Far East Journal of Mathematical Sciences (FJMS)
Volume 62, Number 2, 2012, Pages 177-188
Published Online: February 2012
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

ON DECOMPOSITION OF FUZZY PRIME SUBMODULES

Kukil Kalpa Rajkhowa and Helen K. Saikia
Department of Mathematics
Girijananda Chowdhury Institute of Management and Technology
Guwahati-781017, India
e-mail: kukilrajkhowa@yahoo.com

Department of Mathematics
Gauhati University
Guwahati-781014, India
e-mail: hsaikia@yahoo.com

Abstract

Our objective is to investigate the fuzzy aspects of prime submodules of an R-module M. Using the concepts of fuzzy prime ideals and fuzzy residual quotients, we establish various results on fuzzy prime submodules of M. Finally, we define fuzzy prime decomposition, irredundant fuzzy prime decomposition and normal fuzzy prime decomposition, and obtain some related theorems.

1. Introduction

In 1965, Zadeh introduced the concept of fuzzy set [13], and it was a new episode towards the development of science and technology. Rosenfeld used the concept of Zadeh in abstract algebra [11] and opened up a new insight in

2010 Mathematics Subject Classification: 03E72, 16D10.

Keywords and phrases: fuzzy prime ideal, fuzzy prime submodule, fuzzy residual quotient, fuzzy prime decomposition, normal fuzzy prime decomposition.

Received July 22, 2011; Revised November 12, 2011

the field of Mathematical Science. Since then, many researchers have been working on the concepts like fuzzy semigroups, fuzzy groups, fuzzy rings, fuzzy semi-rings, fuzzy near-rings, fuzzy sub-rings, fuzzy ideals and so on. Fuzzy submodules of an *R*-module *M* were first introduced by Negoita and Ralescu [9]. Pan [10] studied fuzzy finitely generated modules and quotient modules. In [12], Sidky introduced the notion of radical of a fuzzy submodule and also defined primary fuzzy submodules and obtained some important properties. Recently, the notion of fuzzy prime submodules and Zariski topology on *Spec*(*M*), the set of fuzzy prime submodules of an *R*-module *M*, have been studied by many authors (for example, see [1, 5, 6]). In this paper, our attempt is to investigate the fuzzy aspects of prime submodules of an *R*-module *M* and their relation with fuzzy prime ideals of *R*. Using the concepts of fuzzy prime submodules, fuzzy prime ideals and fuzzy residual quotients; it is proved that if μ is a fuzzy prime submodule of an *R*-module *M* and ν is any fuzzy submodule of *M*, with $\mu : \nu \neq 1_R$, then $\mu : \nu$ is a fuzzy prime ideal of *R*. If *f* is an epimorphism from an *R*-module *M* to an *R*-module *N*, μ is a fuzzy prime submodule of *M* which is *f*-invariant, then $\mu : 1_M = f(M) : 1_N$. It is shown that the finite intersection of fuzzy prime submodules of *M* is also a fuzzy prime submodule of *M*. Finally, we define fuzzy prime decomposition, irredundant fuzzy prime decomposition and normal fuzzy prime decomposition and establish some related theorems.

2. Definitions and Notation

Throughout this paper, *R* denotes a commutative ring with unity and *M* denotes a module over *R*.

Definition 2.1. A map $\mu : X \to [0, 1]$ is called a *fuzzy subset* of the set *X*. The set of all fuzzy subsets of *X* is denoted by $[0, 1]^X$.

Definition 2.2. Let $\mu, \nu \in [0, 1]^X$. Then μ is contained in ν if $\mu(x) \leq \nu(x), \forall x \in X$ and is denoted by $\mu \subseteq \nu$.

Definition 2.3. Let $\mu, \nu \in [0, 1]^X$. The *union* and *intersection* of μ and ν, denoted by $\mu \bigcup \nu$ and $\mu \bigcap \nu$, are defined as $(\mu \bigcup \nu)(x) = \mu(x) \vee \nu(x)$ and $(\mu \bigcap \nu)(x) = \mu(x) \wedge \nu(x)$ for all $x \in X$, respectively.

