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INTUITIONISTIC FUZZY CONGRUENCES

KUL HUR, SU YOUN JANG

and

YOUNG BAE JUN

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Abstract

We introduce the concept of an intuitionistic fuzzy congruence on a semigroup and investigate some of its properties. And we study some properties under semigroup homomorphisms.

0. Introduction

The subject of fuzzy sets as an approach to a mathematical representation of vagueness in everyday language was introduced by Zadeh [23] in 1965. He generalized the idea of the characteristic function of a subset of a set X by defining a fuzzy subset of X as a map from X into $[0, 1]$. After that time, many researchers [1, 16-18, 20-22] introduced the concept of a fuzzy congruence which plays an important role in the theory of fuzzy sets and their applications. And they studied some of its properties.

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy

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sets was introduced by Atanassov [2]. Recently, Çoker and his colleagues [6, 7, 9], and Lee and Lee [19] introduced the concept of intuitionistic fuzzy topological spaces. Also, Banerjee and Basnet [3], Biswas [4], Hur and his colleagues [11, 12, 15] applied to group theory using intuitionistic fuzzy sets. In 1996, Bustince and Burillo [5] introduced the concept of intuitionistic fuzzy relations and studied some of its properties. In 2003, Deschrijver and Kerre [8] investigated some properties of the composition of intuitionistic fuzzy relations. In particular, Hur and his colleagues [13, 14] studied various properties of intuitionistic fuzzy equivalence relations on a set and intuitionistic fuzzy congruences on a lattice.

In this paper, we introduce the concept of an intuitionistic fuzzy congruence and investigate some of its properties. Also we study semigroup homomorphisms.

1. Preliminaries

In this section, we list some basic concepts and well-known results which are needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I and for any ordinary relation R on a set X, we will denote the characteristic function of R as χ_R.

Definition 1.1 [2]**.** Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, IFS) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as $\mathrm{IFS}(X)$.

Definition 1.2 [2]. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ iff $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\ ]A = (\mu_A, 1 - \mu_A)$, $\langle\ \rangle A = (1 - \nu_A, \nu_A)$.

Definition 1.3 [6]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(1) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.

(2) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 1.4 [5]. Let X be a set. Then a complex mapping $R = (\mu_R, \nu_R) : X \times X \to I \times I$ is called an *intuitionistic fuzzy relation* (in short, IFR) on X if $\mu_R(x, y) + \nu_R(x, y)$ for each $(x, y) \in X \times X$, i.e., $R \in$ IFS$(X \times X)$.

We will denote the set of all IFRs on a set X as IFR(X).

Definition 1.5 [5]. Let $R \in$ IFR(X). Then the *inverse* of R, R^{-1} is defined by $R^{-1}(x, y) = R(y, x)$ for any $x, y \in X$.

Definition 1.6 [8]. Let X be a set and let $R, Q \in$ IFR(X). Then the *composition* of R and Q, $Q \circ R$, is defined as follows: for any $x, y \in X$,

$$\mu_{Q \circ R}(x, y) = \bigvee_{z \in X} [\mu_R(x, z) \wedge \mu_Q(z, y)]$$

and

$$\nu_{Q \circ R}(x, y) = \bigwedge_{z \in X} [\nu_R(x, z) \vee \nu_Q(z, y)].$$

Result 1.A [13, Proposition 2.4]**.** Let X be a set and let $R_1, R_2, R_3, Q_1, Q_2 \in \text{IFR}(X)$. Then

(1) $(R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3)$.

(2) If $R_1 \subset R_2$ and $Q_1 \subset Q_2$, then $R_1 \circ Q_1 \subset R_2 \circ Q_2$. In particular, if $Q_1 \subset Q_2$, then $R_1 \circ Q_1 \subset R_1 \circ Q_2$.

(3) $R_1 \circ (R_2 \cup R_3) = R_1 \circ R_2 \cup R_1 \circ R_3$.

(4) $R_1 \circ (R_2 \cap R_3) = R_1 \circ R_2 \cap R_1 \circ R_3$.

(5) If $R_1 \subset R_2$, then $R_1^{-1} \subset R_2^{-1}$.

