Far East Journal of Mathematical Sciences (FJMS)
Volume 62, Number 2, 2012, Pages 257-268
Published Online: February 2012
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

STRONG CONVERGENCE FOR EQUILIBRIUM PROBLEM AND FIXED POINT PROBLEMS OF NON-SELF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN HILBERT SPACE

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Abstract

In this paper, an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of non-self asymptotically nonexpansive mappings is introduced in Hilbert spaces, under some suitable conditions, a strong convergence theorem is proved.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $\Phi : C \times C \to R$ be a bifunction, where R is the set of real numbers. The equilibrium problem (for short, EP) for Φ is to find $x^* \in C$ such that

2010 Mathematics Subject Classification: 47H09, 47J25.

Keywords and phrases: iteration method, equilibrium problem, common element, non-self asymptotically nonexpansive mappings, fixed point.

Received September 30, 2011

$$\Phi(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of $EP(1.1)$ is denoted by $EP(\Phi)$, this is

$$EP(\Phi) = \{x^* \in C : \Phi(x^*, y) \geq 0, \forall y \in C\}. \tag{1.2}$$

Let E be a real Banach space. Then a mapping $T : E \to E$ is said to be *L-Lipschitzian* if there exists an $L > 0$ such that

$$\| Tx - Ty \| \leq L \| x - y \|, \quad \forall x, y \in E. \tag{1.3}$$

T is said to be *nonexpansive* if $L = 1$ in (1.3). The set of all fixed points of T is denoted by $F(T)$, that is $F(T) = \{x \in K : Tx = x\}$. If C is a bounded nonempty closed convex subset of H and T is a nonexpansive mapping from C into itself, then $F(T)$ is nonempty [13]. A mapping T is said to be an *asymptotically nonexpansive* from C into itself if there exists $\{k_n\} \subset [1, +\infty)$ satisfying $\lim_{n\to\infty} k_n = 1$ such that $\| T^n x - T^n y \| \leq k_n \| x - y \|$ for all $x, y \in C$.

Being an important generalization of the concept of nonexpansive mapping, the concept of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [6] in 1972. They proved that an asymptotically nonexpansive mapping has a fixed point if C is a nonempty closed convex subset of a real uniformly convex Banach space.

It is well known that $EP(1.1)$ contains as special cases, for instance, optimization problems, Nash equilibrium problems, complementarity problems, fixed point problems, variational inequalities and some problems arising from physics and engineering. The solutions of $EP(1.1)$ has been widely studied by many authors (see, e.g., [4, 7-9, 11, 12] and references therein). They constructed many iteration methods to approximate a common element of the set of solutions of $EP(1.1)$ and the set of fixed points of nonexpansive mappings in Hilbert spaces. But in some of iteration methods, they have to compute project subsets at each step of iteration processes (see, e.g., [7, 12]). It is a very difficult work. For example, in [12], for finding a common element of $EP(\Phi)$ and $F(T)$, Tada and Takahashi [12] introduced the following iterative scheme for nonexpansive mapping in a Hilbert space: For any given $x_0 \in H$ and let

$$\begin{cases} u_n \in C \text{ such that } F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in C, \\ w_n = (1 - \alpha_n)x_n + \alpha_n T u_n, \\ C_n = \{z \in H : \| w_n - z \| \leq \| x_n - z \|\}, \\ Q_n = \{z \in H : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases}$$

for each nonnegative integer n, under some mild assumptions on parameters $\{\alpha_n\}$ and $\{r_n\}$, they proved that $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(T) \cap EP(\Phi)$, where $z = P_{F(T)\cap EP(\Phi)} x_0$.