Definition 2.4. A fuzzy subset μ of X of the form

$$\mu(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

Definition 2.5. Let f be a mapping from a non-empty set X into a non-empty set Y and let $\mu \in [0, 1]^X$, $\nu \in [0, 1]^Y$. Then $f(\mu) \in [0, 1]^Y$ and $f^{-1}(\nu) \in [0, 1]^X$ are defined as follows:

$$f(\mu)(y) = \begin{cases} \bigvee\{\mu(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

and

$$f^{-1}(\nu)(x) = \nu(f(x)), \ \forall x \in X.$$

Definition 2.6. A fuzzy subset μ of R is called a *fuzzy ideal* if it satisfies the following properties:

(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in R$.

(ii) $\mu(xy) \geq \mu(x) \vee \mu(y)$, for all $x, y \in R$.

The set of all fuzzy ideals of R is denoted by $LI(R)$.

Definition 2.7. Let $\mu, \nu \in LI(R)$. We define $\mu\nu \in LI(R)$ as follows:

$$(\mu\nu)(x) = \bigvee\{\mu(y) \wedge \nu(z) : y, z \in R, yz = x\}, \ \forall x \in R.$$

Definition 2.8. Let $\zeta \in LI(R)$. Then ζ is called a *fuzzy prime ideal* of R if ζ is non-constant and for every $\mu, \nu \in LI(R)$, $\mu\nu \subseteq \zeta$ implies that $\mu \subseteq \zeta$ or $\nu \subseteq \zeta$.

Definition 2.9. Let $c \in [0, 1)$. Then c is called a *prime element* of $[0, 1]$ if $a \wedge b \leq c$, implies that either $a \leq c$ or $b \leq c$, for all $a, b \in [0, 1]$.

Definition 2.10. Let $\zeta \in [0, 1]^R$ and $\mu \in [0, 1]^M$. We define $\zeta\mu \in [0, 1]^M$ as follows:

$$(\zeta\mu)(x) = \vee\{\zeta(r) \wedge \mu(y) : r \in R,\ y \in M,\ ry = x\},\ \forall x \in M.$$

Definition 2.11. A fuzzy submodule of M is a fuzzy subset $\mu \in [0, 1]^M$ such that

(i) $\mu(\theta) = 1$, where θ is the zero element of M;

(ii) $\mu(rx) \geq \mu(x)$, for all $r \in R$ and $x \in M$ and

(iii) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in M$.

The set of all fuzzy submodules of M is denoted by $L(M)$.

Definition 2.12. Let μ be a fuzzy submodule of M. Then $\mu_* = \{x \in M : \mu(x) = 1\}$.

Definition 2.13. For $\mu, \nu \in [0, 1]^M$ and $\zeta \in [0, 1]^R$, we define $\mu : \nu \in [0, 1]^R$ and $\mu : \zeta \in [0, 1]^M$ as follows:

$$\mu : \nu = \cup\{\eta \in [0, 1]^R : \eta\nu \subseteq \mu\},$$

$$\mu : \zeta = \cup\{\nu \in [0, 1]^M : \zeta\nu \subseteq \mu\}.$$

Definition 2.14. A non-constant fuzzy submodule μ of M is said to be *prime* if for $\zeta \in LI(R)$ and $\nu \in L(M)$ such that $\zeta\nu \subseteq \mu$, then either $\nu \subseteq \mu$ or $\zeta \subseteq \mu : 1_M$. By $F - Spec(M)$, we mean the set of all fuzzy prime submodules of M.

Definition 2.15. Let M, N be R-modules and $f : M \to N$ be a module homomorphism. A fuzzy subset μ of M is called *f-invariant* if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$, for all $x, y \in M$.

Definition 2.16. Let $\mu \in L(M)$. Then a decomposition of μ as a finite intersection, $\mu = \bigcap_{i=1}^{n} \mu_i$, of fuzzy prime submodules μ_i, $i = 1, 2, ..., n$; of M is called *fuzzy prime decomposition* of μ and the set $\{\mu_i : 1_M \mid i = 1, 2, ..., n\}$ of fuzzy prime ideals is called the *set of associated fuzzy prime ideals* of μ. A fuzzy prime decomposition $\mu = \bigcap_{i=1}^{n} \mu_i$ is called *irredundant* if no μ_i contains $\bigcap_{j=1, j \neq i}^{n} \mu_j$ and an irredundant fuzzy prime decomposition of μ is called *normal* if distinct μ_i have distinct associated fuzzy prime ideals.