(6) $(R^{-1})^{-1} = R$ and $(R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1}$.

(7) $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$.

(8) $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$.

Definition 1.7 [5]**.** An intuitionistic fuzzy relation R on a set X is called an *intuitionistic fuzzy equivalence relation* (in short, IFER) on X if it satisfies the following conditions:

(i) it is *intuitionistic fuzzy reflexive*, i.e., $R(x, x) = (1, 0)$ for each $x \in X$.

(ii) it is *intuitionistic fuzzy symmetric*, i.e., $R^{-1} = R$.

(iii) it is *intuitionistic fuzzy transitive*, i.e., $R \circ R \subset R$.

We will denote the set of all IFERs on X as $\text{IFE}(X)$.

Result 1.B [13, Remark 2.8(4)]**.** Let R be an ordinary relation on a set X. Then R is an equivalence relation on X if and only if $(\chi_R, \chi_{R^c}) \in \text{IFE}(X)$.

Result 1.C [13, Proposition 2.14]**.** Let X be a set and let $Q, R \in \text{IFE}(X)$. If $Q \circ R = R \circ Q$, then $R \circ Q \in \text{IFE}(X)$.

Let R be an intuitionistic fuzzy equivalence relation on a set X and let $a \in X$. We define a complex mapping $Ra : X \to I \times I$ as follows: for each $x \in X$,

$$Ra(x) = R(a, x).$$

Then clearly $Ra \in \mathrm{IFS}(X)$. The intuitionistic fuzzy set Ra in X is called an *intuitionistic fuzzy equivalence class* of R containing $a \in X$. The set $\{Ra : a \in X\}$ is called the *intuitionistic fuzzy quotient set of* X *by* R and denoted by S/R.

Result 1.D [13, Theorem 2.15]**.** Let R be an intuitionistic fuzzy equivalence relation on a set X. Then the following hold:

(1) $Ra = Rb$ if and only if $R(a, b) = (1, 0)$ for any $a, b \in X$.

(2) $R(a, b) = (0, 1)$ if and only if $Ra \cap Rb = 0_{\sim}$ for any $a, b \in X$.

(3) $\bigcup_{a \in X} Ra = 1_{\sim}$.

(4) There exists the surjection $p : X \to X/R$ defined by $p(x) = Rx$ for each $x \in X$.

Definition 1.8 [13]**.** Let R be an intuitionistic fuzzy relation on a set X. For each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$, let

$$R^{(\lambda, \mu)} = \{(a, b) \in X \times X : \mu_R(a, b) \geq \lambda \text{ and } \nu_R(a, b) \leq \mu\}.$$

This set is called the (λ, μ)*-level set* of R.

It is clear that $R^{(\lambda, \mu)}$ is a relation on X.

Result 1.E [13, Theorem 2.17]**.** Let X be a set and let $R \in \mathrm{IFR}(X)$. Then $R \in \mathrm{IFE}(X)$ if and only if $R^{(\lambda, \mu)}$ is an equivalence relation on X for each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$.

Definition 1.9 [13]**.** Let X be a set, let $R \in \mathrm{IFR}(X)$ and let $\{R_\alpha\}_{\alpha \in \Gamma}$ be the family of all the IFERs on X containing R. Then $\bigcap_{\alpha \in \Gamma} R_\alpha$ is called the *IFER generated by* R and denoted by R^e.

It is easily seen that R^e is the smallest intuitionistic fuzzy equivalence relation containing R.

Definition 1.10 [13]**.** Let X be a set and let $R \in \mathrm{IFR}(X)$. Then the *intuitionistic fuzzy transitive closure* of R, denoted by R^∞, is defined as follows:

$$R^\infty = \bigcup_{n \in \mathbb{N}} R^n, \text{ where } R^n = R \circ R \circ \cdots \circ R \ (n \text{ factors}).$$

Result 1.F [13, Theorem 3.6]**.** If R is an IFR on a set X, then $R^e = [R \cup R^{-1} \cup \Delta]^\infty$.

