively, open) subset of (Y, v) whenever F is a closed (respectively, open) subset of (X, u).

The *product* [6] of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$, is the closure space $\left(\prod_{\alpha \in I} X_\alpha, u\right)$, where $\prod_{\alpha \in I} X_\alpha$ denotes the Cartesian product of sets $X_\alpha, \alpha \in I$, and u is the closure operator generated by the projections $\pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta, \beta \in I$, i.e., is defined by

$$uA = \prod_{\alpha \in I} u_\alpha \pi_\alpha(A)$$

for each $A \subseteq \prod_{\alpha \in I} X_\alpha$. Clearly, if $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ is a family of closure spaces, then the projection map $\pi_\beta : \prod_{\alpha \in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous for every $\beta \in I$.

Now, we recall the properties of Cartesian products, see in [4].

Theorem 2.1. *Let $\{X_\alpha \mid \alpha \in I\}$ be a family of nonempty sets. For each $\alpha \in I$, let A_α be a subset of X_α. Then*

(1) $\prod_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} \pi_\alpha^{-1}(A_\alpha)$,

(2) $\prod_{\alpha \in I} X_\alpha - \pi_\beta^{-1}(A_\beta) = \pi_\beta^{-1}(X_\beta - A_\beta)$,

(3) $\prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} \pi_\alpha^{-1}(X_\alpha - A_\alpha)$.

3. Main Results

Now, we will give a characterization of closed sets in product closure spaces.

Theorem 3.1. *Let* $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ *be a family of closure spaces and* $F \subseteq \prod_{\alpha \in I} X_\alpha$. *Then F is a closed subset of* $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$ *if and only if* $F = \prod_{\alpha \in I} F_\alpha$, *where* F_α *is a closed subset of* X_α *for each* $\alpha \in I$.

Proof. Assume that F is a closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$. *Then*

$$F = uF = \prod_{\alpha \in I} u_\alpha \pi_\alpha(F).$$

Fixed $\beta \in I$. Thus, $\pi_\beta(F) = u_\beta \pi_\beta(F)$ and so $\pi_\beta(F)$ is a closed subset of X_β. For each $\alpha \in I$, let $F_\alpha = \pi_\alpha(F)$. Hence, F_α is a closed subset of X_α for all $\alpha \in I$ and

$$F = uF = \prod_{\alpha \in I} u_\alpha \pi_\alpha(F) = \prod_{\alpha \in I} u_\alpha F_\alpha = \prod_{\alpha \in I} F_\alpha.$$

Conversely, assume that $F = \prod_{\alpha \in I} F_\alpha$, where F_α is a closed subset of X_α, for each $\alpha \in I$. Since $\pi_\alpha(F) = F_\alpha$ is a closed subset of X_α for all $\alpha \in I$,

$$uF = \prod_{\alpha \in I} u_\alpha \pi_\alpha(F) = \prod_{\alpha \in I} u_\alpha F_\alpha = \prod_{\alpha \in F} F_\alpha = F.$$

Hence, F is a closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$. $\square$

Next, we provide a characterization of open sets in product closure spaces.

Theorem 3.2. *Let* $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ *be a family of closure spaces and G be a subset of* $\prod_{\alpha \in I} X_\alpha$. *Then G is an open subset of* $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$ *if* *and only if* $G = \bigcup_{\alpha \in I} \pi_\alpha^{-1}(G_\alpha)$, *where* G_α *is an open subset of* X_α *for each* $\alpha \in I$.

Proof. Assume that G is an open subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$. Then $\prod_{\alpha \in I} X_\alpha - G$ is a closed subset of $\prod_\alpha X_\alpha$. By Theorem 3.1, we obtain that $\prod_{\alpha \in I} X_\alpha - G = \prod_{\alpha \in F} F_\alpha$, where F_α is a closed subset of X_α for each $\alpha \in I$. By Theorem 2.1 (3), we have

$$G = \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} F_\alpha = \bigcup_{\alpha \in I} \pi_\alpha^{-1}(X_\alpha - F_\alpha).$$

For each $\alpha \in I$, let $G_\alpha = X_\alpha - F_\alpha$. Then G_α is an open subset of X_α for each $\alpha \in I$ and $G = \bigcup_{\alpha \in I} \pi_\alpha^{-1}(G_\alpha)$.