In 2003, Chidume et al. [2] introduced the concept of non-self asymptotically nonexpansive mappings to be a generalization of asymptotically nonexpansive self-mappings as follows: Let C be a nonempty closed convex subset of a real normed linear space E, P be a nonexpansive retraction from E onto C, a non-self mapping $T : C \to E$ is said to be *non-self asymptotically nonexpansive* if there exists $\{k_n\} \subset [1, +\infty)$ satisfying $\lim_{n\to\infty} k_n = 1$ such that $\| T(PT)^{n-1}x - T(PT)^{n-1}y \| \leq k_n \| x - y \|$ for all $x, y \in C$ and each integer $n \geq 1$. Since then some authors (see, e.g., [2, 3, 14-17]) have obtained some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.

Inspired and motivated by the work of [2, 7, 12] and [15], we introduce an iteration scheme, which reduces the burden of computation task to compute C_n and Q_n in iterative processes, and show that the iteration schemes $\{x_n\}$ and $\{u_n\}$ converge strongly to a common element of the set of solutions of EP(1.1) and the set of fixed points of a non-self asymptotically nonexpansive mapping in Hilbert spaces.

2. Preliminaries

Throughout this paper, let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, C be a nonempty closed convex subset of H. We denote the strong convergence, weak convergence of a sequence $\{x_n\}$ to a point $x \in X$ by $x_n \to x$, $x_n \rightharpoonup x$ as $n \to \infty$, respectively.

In this paper, we always assume that bifunction $\Phi : C \times C \to R$ satisfies the following conditions:

(A1) $\Phi(x, x) = 0$, $\forall x \in C$;

(A2) $\Phi(x, y) + \Phi(y, x) \leq 0$, $\forall x, y \in C$;

(A3) For all $x, y, z \in C$, $\lim_{t \downarrow 0} \Phi(tz + (1 - t)x, y) \leq \Phi(x, y)$;

(A4) For each $x \in C$, the function $y \mapsto \Phi(x, y)$ is convex and lower semicontinuous.

By using the conditions above, Combettes and Hirstoaga [5] obtained the following results:

Lemma 2.1 [5]. *Let C be a nonempty closed convex subset of Hilbert space H and let $\Phi : C \times C \to R$ be a bifunction satisfying* (A1)-(A4). *Let $r > 0$ and $x \in H$. Then there exists a $z \in K$ such that*

$$\Phi(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

Lemma 2.2 [5]. *Assume that $\Phi : C \times C \to R$ satisfies* (A1)-(A4). *For $r > 0$ and $x \in H$, define a mapping $T_r(x) : H \to H$ as follows*:

$$T_r(x) = \left\{ z \in C : \Phi(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}, \forall x \in H. \tag{2.2}$$

Then,

(1) T_r *is single-valued*;

(2) T_r *is firmly nonexpansive, that is, for all* $x, y \in H$,

$$\| T_r x - T_r y \|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

(3) $F(T_r) = EP(\Phi)$;

(4) $EP(\Phi)$ *is nonempty, closed and convex.*

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *demiclosed* at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

Lemma 2.3 [2]**.** *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E, and $T : C \to E$ be a non-self asymptotically nonexpansive mapping. If T has a fixed point, then $I - T$ is demiclosed at zero.*

Lemma 2.4 [10]**.** *Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a real uniformly convex Banach space E. If $0 < a \leq \alpha_n \leq b < 1$, $\limsup_{n\to\infty} \| x_n \| \leq r$, $\limsup_{n\to\infty} \| y_n \| \leq r$ and $\lim_{n\to\infty} \| \alpha_n x_n + (1 - \alpha_n) y_n \| = r$, then $\lim_{n\to\infty} \| x_n - y_n \| = 0$.*

Lemma 2.5 [18]**.** *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two nonnegative real number sequences satisfying*

$$\alpha_{n+1} \leq \alpha_n + \beta_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists.

3. Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H.* Let $\Phi : C \times C \to R$ *be a bifunction satisfying* (A1)-(A4), *and $T : C \to E$ be a non-self asymptotically nonexpansive mapping with* *sequence* $\{k_n\} \subset [1, \infty)$ *such that* $\Omega = F(T) \bigcap EP(\Phi) \neq \varnothing$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *For any given* $x_1 \in C$, $\{x_n\}$ *and* $\{u_n\}$ *are defined by*

$$\begin{cases} \Phi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ y_n = P((1-\beta_n)u_n + \beta_n T(PT)^{n-1}u_n), \ n \geq 1, \\ x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}y_n), \end{cases} \tag{3.1}$$

where P *is a projection operator from* H *onto* C. *If* T *is completely continuous, and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{r_n\}$ *satisfy the following conditions*:

(1) $0 < a \leq \alpha_n \leq 1 - a < 1$, $n \geq 1$;

(2) $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$,

then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a common element* $q \in \Omega$.

Proof. For any $p \in \Omega$, it follows from Lemma 2.2 that $u_n = T_{r_n}x_n$ and $\| u_n - p \| = \| T_{r_n}x_n - p \| \leq \| x_n - q \|$. Put $k_n = 1 + \delta_n$, from (3.1), we have

$$\| y_n - p \| \leq (1-\beta_n)\| u_n - p \| + \beta_n(1+\delta_n)\| u_n - p \| \leq (1+\delta_n)\| x_n - p \|, \tag{3.2}$$

in addition,

$$\begin{aligned} \| x_{n+1} - p \| &\leq (1-\alpha_n)\| x_n - p \| + \alpha_n(1+\delta_n)\| y_n - p \| \\ &\leq (1-\alpha_n)\| x_n - p \| + \alpha_n(1+\delta_n)^2\| x_n - p \| \\ &\leq (1 + 2\delta_n + \delta_n^2)\| x_n - p \| \\ &\leq e^{\sum_{k=1}^{n}(2\delta_k + \delta_k^2)}\| x_1 - p \|. \end{aligned} \tag{3.3}$$

Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, we have $\sum_{n=1}^{\infty}\delta_n < \infty$. Thus, it follows from (3.3) that $\{x_n\}$ is bounded. Further, $\{u_n\}$, $\{y_n\}$, $\{T(PT)^{n-1}u_n\}$ and $\{T(PT)^{n-1}y_n\}$ are bounded, too.

Set $M = \sup\{\| x_n - p \|, n \geq 1\}$, from (3.3), we also have

$$\| x_{n+1} - p \| \leq \| x_n - p \| + (2\delta_n + \delta_n^2)M. \tag{3.4}$$

It follows from Lemma 2.5 that $\lim_{n\to\infty}\| x_n - p \|$ exists for any $p \in \Omega$.

We now show that $\lim_{n\to\infty}\| x_n - Tx_n \| = 0$. Assume that $\lim_{n\to\infty}\| x_n - p \| = r$. Since $\| y_n - p \| \leq (1 + \delta_n)\| x_n - p \|$, we may obtain

$$\limsup_{n\to\infty}\| y_n - p \| \leq r. \tag{3.5}$$

On the other hand, since $\| T(PT)^{n-1}y_n - p \| \leq (1 + \delta_n)\| y_n - p \|$, we have

$$\limsup_{n\to\infty}\| T(PT)^{n-1}y_n - p \| \leq r. \tag{3.6}$$

Since (from (3.3))

$$\begin{aligned} \| x_{n+1} - p \| &\leq \| (1 - \alpha_n)(x_n - p) + \alpha_n(T(PT)^{n-1}y_n - p) \| \\ &\leq \| x_n - p \| + (2\delta_n + \delta_n^2)\| x_n - p \|, \end{aligned}$$

we have

$$\begin{aligned} &\limsup_{n\to\infty}\| x_{n+1} - p \| \\ &\leq \liminf_{n\to\infty}\| (1 - \alpha_n)(x_n - p) + \alpha_n(T(PT)^{n-1}y_n - p) \| \\ &\leq \limsup_{n\to\infty}\| (1 - \alpha_n)(x_n - p) + \alpha_n(T(PT)^{n-1}y_n - p) \| \\ &\leq \limsup_{n\to\infty}\| x_n - p \| + \limsup_{n\to\infty}(2\delta_n + \delta_n^2)\| x_n - p \| \\ &= \limsup_{n\to\infty}\| x_n - p \|. \end{aligned}$$