Definition 1.11 [11]**.** Let $(X, \cdot)$ be a groupoid and let $A, B \in \mathrm{IFS}(X)$. Then the *intuitionistic fuzzy product* of A and B, $A \circ B$ is defined as follows: for any $x \in X$,

$$(A \circ B)(x) = \begin{cases} (\bigvee_{yz=x} [\mu_A(y) \wedge \mu_B(z)], \bigwedge_{yz=x} [\nu_A(y) \vee \nu_B(z)]), \\ (0, 1) \qquad \text{if } x \text{ is not expressible as } x = yz. \end{cases}$$

Definition 1.12 [11]**.** Let $(X, \cdot)$ be a groupoid and let $A \in \mathrm{IFS}(X)$. Then A is called an *intuitionistic fuzzy subgroupoid* (in short, IFGP) of X if for any $x, y \in X$,

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \text{ and } \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y).$$

We will denote the set of all IFGPs of a groupoid X as $\mathrm{IFGP}(X)$. Then it is clear that $0_\sim$ and $1_\sim \in \mathrm{IFGP}(X)$.

Definition 1.13 [15]**.** Let G be a group and let $A \in \mathrm{IFGP}(G)$. Then A is called an *intuitionistic fuzzy subgroup* (in short, IFG) of G if $A(x^{-1}) \geq A(x)$, i.e., $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$, for each $x \in G$.

We will denote the set of all IFGs of G as $\mathrm{IFG}(G)$.

Definition 1.14 [15]**.** Let G be a group and let $A \in \mathrm{IFG}(G)$. Then A is said to be *normal* if $A(xy) = A(yx)$ for any $x, y \in G$.

We will denote the family of all intuitionistic fuzzy normal subgroups of a group G as $\text{IFNG}(G)$. In particular, we will denote the set $\{N \in \text{IFNG}(G) : N(e) = (1, 0)\}$ as $\text{IFN}(G)$.

Result 1.G [15, Proposition 3.2]**.** Let A be an IFS of a group G and let $B \in \text{IFNG}(G)$. Then $A \circ B = B \circ A$.

Definition 1.15 [12]**.** Let G be a group, let $A \in \text{IFG}(G)$ and let $x \in G$. We define two complex mappings

$$Ax = (\mu_{Ax}, \nu_{Ax}) : G \to I \times I$$

and

$$xA = (\mu_{xA}, \nu_{xA}) : G \to I \times I$$

as follows respectively: for each $g \in G$,

$$Ax(g) = A(gx^{-1}) \text{ and } xA(g) = A(x^{-1}g).$$

Then Ax [resp. xA] is called the *intuitionistic fuzzy right* [resp. *left*] *coset* of G determined by x and A.

It is clear that if $A \in \text{IFNG}(G)$, then the intuitionistic fuzzy left coset and the intuitionistic fuzzy right coset of A on G coincide and in this case, we call *intuitionistic fuzzy coset* instead of intuitionistic fuzzy left coset or intuitionistic fuzzy right coset.

2. Intuitionistic Fuzzy Congruences

Definition 2.1 [10]**.** A relation R on a groupoid S is said to be

(1) *left compatible* if $(a, b) \in R$ implies $(xa, xb) \in R$ for any a, b, $x \in S$,

(2) *right compatible* if $(a, b) \in R$ implies $(ax, bx) \in R$ for any a, b, $x \in S$,

(3) *compatible* if $(a, b) \in R$ and $(c, d) \in R$ imply $(ab, cd) \in R$ for any a, b, c, $d \in R$,

(4) a *left* [resp. *right*] *congruence* on S if it is a left [resp. right] compatible equivalence relation.

(5) a *congruence* on S if it is a compatible equivalence relation.

It is well known [10, Proposition I.5.1] that a relation R on a groupoid S is a congruence if and only if it is both a left and a right congruence on S.

Now we will introduce the notion of an intuitionistic fuzzy compatible relation on a groupoid.

Definition 2.2. An IFR R on a groupoid S is said to be

(1) *intuitionistic fuzzy left compatible* if $\mu_R(x, y) \leq \mu_R(zx, zy)$ and $\nu_R(x, y) \geq \nu_R(zx, zy)$, for any $x, y, z \in S$.