Definition 2.17. A fuzzy prime submodule μ_i in the normal prime decomposition $\mu = \bigcap_{i=1}^{n} \mu_i$ is *isolated* if the associated fuzzy prime ideal $\mu_i : 1_M$ is minimal in the set of associated fuzzy prime ideals of μ.

3. Preliminaries

This section contains some basic results needed in the sequel.

Lemma 3.1 [1]**.** *Let $c \in [0, 1]$ and N be a submodule of M. Then* $(1_N \cup c_M) : 1_M = 1_{N:M} \cup c_R$.

Lemma 3.2 [1]**.** *Let $\mu \in L(M)$. Then μ is a fuzzy prime submodule of M if and only if $\mu = 1_{\mu_*} \cup c_M$ such that μ_* is a prime submodule of M and c is a prime element of* $[0, 1]$.

Lemma 3.3 [1]**.** *If μ is a fuzzy prime submodule of M, then $\mu : 1_M$ is a fuzzy prime ideal of R.*

Lemma 3.4 [1]**.** *Let $\nu \in L(M)$ and $\mu \in F - Spec(M)$. Then*

(i) *If $\nu \subseteq \mu$, then $\mu : \nu = 1_R$ and*

(ii) *If $\nu \nsubseteq \mu$, then $\mu : \nu = \mu : 1_M$.*

Lemma 3.5 [1]. *Let* $\mu \in L(M)$ *and* $\zeta \in LI(R)$. *If* μ *is a fuzzy prime submodule of M, then the following statements are satisfied*:

(i) *if* $\zeta \nsubseteq (\mu : 1_M)$, *then* $\mu : \zeta = \mu$ *and*

(ii) *if* $\zeta \subseteq (\mu : 1_M)$, *then* $\mu : \zeta = 1_M$.

Lemma 3.6 [1]. *Let M and N be R-modules and f be a homomorphism from M onto N. Then the following statements are satisfied*:

(i) *Let* $\mu \in F - Spec(M)$ *be f-invariant. Then* $f(\mu) \in F - Spec(N)$.

(ii) *If* $\nu \in F - Spec(N)$, *then* $f^{-1}(\nu) \in F - Spec(M)$.

Lemma 3.7 [7]. *Let* $\mu, \nu \in [0, 1]^M$ *and* $\zeta \in [0, 1]^R$. *Then*

(i) $(\mu : \nu)\nu \subseteq \mu$,

(ii) $\zeta(\mu : \zeta) \subseteq \mu$,

(iii) $\zeta\nu \subseteq \mu \Leftrightarrow \zeta \subseteq \mu : \nu \Leftrightarrow \nu \subseteq \mu : \zeta$.

Lemma 3.8 [7]. *Let* μ_i $(i \in I(index\ set))$, $\nu \in [0, 1]^M$ *and* $\zeta \in [0, 1]^R$. *Then*

(i) $(\bigcap_{i\in I} \mu_i : \nu) = \bigcap_{i\in I}(\mu_i : \nu)$,

(ii) $(\bigcap_{i\in I} \mu_i : \zeta) = \bigcap_{i\in I}(\mu_i : \zeta)$.

Lemma 3.9 [7]. *Let* $\mu \in L(M)$, $\mu_i(i \in I(index\ set))$, $\nu \in [0, 1]^M$ *and* $\zeta \in [0, 1]^R$. *Then*

(i) *If* $\nu_i \in [0, 1]^M$, $i \in I$, *then* $\mu : (\bigcup_{i\in I} \nu_i) = \bigcap_{i\in I}(\mu : \nu_i)$.

(ii) *If* $\zeta_i \in [0, 1]^R$, $i \in I$, *then* $\mu : (\bigcup_{i\in I} \zeta_i) = \bigcap_{i\in I}(\mu : \zeta_i)$.