Thus we have

$$\lim_{n\to\infty}\| (1 - \alpha_n)(x_n - p) + \alpha_n(T(PT)^{n-1}y_n - p) \| = r. \tag{3.7}$$

It follows from Lemma 2.4 and (3.6) that

$$\lim_{n\to\infty}\| x_n - T(PT)^{n-1}y_n \| = 0. \tag{3.8}$$

From (3.1), we may obtain that

$$\begin{aligned}\| x_{n+1} - x_n \| &= \| P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1} y_n) - Px_n \| \\ &\le \| ((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1} y_n) - x_n \| \\ &= \alpha_n \| T(PT)^{n-1} y_n - Px_n \|,\end{aligned}$$

hence, it follows from (3.8) that

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0 \tag{3.9}$$

and

$$\lim_{n\to\infty} \| x_{n+1} - T(PT)^{n-1} y_n \| = 0. \tag{3.10}$$

In addition,

$$\| x_n - p \| \le \| x_n - T(PT)^{n-1} y_n \| + (1+\delta_n) \| y_n - p \|,$$

taking $\liminf$ on the both sides in inequality above and using (3.8), then $\liminf_{n\to\infty} \| y_n - p \| \ge r$. Thus, since $\limsup_{n\to\infty} \| y_n - p \| \le r$, we have $\lim_{n\to\infty} \| y_n - p \| = r$. Using the same proof method of the equality (3.7), we can also obtain

$$\lim_{n\to\infty} \| (1-\beta_n)(u_n - p) + \beta_n (T(PT)^{n-1} u_n - p) \| = r.$$

Notice that $\limsup_{n\to\infty} \| u_n - p \| \le \lim_{n\to\infty} \| x_n - p \| = r$ and

$$\limsup_{n\to\infty} \| T(PT)^{n-1} u_n - p \| \le \limsup_{n\to\infty} (1+\delta_n) \| u_n - p \| \le r.$$

So, it follows from Lemma 2.4 that

$$\lim_{n\to\infty} \| u_n - T(PT)^{n-1} u_n \| = 0. \tag{3.11}$$

Further, from (3.1) and (3.11), we have

$$\lim_{n\to\infty} \| y_n - u_n \| = 0. \tag{3.12}$$

Since non-self asymptotically nonexpansive mappings are L-Lipschitzian, there exists a constant $L > 0$ such that $\| T(PT)^{n-1}x - T(PT)^{n-1}y \| \le L\| x - y \|$ for all $x, y \in C$ and each $n \ge 1$, so

$$\begin{aligned} \| x_n - u_n \| &\le \| x_n - T(PT)^{n-1}y_n \| + \| T(PT)^{n-1}y_n - T(PT)^{n-1}u_n \| \\ &\quad + \| T(PT)^{n-1}u_n - u_n \| \\ &\le \| x_n - T(PT)^{n-1}y_n \| + L\| y_n - u_n \| + \| T(PT)^{n-1}u_n - u_n \|. \end{aligned}$$

From (3.8), (3.11) and (3.12), we have

$$\lim_{n\to\infty}\| x_n - u_n \| = 0, \quad \lim_{n\to\infty}\| x_n - y_n \| = 0. \tag{3.13}$$

Furthermore, since

$$\begin{aligned} \| x_n - T(PT)^{n-1}x_n \| &\le \| x_n - T(PT)^{n-1}y_n \| + \| T(PT)^{n-1}y_n - T(PT)^{n-1}x_n \| \\ &\le \| x_n - T(PT)^{n-1}y_n \| + L\| y_n - x_n \|, \end{aligned}$$