(2) *intuitionistic fuzzy right compatible* if $\mu_R(x, y) \leq \mu_R(xz, yz)$ and $\nu_R(x, y) \geq \nu_R(xz, yz)$, for any $x, y, z \in S$.

(3) *intuitionistic fuzzy compatible* if $\mu_R(x, y) \wedge \mu_R(z, t) \leq \mu_R(xz, yt)$ and $\nu_R(x, y) \vee \nu_R(z, t) \geq \nu_R(xz, yt)$, for any $x, y, z, t \in S$.

Example 2.3. Let $S = \{e, a, b\}$ be the groupoid with multiplication table:

	e	a	b
e	e	a	b
a	a	b	a
b	b	b	a

(1) Let $R_1 = (\mu_{R_1}, \nu_{R_1}) : S \times S \to I \times I$ be the complex mapping defined as the following matrix:

R_1	e	a	b
e	(λ_{11}, μ_{11})	(λ_{12}, μ_{12})	(λ_{13}, μ_{13})
a	(λ_{21}, μ_{21})	(λ_{22}, μ_{22})	(λ_{23}, μ_{23})
b	(λ_{31}, μ_{31})	(λ_{32}, μ_{32})	(λ_{33}, μ_{33})

where $(\lambda_{ij}, \mu_{ij}) \in I \times I$ with $\lambda_{ij} + \mu_{ij} \leq 1$, $1 \leq i, j \leq n$ such that (λ_{11}, μ_{11}) and (λ_{21}, μ_{21}) are arbitrary, and

$$\lambda_{22} \geq S_1, \mu_{23} \leq \mu_{12}, \lambda_{22} \geq \lambda_{13}, \mu_{22} \leq \mu_{13},$$

$$\lambda_{33} \geq S_1,\ \mu_{33} \leq \mu_{12},\ \lambda_{22} \geq \lambda_{31},\ \mu_{22} \leq \mu_{31},$$

$$\lambda_{32} \geq \lambda_{13},\ \mu_{32} \leq \mu_{13}.$$

Then we can see that R_1 is an intuitionistic fuzzy left compatible relation in S.

(2) Let $R_2 = (\mu_{R_2}, \nu_{R_2}) : S \times S \to I \times I$ be the complex mapping defined as the following matrix:

R_2	e	a	b
e	(λ_{11}, μ_{11})	(λ_{12}, μ_{12})	(λ_{13}, μ_{13})
a	(λ_{21}, μ_{21})	(λ_{22}, μ_{22})	(λ_{23}, μ_{23})
b	(λ_{31}, μ_{31})	(λ_{32}, μ_{32})	(λ_{33}, μ_{33})

where $(\lambda_{ij}, \mu_{ij}) \in I \times I$ with $\lambda_{ij} + \mu_{ij} \leq 1,\ 1 \leq i, j \leq n$ such that (λ_{11}, μ_{11}) and (λ_{21}, μ_{21}) are arbitrary, and

$$\lambda_{12} \leq \lambda_{23},\ \mu_{12} \geq \mu_{23},\ \lambda_{12} \leq \lambda_{32},\ \mu_{12} \geq \mu_{32},$$

$$\lambda_{13} \leq \lambda_{23},\ \mu_{13} \geq \mu_{23},\ \lambda_{13} \leq \lambda_{32},\ \mu_{13} \geq \mu_{32},$$

$$\lambda_{23} \leq \lambda_{33},\ \mu_{23} \geq \mu_{33},\ \lambda_{23} \leq \lambda_{22},\ \mu_{23} \geq \mu_{22}.$$

Then we can see that R_2 is an intuitionistic fuzzy right compatible relation in S.