Lemma 3.10 [7]. *Let* $\mu, \nu \in LI(R)$. *Then* $\mu\nu \subseteq \mu \bigcap \nu$.

4. Main Results

We now present our main results.

Theorem 4.1. *Let* μ *be a fuzzy prime submodule of M and* $\nu \in L(M)$. *If* $\mu : \nu \neq 1_R$, *then* $\mu : \nu$ *is a fuzzy prime ideal of R.*

Proof. We assume $\mu : \nu \neq 1_R$. Then $\nu \nsubseteq \mu$ and $\mu : \nu = \mu : 1_M$, by Lemma 3.4. Let $\eta, \zeta \in LI(R)$ be such that $\eta\zeta \subseteq \mu : \nu$ and $\eta \nsubseteq \mu : \nu$. Now $\eta\zeta \subseteq \mu : \nu$ gives $(\eta\zeta)\nu \subseteq \mu$, from Lemma 3.7. This implies $\eta(\zeta\nu) \subseteq \mu$, by [7, Theorem 3.1.4]. So, $\eta \subseteq \mu : 1_M$ or $\zeta\nu \subseteq \mu$, as μ is a fuzzy prime submodule of *M*. If $\eta \subseteq \mu : 1_M$, then $\eta 1_M \subseteq \mu$, by Lemma 3.7. Also, $\nu \subseteq 1_M$ gives $\eta\nu \subseteq \eta 1_M$, which gives $\eta\nu \subseteq \mu$ as $\eta 1_M \subseteq \mu$. Again using Lemma 3.7, we get $\eta \subseteq \mu : \nu$, a contradiction. So $\zeta\nu \subseteq \mu$ implies $\zeta \subseteq \mu : \nu$, from Lemma 3.7. Hence $\mu : \nu$ is a fuzzy prime ideal of *R*. □

Theorem 4.2. *Let* $\mu \in L(M)$ *and* $\zeta \in LI(R)$. *If* μ *is a fuzzy prime ideal of R and* $\mu : \zeta \neq 1_M$, *then* $\mu : \zeta$ *is a fuzzy prime submodule of M.*

Proof. If $\eta \in LI(R), \nu \in L(M)$ and $\eta\nu \subseteq \mu : \zeta$, then $\eta(\zeta\nu) = \zeta(\eta\nu) \subseteq \zeta(\mu : \zeta) \subseteq \mu$, by Lemma 3.7. This gives $\zeta\nu \subseteq \mu$ or $\eta \subseteq \mu : 1_M$, as μ is a fuzzy prime ideal of *R*. Therefore, we have $\nu \subseteq \mu : \zeta$, from Lemma 3.7. From $\mu : \zeta \neq 1_M$, we get $\zeta 1_M \nsubseteq \mu$. As $(\mu : \zeta) : 1_M = \mu : (\zeta : 1_M)$, thus $(\mu : \zeta) : 1_M = \mu : 1_M$, by Lemma 3.4. This gives $\nu \subseteq \mu : \zeta$ or $\eta \subseteq \mu : 1_M = (\mu : \zeta) : 1_M$. Hence, $\mu : \zeta$ is a fuzzy prime submodule of *M*. □

Theorem 4.3. *Let M and N be two modules over R and f be an epimorphism of M onto N.*

(i) *If* μ *is a fuzzy prime submodule of M and is f-invariant, then* $\mu : 1_M = f(\mu) : 1_N$.

(ii) *If* ν *is a fuzzy prime submodule of N, then* $\nu : 1_N = f^{-1}(\nu) : 1_M$.

Proof.

(i) We assume that μ is a fuzzy prime submodule of M and is f-invariant. Then $f(\mu)$ is a fuzzy prime submodule of N, by Lemma 3.6. Also, $\mu = 1_{\mu_*} \cup c_M$, where μ_* is a prime submodule of M and c is a prime element of $[0, 1]$, from Lemma 3.2. Now

$$(1_{f(\mu_*)} \cup c_N)(y) = 1_{f(\mu_*)}(y) \vee c_N(y)$$

$$= \begin{cases} 1 & \text{if } y \in f(\mu_*), \\ c & \text{if } y \notin f(\mu_*). \end{cases}$$