it follows from (3.8), (3.9) and (3.13) that

$$\lim_{n\to\infty}\| x_n - T(PT)^{n-1}x_n \| = 0$$

and

$$\lim_{n\to\infty}\| x_{n+1} - T(PT)^{n-1}x_n \| = 0. \tag{3.14}$$

For $n \ge 2$, we have

$$\begin{aligned} \| x_n - Tx_n \| &\le \| x_n - T(PT)^{n-1}x_n \| + \| T(PT)^{n-1}x_n - Tx_n \| \\ &\le \| x_n - T(PT)^{n-1}x_n \| + L\| T(PT)^{n-2}x_n - x_n \| \\ &\le \| x_n - T(PT)^{n-1}x_n \| + L\| T(PT)^{n-2}x_n - T(PT)^{n-2}x_{n-1} \| \\ &\quad + L\| T(PT)^{n-2}x_{n-1} - x_n \| \\ &\le \| x_n - T(PT)^{n-1}x_n \| + L^2\| x_n - x_{n-1} \| + L\| T(PT)^{n-2}x_{n-1} - x_n \|. \end{aligned}$$

Therefore, from (3.9) and (3.14), we obtain

$$\lim_{n\to\infty}\| x_n - Tx_n \| = 0. \tag{3.15}$$

Finally, we show that $\{x_n\}$ and $\{u_n\}$ converge strongly to a point in Ω.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to a $q \in C$. It follows from (3.15) and Lemma 2.3 that $q \in F(T)$. Since $\lim\| x_n - u_n \| = 0$, we know that $\{u_{n_j}\} \rightharpoonup q$ as $j \to \infty$. We now show that $q \in EP(\Phi)$.

As a matter of fact, from $u_n = T_{r_n} x_n$, we have

$$\Phi(u_{n_j}, y) + \frac{1}{r_{n_j}} \langle y - u_{n_j}, u_{n_j} - x_{n_j} \rangle \geq 0, \quad \forall y \in C. \tag{3.16}$$

From (A2), we have

$$\frac{1}{r_{n_j}} \langle y - u_{n_j}, u_{n_j} - x_{n_j} \rangle \geq \Phi(y, u_{n_j}), \quad \forall y \in C. \tag{3.17}$$

Since $\{x_n\}$, $\{u_n\}$ are bounded, $\{u_{n_j}\} \rightharpoonup q$ as $j \to \infty$ and $\Phi(x, y)$ is lower semicontinuous for the second variable, thus as $j \to \infty$ in (3.17), we have

$$\Phi(y, q) \leq 0, \quad \forall y \in C. \tag{3.18}$$

For all $y \in C$, let $t \in (0, 1)$, put $y_t = ty + (1-t)q$. Then $y_t \in C$ and $\Phi(y_t, q) \leq 0$. It follows from (A1) and (A4) that

$$0 = \Phi(y_t, y_t) \leq t\Phi(y_t, y) + (1-t)\Phi(y_t, q) \leq t\Phi(y_t, y), \quad \forall y \in C,$$

this yields that $\Phi(y_t, y) \geq 0$ for any $y \in C$. Further, as $t \to 0^+$, we have $\Phi(q, y) \geq 0$ for all $y \in C$. This implies that $q \in EP(\Phi)$.

Since T is completely continuous, $Tx_{n_j} \to q$ as $j \to \infty$. From (3.15), we know that $\lim_{n\to\infty}\| x_{n_j} - q \| = 0$. Further, since $\lim_{n\to\infty}\| x_n - q \|$

exists, we may conclude that $\lim_{n\to\infty}\| x_n - q \| = 0.$ This implies that $\{x_n\}$ converges strongly to $q \in \Omega.$ On the other hand, since $\| u_n - q \| \leq \| x_n - q \|,$ we also know that $\{u_n\}$ converges strongly to $q \in \Omega.$ The proof is completed.

Remark. In fact, the sequences $\{x_n\}$ and $\{u_n\}$ defined in (3.1) also converge strongly to a common element $q \in \Omega$ when the non-self asymptotically nonexpansive mapping T is demi-compact, since we have obtained the inequality (3.15). In addition, when $\Phi(x, y) = 0$ for all $x, y \in C,$ the sequence $\{x_n\}$ converges strongly to a fixed point of T.

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