(3) Let $R_3 = (\mu_{R_3}, \nu_{R_3}) : S \times S \to I \times I$ be the complex mapping defined as the following matrix:

R_3	e	a	b
e	(λ_{11}, μ_{11})	(λ_{12}, μ_{12})	(λ_{13}, μ_{13})
a	(λ_{21}, μ_{21})	(λ_{22}, μ_{22})	(λ_{23}, μ_{23})
b	(λ_{31}, μ_{31})	(λ_{32}, μ_{32})	(λ_{33}, μ_{33})

where $(\lambda_{ij}, \mu_{ij}) \in I \times I$ with $\lambda_{ij} + \mu_{ij} \leq 1,\ 1 \leq i, j \leq n$ such that

$$\lambda_{11} \wedge \lambda_{12} \leq \lambda_{12},\ \mu_{11} \vee \mu_{12} \geq \mu_{12},$$

$$\lambda_{11} \wedge \lambda_{13} \leq \lambda_{13},\ \mu_{11} \vee \mu_{13} \geq \mu_{13},$$

$$\lambda_{12} \wedge \lambda_{13} \leq \lambda_{12},\ \mu_{12} \vee \mu_{13} \geq \mu_{12},$$

$$\lambda_{21} \wedge \lambda_{22} \leq \lambda_{32},\ \mu_{21} \vee \mu_{22} \geq \mu_{32},$$

$$\lambda_{21} \wedge \lambda_{23} \leq \lambda_{33},\ \mu_{21} \vee \mu_{23} \geq \mu_{33},$$

$$\lambda_{22} \wedge \lambda_{23} \leq \lambda_{32},\ \mu_{22} \vee \mu_{33} \geq \mu_{32},$$

$$\lambda_{31} \wedge \lambda_{32} \leq \lambda_{22},\ \mu_{31} \vee \mu_{32} \geq \mu_{22},$$

$$\lambda_{31} \wedge \lambda_{33} \leq \lambda_{23},\ \mu_{31} \vee \mu_{33} \geq \mu_{23},$$

$$\lambda_{32} \wedge \lambda_{33} \leq \lambda_{22},\ \mu_{32} \vee \mu_{33} \geq \mu_{22}.$$

Then we can see that R_3 is an intuitionistic fuzzy compatible relation in S.

Lemma 2.4. *Let R be a relation on a groupoid S. Then R is left compatible if and only if (χ_R, χ_{R^c}) is intuitionistic fuzzy left compatible.*

Proof. ($\Rightarrow$) Suppose R is left compatible. Let $a, b, x \in S$.

Case (i). Suppose $(a, b) \in R$. Then $\chi_R(a, b) = 1$ and $\chi_{R^c}(a, b) = 0$. Since R is left compatible, $(xa, xb) \in R$. Thus $\chi_R(xa, xb) = 1 = \chi_R(a, b)$ and $\chi_{R^c}(xa, xb) = 0 = \chi_{R^c}(a, b)$.

Case (ii). Suppose $(a, b) \notin R$. Then $\chi_R(a, b) = 0 \leq \chi_R(xa, xb)$ and $\chi_{R^c}(a, b) = 1 \geq \chi_{R^c}(xa, xb)$. Hence, in either cases, (χ_R, χ_{R^c}) is intuitionistic fuzzy left compatible.

($\Leftarrow$) Suppose (χ_R, χ_{R^c}) is intuitionistic fuzzy left compatible. Let $a, b, x \in S$ and suppose $(a, b) \in R$. Since (χ_R, χ_{R^c}) is intuitionistic fuzzy left compatible, $\chi_R(xa, xb) \geq \chi_R(a, b) = 1$ and $\chi_{R^c}(xa, xb) \leq \chi_{R^c}(a, b) = 0$. Thus $\chi_R(xa, xb) = 1$ and $\chi_{R^c}(xa, xb) = 0$. So $(xa, xb) \in R$. Hence R is left compatible.

Lemma 2.4′ [the dual of Lemma 2.4]**.** *Let R be a relation on a groupoid S. Then R is right compatible if and only if (χ_R, χ_{R^c}) is intuitionistic fuzzy right compatible.*

Definition 2.5. An IFER R on a groupoid S is called an:

(1) *intuitionistic fuzzy left congruence* (in short, IFLC) if it is intuitionistic fuzzy left compatible.

(2) *intuitionistic fuzzy right congruence* (in short, IFRC) if it is intuitionistic fuzzy right compatible.

(3) *intuitionistic fuzzy congruence* (in short, IFC) if it is intuitionistic fuzzy compatible.

We will denote the set of all IFCs [resp. IFLCs and IFRCs] on a groupoid S as IFC(S) [resp. IFLC(S) and IFRC(S)]. It is clear that $\Delta, \nabla \in \mathrm{IFC}(S)$.