Also,

$$f(\mu)(y) = \vee\{\mu(x) : x \in f^{-1}(y)\} \text{ if } f^{-1}(y) \neq \phi$$

$$= \vee\{(1_{\mu_*} \cup c_M)(x) : x \in f^{-1}(y)\}$$

$$= \begin{cases} 1 & \text{if } x \in \mu_*, x \in f^{-1}(y), \\ c & \text{if } x \notin \mu_*, x \in f^{-1}(y). \end{cases}$$

But

$$x \in f^{-1}(y) \Rightarrow f(x) = y \Rightarrow \begin{cases} y \in f(\mu_*) & \text{if } x \in \mu_*, \\ y \notin f(\mu_*) & \text{if } x \notin \mu_*. \end{cases}$$

This gives $f(\mu) = 1_{f(\mu_*)} \cup c_N$. Again, $\mu : 1_M = (1_{\mu_*} \cup c_M) : 1_M = 1_{\mu_* : M} \cup c_R$ by Lemma 3.1. Similarly, $f(\mu) : 1_N = 1_{f(\mu_*) : N} \cup c_R$. Now, let $r \in f(\mu_*) : N$. Then $rN \subseteq f(\mu_*)$. This gives $rf(M) \subseteq f(\mu_*)$, as f is an epimorphism, which implies $f(rM) \subseteq f(\mu_*)$. Let $x = rm \in rM, m \in M$. Then $f(rm) \in f(rM)$. Thus $f(rm) \in f(\mu_*)$. This implies $f(rm) = f(z)$, for some $z \in \mu_*$. As μ is f-invariant, so $\mu(rm) = \mu(z) = 1$, i.e., $rm \in \mu_*$. From this, we get $rM \subseteq \mu_*$ and this implies $r \in \mu_* : M$. So, $f(\mu_*) : N \subseteq \mu_* : M$. Again, let $p \in \mu_* : M$. Then $pM \subseteq \mu_*$. From this, $f(pM) \subseteq$

$f(\mu_*)$. So, $pf(M) \subseteq f(\mu_*)$, i.e., $pN \subseteq f(\mu_*)$, as f is an epimorphism. This implies $p \in f(\mu_*) : N$. Thus $\mu_* : M \subseteq f(\mu_*) : N$. Therefore, $f(\mu_*) : N = \mu_* : M$. Hence $\mu : 1_M = f(\mu) : 1_N$.

(ii) Similar to (i). □

Theorem 4.4. *The intersection of a finite number of fuzzy prime submodules of M is a fuzzy prime submodule of M.*

Proof. It is sufficient to prove this for two fuzzy prime submodules of *M*. Let μ, ν be two fuzzy prime submodules of *M*. Now, we have $\mu = 1_{\mu_*} \cup a_M$ and $\nu = 1_{\nu_*} \cup b_M$, where μ_*, ν_* are fuzzy prime submodules of *M*, *a*, *b* are prime elements in $[0, 1]$, from Lemma 3.2. So, using Lemma 3.1, $\mu : 1_M = (1_{\mu_*} \cup a_M) : 1_M = 1_{\mu_* : M} \cup a_R = 1_{\nu_* : M} \cup b_R$. Thus $a = b$ and $\mu_* : M = \nu_* : M$. Clearly, $\mu \cap \nu = 1_{\mu_* \cap \nu_*} \cup a_M$. Since μ_*, ν_* are fuzzy prime submodules of *M*, it follows that $\mu_* \cap \nu_*$ is also a fuzzy prime submodule of *M*. Therefore, $\mu \cap \nu$ is a fuzzy prime submodule of *M*. □

Theorem 4.5. *If* $\mu_i (1 \leq i \leq n)$ *are fuzzy prime submodules of M, then* $(\bigcap_{i=1}^n \mu_i) : 1_M$ *is a fuzzy prime ideal of R.*

Proof. Since μ_i is a fuzzy prime submodule of *M*, for $1 \leq i \leq n$. So, $\bigcap_{i=1}^n \mu_i$ is a fuzzy prime submodule of *M*, by Theorem 4.4. Thus, using Lemma 3.3, we get $(\bigcap_{i=1}^n \mu_i) : 1_M$ is a fuzzy prime ideal of *R*. □