Conversely, assume that $G = \bigcup_{\alpha \in I} \pi_\alpha^{-1}(G_\alpha)$, where G_α is an open subset of X_α for each $\alpha \in I$. Then $X_\alpha - G_\alpha$ is a closed subset of X_α for each $\alpha \in I$ and, by Theorem 2.1 (3),

$$\prod_{\alpha \in I} X_\alpha - G = \prod_{\alpha \in I} X_\alpha - \bigcup_{\alpha \in I} \pi_\alpha^{-1}(G_\alpha) = \prod_{\alpha \in I} (X_\alpha - G_\alpha).$$

Hence, $\prod_{\alpha \in I} X_\alpha - G$ is a closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$, and so G is open in $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$. $\square$

Next, we prove that all projection maps are open.

Theorem 3.3. *Let* $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ *be a family of closure spaces. Then the projection map* $\pi_\beta : \prod_{\alpha \in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ *is open for all* $\beta \in I$.

Proof. Fixed $\beta \in I$ and let G be an open subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$. Then $G = \bigcup_{\alpha \in I} \pi_\alpha^{-1}(G_\alpha)$, where G_α is an open subset of X_α for each $\alpha \in I$. We see that

$$\pi_\gamma^{-1}(G_\gamma) = G_\gamma \times \prod_{\substack{\alpha \in I \\ \alpha \neq \gamma}} X_\alpha$$

for all $\gamma \in I$. If there exists $\gamma \in I$ such that $\gamma \neq \beta$ and $G_\gamma \neq \varnothing$, then

$$\pi_\beta(\pi_\gamma^{-1}(G_\gamma)) = \pi_\beta\left(G_\gamma \times \prod_{\substack{\alpha \in I \\ \alpha \neq \gamma}} X_\alpha\right) = X_\beta,$$

and so $\pi_\beta(G) = \bigcup_{\alpha \in I} \pi_\beta(\pi_\alpha^{-1}(G_\alpha)) = X_\beta$ is an open subset of (X_β, u_β).

Suppose that $G_\alpha = \varnothing$ for all $\alpha \neq \beta$. Then $\pi_\alpha^{-1}(G_\alpha) = \varnothing$ for all $\alpha \neq \beta$. Thus, $G = \bigcup_{\alpha \in I} \pi_\alpha^{-1}(G_\alpha) = \pi_\beta^{-1}(G_\beta)$. Hence, $\pi_\beta(G) = \pi_\beta(\pi_\beta^{-1}(G_\beta)) = G_\beta$ is an open subset of (X_β, u_β). Thus, the proof is completed. □

Finally, we give a property of the closure operator of the product of a family of closure spaces.

Theorem 3.4. *Let* $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ *be a family of closure spaces and* w *be a closure operator on* $\prod_{\alpha \in I} X_\alpha$ *such that* $\pi_\beta : \left(\prod_{\alpha \in I} X_\alpha, w\right) \to (X_\beta, u_\beta)$ *is continuous for all* $\beta \in I$. *Then* $w \leq u$, *where* u *is the closure operator on the product of a family* $\{(X_\alpha, u_\alpha) : \alpha \in I\}$, *i.e.,* $uA = \prod_{\alpha \in I} u_\alpha \pi_\alpha(A)$ *for all* $A \subseteq \prod_{\alpha \in I} X_\alpha$.

Proof. Let A be a subset of $\prod_{\alpha \in I} X_\alpha$. Since, for each $\beta \in I$, $\pi_\beta : \left(\prod_{\alpha \in I} X_\alpha, w\right) \to (X_\beta, u_\beta)$ is continuous, $\pi_\beta(wA) \subseteq u_\beta \pi_\beta A$. Then

$$\prod_{\alpha \in I} \pi_\alpha(wA) \subseteq \prod_{\alpha \in I} u_\alpha \pi_\alpha(A) = uA.$$

By Theorem 2.1 (1), we obtain that

$$wA \subseteq \bigcap_{\alpha \in I} \pi_\alpha^{-1}(\pi_\alpha(wA)) = \prod_{\alpha \in I} \pi_\alpha(wA).$$

Thus, $wA \subseteq uA$. Hence $w \leq u$. □

References

[1] C. Boonpok, On continuous maps in closure spaces, Gen. Math. 17 (2009), 127-134.

[2] C. Boonpok and J. Khampakdee, Between closed sets and generalized closed sets in closure spaces, Acta Math. Univ. Ostra. 16 (2008), 3-14.

[3] E. Čech, Topological spaces, Topological Paper of Eduard Čech, Academia, Prague (1968), 436-472.

[4] J. Dujunji, Topology, Allyn and Bacon, Boston, 1966.

[5] J. Šlapal, Closure operations for digital topology, Theoret. Comput. Sci. 305 (2003), 457-471.

[6] J. Šlapal, A digital analogue of the Jordan curve theorem, Discr. Appl. Math. 139 (2004), 231-251.