Example 2.6. Let $S = \{e, a, b\}$ be the groupoid defined in Example 2.3. Let $R = (\mu_R, \nu_R) : S \times S \to I \times I$ defined as the following matrix:

R	e	a	b
e	(1, 0)	(0.6, 0.4)	(0.6, 0.4)
a	(0.6, 0.4)	(1, 0)	(0.7, 0.2)
b	(0.6, 0.4)	(0.7, 0.2)	(1, 0)

Then it can easily be checked that $R \in \mathrm{IFE}(S)$. Moreover we can see that $R \in \mathrm{IFC}(S)$.

Proposition 2.7. *Let S be a groupoid and let $R \in \mathrm{IFE}(S)$. Then $R \in \mathrm{IFC}(S)$ if and only if it is both an IFLC and an IFRC.*

Proof. ($\Rightarrow$) Suppose $R \in \mathrm{IFC}(S)$ and let $x, y, z \in S$. Then

$$\mu_R(x, y) = \mu_R(x, y) \wedge \mu_R(z, z) \leq \mu_R(xz, yz)$$

and

$$\nu_R(x, y) = \nu_R(x, y) \vee \nu_R(z, z) \geq \nu_R(xz, yz).$$

Also,

$$\mu_R(x, y) = \mu_R(z, z) \wedge \mu_R(x, y) \leq \mu_R(zx, zy)$$

and

$$\nu_R(x, y) = \nu_R(z, z) \vee \nu_R(x, y) \geq \nu_R(zx, zy).$$

Hence R is both an IFLC and an IFRC.

($\Leftarrow$) Suppose R is both an IFLC and an IFRC. Let $x, y, z, t \in S.$ Then

$$\begin{aligned}\mu_R(x, y) \wedge \mu_R(z, t) &= \mu_R(x, y) \wedge \mu_R(z, z) \wedge \mu_R(y, y) \wedge \mu_R(z, t)\\ &\leq \mu_R(xz, yz) \wedge \mu_R(yz, yt)\\ &\leq \mu_R(xz, yt) \quad (\text{since } R \circ R \subset R)\end{aligned}$$

and

$$\begin{aligned}\nu_R(x, y) \vee \nu_R(z, t) &= \nu_R(x, y) \vee \nu_R(z, z) \vee \nu_R(y, y) \vee \nu_R(z, t)\\ &\geq \nu_R(xz, yz) \vee \nu_R(yz, yt) \geq \nu_R(xz, yt).\end{aligned}$$

So R is intuitionistic fuzzy compatible. Hence $R \in \mathrm{IFC}(S)$.

We will denote the set of all ordinary congruences on a groupoid S as $C(S)$.

The following is the immediate result of Result 1.B, Lemmas 2.4 and 2.4′, and Proposition 2.7.

Theorem 2.8. *Let R be relation on a groupoid S. Then $R \in C(S)$ if and only if* $(\chi_R, \chi_{R^c}) \in \mathrm{IFC}(S)$.

For any intuitionistic fuzzy left [resp. right] compatible relation R, it is clear that if G is a group, then $R(x, y) = R(tx, ty)$, i.e., $\mu_R(x, y) = \mu_R(tx, ty)$ and $\nu_R(x, y) = \nu_R(tx, ty)$ [resp. $R(x, y) = R(xt, yt)$, i.e., $\mu_R(x, y) = \mu_R(xt, yt)$ and $\nu_R(x, y) = \nu_R(xt, yt)$] for any $x, y, t \in G$. Hence we have the following result.

Lemma 2.9. *Let R be an IFC on a group G. Then*

$$R(xay, xby) = R(xa, xb) = R(ay, by) = R(a, b)$$

for any $a, b, x, y \in G$.