Theorem 4.6. *Let* $\mu \in L(M)$, *and* $\mu = \bigcap_{i=1}^n \mu_i$ *be an irredundant fuzzy prime decomposition of* μ, *where* μ_i *are fuzzy prime submodules of M. Then a fuzzy prime ideal* $\alpha \in \{\mu_i : 1_M \mid i = 1, 2, ..., n\}$ *if and only if there exists* $\nu \in L(M)$ *such that* $\mu : \nu = \alpha$. *Hence, the set of fuzzy prime ideals* $\{\mu_i : 1_M \mid i = 1, 2, ..., n\}$ *is independent of the particular irredundant fuzzy prime decomposition of* μ.

Proof. Let $\mu \in L(M)$, and $\mu = \bigcap_{i=1}^{n} \mu_i$ be an irredundant fuzzy prime decomposition of μ, where μ_i are fuzzy prime submodules of *M*. Now, for $\nu \in L(M)$, we have $\mu : \nu = (\bigcap_{i=1}^{n} \mu_i) : \nu = \bigcap_{i=1}^{n} (\mu_i : \nu)$ by Lemma 3.8. Again, from Lemma 3.4 and Theorem 4.2, we get $\mu_i : \nu = 1_R$ if $\nu \subseteq \mu_i$ and $\mu_i : \nu$ is a fuzzy prime ideal of *R* if $\nu \nsubseteq \mu_i$. Hence $\mu_i : \nu \in LI(R)$. Thus $\mu : \nu = \bigcap_{i=1}^{n}(\mu_i : \nu) = \bigcap_{j=1}^{m}(\mu_{s_j} : 1_\mu)$, where the intersection is taken over those s_j such that $\nu \nsubseteq \mu_{s_j}$. Also, $\mu : \nu$ is a fuzzy prime ideal of *R*. We suppose that $\mu : \nu = \alpha$. From this, we get $\alpha \supseteq (\mu_{s_1} : 1_M)(\mu_{s_2} : 1_M) \cdots (\mu_{s_m} : 1_M)$ by Lemma 3.10 and this gives $\alpha \supseteq \mu_{s_j} : 1_M$ for some s_j. Also, $\alpha \subseteq \mu_{s_j} : 1_M$, so $\alpha = \mu_{s_j} : 1_M$. Next, consider any one of the associated fuzzy prime ideals $\mu_i : 1_M$ of $\mu = \bigcap_{i=1}^{n} \mu_i$. Let $\nu = \bigcap_{j=1,\, j \neq 1}^{n} \mu_j$. Then by Lemma 3.8, we have $\mu : \nu = (\bigcap_{k=1}^{n} \mu_k) : (\bigcap_{j=1,\, j\neq i}^{n} \mu_j) = \bigcap_{k=1}^{n}(\mu_k : \bigcap_{j=1,\, j\neq i}^{n} \mu_j) = \mu_i : (\bigcap_{j=1,\, j\neq i}^{n} \mu_j)$. Also, $\bigcap_{j=1,\, j\neq i}^{n} \mu_j \nsubseteq \mu_i$. This gives $\mu : \nu = \mu_i : 1_M$, from Lemma 3.4. Hence the theorem. □

Theorem 4.7. *Let* $\mu \in L(M)$. *If* μ *has a fuzzy prime decomposition, then* μ *has a normal fuzzy prime decomposition.*

Proof. We assume that μ has a fuzzy prime decomposition $\mu = \bigcap_{i=1}^{n} \mu_i$. If $\mu_{i_1}, ..., \mu_{i_k} \in \{\mu_1, \mu_2, ..., \mu_n\}$ are such that $\mu_{i_1} : 1_M = \cdots = \mu_{i_k} : 1_M = \mu_i : 1_M$, let $\mu_i' = \bigcap_{j=1}^{k} \mu_{i_j}$. Then μ_i' is a fuzzy prime submodule of *M* and $\mu_i' : 1_M = \mu_i : 1_M$, by Theorem 4.4. Thus $\mu = \mu_1' \cap \cdots \cap \mu_m'$, where the μ_i' have distinct associated fuzzy prime ideals. If $\mu_i' \supseteq \bigcap_{j=1,\, j\neq i}^{m} \mu_j'$ for some *i*, then μ_i' is deleted. Therefore, μ has a normal fuzzy prime decomposition. □