Example 2.10. Let V be the *klein* 4-*group* with the following operation table:

R	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Let $R = (\mu_R, \nu_R) : V \times V \to I \times I$ be the complex mapping defined as the following matrix:

R	e	a	b	c
e	(1, 0)	(0.6, 0.3)	(0.9, 0.1)	(0.6, 0.3)
a	(0.6, 0.3)	(1, 0)	(0.6, 0.3)	(0.9, 0.1)
b	(0.9, 0.1)	(0.6, 0.3)	(1, 0)	(0.6, 0.3)
c	(0.6, 0.3)	(0.9, 0.1)	(0.6, 0.3)	(1, 0)

Then we can see that $R \in \mathrm{IFC}(V)$. Moreover, it is easily checked that Lemma 2.9 holds: for any $s, t, x, y \in V$,

$$R(xsy, xty) = R(xs, xt) = R(sy, ty) = R(s, t).$$

The following is the immediate result of Proposition 2.7 and Lemma 2.9.

Theorem 2.11. *Let R be an intuitionistic fuzzy relation on a group G. Then $R \in \mathrm{IFC}(G)$ if and only if it is an intuitionistic fuzzy left (right) compatible equivalence relation.*

Proposition 2.12. *Let R be an intuitionistic fuzzy compatible relation on a groupoid S. Then, for each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$, the set $R^{(\lambda, \mu)}$ is a compatible relation on S.*

Proof. Let $a, b, c, d \in S$ and suppose $(a, b) \in R^{(\lambda, \mu)}$ and $(c, d) \in R^{(\lambda, \mu)}$. Then

$$\mu_R(a, b) \geq \lambda,\ \nu_R(a, b) \leq \mu \text{ and } \mu_R(c, d) \geq \lambda,\ \nu_R(c, d) \leq \mu.$$

Since R is intuitionistic fuzzy compatible,

$$\mu_R(ac, bd) \geq \mu_R(a, b) \wedge \mu_R(c, d) \geq \lambda$$

and

$$\nu_R(ac, bd) \leq \nu_R(a, b) \vee \nu_R(c, d) \leq \mu.$$

Thus $(ac, bd) \in R^{(\lambda, \mu)}$. Hence $R^{(\lambda, \mu)}$ is compatible.

The following is the immediate result of Result 1.E and Proposition 2.12.

Proposition 2.13. *Let R be an IFC on a groupoid S. Then, for each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$, the set $R^{(\lambda, \mu)}$ is a congruence on S.*

Lemma 2.14. *Let P and Q be intuitionistic fuzzy compatible relations on a groupoid S. Then $Q \circ P$ is also an intuitionistic fuzzy compatible relation on S.*

Proof. Let $a, b, x \in S$. Then

$$\begin{aligned}\mu_{Q \circ P}(ax, bx) &= \bigvee_{t \in S}[\mu_P(ax, t) \wedge \mu_Q(t, xb)] \\ &\geq \mu_P(xa, xc) \wedge \mu_Q(xc, xb) \text{ for each } c \in S \\ &\geq \mu_P(a, c) \wedge \mu_Q(c, b) \text{ for each } c \in S \\ &\qquad \text{(since } P \text{ and } Q \text{ are compatible)}\end{aligned}$$

and

$$\begin{aligned}\nu_{Q \circ P}(xa, xb) &= \bigwedge_{t \in S}[\nu_P(xa, t) \vee \nu_Q(t, xb)] \\ &\leq \nu_P(xa, xc) \vee \nu_Q(xc, xb) \text{ for each } c \in S \\ &\leq \nu_P(a, c) \vee \nu_Q(c, b) \text{ for each } c \in S.\end{aligned}$$

Thus

$$\mu_{Q \circ P}(ax, bx) \leq \bigvee_{c \in S}[\mu_P(a, c) \wedge \mu_Q(c, b)] = \mu_{Q \circ P}(a, b)$$

and

$$\nu_{Q \circ P}(ax, bx) \leq \bigwedge_{c \in S}[\nu_P(a, c) \vee \nu_Q(c, b)] = \nu_{Q \circ P}(a, b).$$

So $Q \circ P$ is intuitionistic fuzzy right compatible. By the similar arguments, we can see that $Q \circ P$ is intuitionistic fuzzy left compatible. Hence $Q \circ P$ is intuitionistic fuzzy compatible.

Theorem 2.15. *Let P and Q be intuitionistic fuzzy congruences on a groupoid S. Then the following conditions are equivalent*:

(1) $Q \circ P \in \text{IFC}(S)$.