Theorem 4.8. *Let* $\mu \in L(M)$. *Suppose that* $\mu = \bigcap_{i=1}^{n} \mu_i$ *is a normal fuzzy prime decomposition of* μ. *Then there exists a finite set* $\{\mu_1 : 1_M, ...,$

$\mu_m : 1_M\}$, *$m \leq n$, where the* $\mu_i : 1_M$ *are minimal in the set of associated fuzzy prime ideals of* $\mu = \bigcap_{i=1}^{n} \mu_i$, *such that* $\mu : 1_M = \bigcap_{i=1}^{m} (\mu_i : 1_M)$ *and* $\mu : (\bigcup_{i=1}^{m} (\mu_i : 1_M)) = \mu$ *when* $m \geq 2$.

Proof. Suppose that $\mu = \bigcap_{i=1}^{n} \mu_i$ is a normal fuzzy prime decomposition of μ. Then $\mu : 1_M = (\bigcap_{i=1}^{n} \mu_i) : 1_M = \bigcap_{i=1}^{n} (\mu_i : 1_M)$, by Lemma 3.8. Let ζ be any fuzzy prime ideal of R such that $\zeta \supseteq \mu : 1_M$. Then $\zeta \supseteq \bigcap_{i=1}^{n} (\mu_i : 1_M) \supseteq (\mu_1 : 1_M) \cdots (\mu_n : 1_M)$, by Lemma 3.10. So, $\zeta \supseteq (\mu_i : 1_M)$ for some i. Thus ζ contains some $(\mu_i : 1_M)$ that are minimal among $(\mu_1 : 1_M), \ldots, (\mu_n : 1_M)$. Hence, if we select those $(\mu_i : 1_M)$ in $\{(\mu_1 : 1_M), \ldots, (\mu_n : 1_M)\}$ that are minimal and reindex, then we have $\mu : 1_M = \bigcap_{i=1}^{m} (\mu_i : 1_M)$. If $m \geq 2$, then $\mu : (\bigcup_{i=1}^{m} (\mu_i : 1_M)) = \bigcap_{i=1}^{m} (\mu : (\mu_i : 1_M))$, from Lemma 3.9. As $(\mu_i : 1_M) \nsubseteq (\mu : 1_M) = \bigcap_{i=1}^{m} (\mu_i : 1_M)$, therefore by Lemma 3.5, we have $\mu : (\mu_i : 1_M) = \mu$, $\forall i \in \{1, 2, \ldots, n\}$. Hence $\mu : (\bigcup_{i=1}^{m} (\mu_i : 1_M)) = \mu$. □

Theorem 4.9. *Let* $\mu = \bigcap_{i=1}^{n} \mu_i$ *be a normal fuzzy prime decomposition of* μ, *and* μ_i *be isolated fuzzy prime submodules of M. Then*

$$\mu = \mu : (\bigcap_{j=1,\, j \neq i}^{n} (\mu_j : 1_M)), \ \forall i \in \{1, 2, \ldots, n\}.$$

Proof. Since $(\mu_1 : 1_M) \cdots (\mu_{i-1} : 1_M)(\mu_{i+1} : 1_M) \cdots (\mu_n : 1_M) \subseteq \bigcap_{j=1,\, j \neq i}^{n} (\mu_j : 1_M)$, it follows from the minimality of $\mu_i : 1_M$ that $\bigcap_{j=1,\, j \neq i}^{n} (\mu_j : 1_M) \nsubseteq (\mu_i : 1_M)$, and hence $\bigcap_{j=1,\, j \neq i}^{n} (\mu_j : 1_M) \nsubseteq \bigcap_{j=1}^{n} (\mu_j : 1_M) = (\mu : 1_M)$. Thus by Lemma 3.5, we get $\mu : (\bigcap_{j=1,\, j \neq i}^{n} (\mu_j : 1_M)) = \mu$, $\forall i \in \{1, 2, \ldots, n\}$. Hence the theorem. □

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