(2) $Q \circ P \in \text{IFE}(S)$.

(3) $Q \circ P$ *is intuitionistic fuzzy symmetric*.

(4) $Q \circ P = P \circ Q$.

Proof. It is clear that (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4) Suppose the condition (3) holds and let $a, b \in S$. Then

$$\begin{aligned}\mu_{Q \circ P}(a, b) &= \bigvee_{t \in S}[\mu_P(a, t) \wedge \mu_Q(t, b)] \\ &= \bigvee_{t \in S}[\mu_Q(b, t) \wedge \mu_P(t, a)] \\ &\qquad \text{(since } P \text{ and } Q \text{ are intuitionistic fuzzy symmetric)} \\ &= \mu_{P \circ Q}(b, a)\end{aligned}$$

and

$$\nu_{Q \circ P}(a, b) = \bigwedge_{t \in S}[\nu_P(a, t) \vee \nu_Q(t, b)] = \bigwedge_{t \in S}[\nu_Q(b, t) \vee \nu_P(t, a)] = \nu_{P \circ Q}(b, a).$$

Hence $Q \circ P = P \circ Q$.

(4) $\Rightarrow$ (1) Suppose the condition (4) holds. Then, by Result 1.C, $Q, P \in \text{IFE}(S)$. Thus, by Lemma 2.14, $Q \circ P$ is intuitionistic fuzzy compatible. Hence $Q \circ P \in \text{IFC}(S)$ on S. This completes the proof.

Proposition 2.16. *Let S be a groupoid and let $Q, P \in \text{IFC}(S)$. If $P \circ Q = Q \circ P$, then $P \circ Q \in \text{IFC}(S)$.*

Proof. By Result 1.C, it is clear that $P \circ Q \in \text{IFE}(S)$. Let $x, y, t \in S$. Then

$$\begin{aligned}\mu_{P \circ Q}(x, y) &= \bigvee_{z \in S}[\mu_Q(x, z) \wedge \mu_P(z, y)] \leq \bigvee_{z \in S}[\mu_Q(xt, zt) \wedge \mu_P(zt, yt)] \\ &\qquad \text{(since } P \text{ and } Q \text{ are intuitionistic fuzzy right compatible)} \\ &\leq \bigvee_{a \in S}[\mu_Q(xt, a) \wedge \mu_P(a, yt)] = \mu_{P \circ Q}(xt, yt)\end{aligned}$$

and

$$\begin{aligned}\nu_{P\circ Q}(x,\ y) &= \bigwedge_{z\in S}[\nu_Q(x,\ z)\vee\nu_P(z,\ y)]\\ &\geq \bigwedge_{z\in S}[\nu_Q(xt,\ zt)\vee\nu_P(zt,\ yt)]\\ &\geq \bigwedge_{a\in S}[\nu_Q(xt,\ a)\wedge\nu_P(a,\ yt)] = \nu_{P\circ Q}(xt,\ yt).\end{aligned}$$

Similarly, we have $\mu_{P\circ Q}(x, y) \leq \mu_{P\circ Q}(tx, ty)$ and $\nu_{P\circ Q}(x, y) \geq \nu_{P\circ Q}(tx, ty)$. So $P \circ Q$ is intuitionistic fuzzy left and right compatible. Hence $P \circ Q \in$ IFC(S).

Let R be an intuitionistic fuzzy congruence on a semigroup S and let $a \in S$. The intuitionistic fuzzy set Ra in S is called an *intuitionistic fuzzy congruence class of R containing* $a \in S$ and we will denote the set of all intuitionistic fuzzy congruence classes of R as S/R.

Proposition 2.17. *If R is an intuitionistic fuzzy congruence on a groupoid S, then $Ra \circ Rb \subset Rab$, for any $a, b \in S$.*

Proof. Let $x \in S$. Suppose x is not expressible as $x = yz$. Then clearly $(Ra \circ Rb)(x) = (0, 1)$. Thus $Ra \circ Rb \subset Rab$. Suppose x is expressible as $x = yz$. Then

$$\begin{aligned}\mu_{Ra\circ Rb}(x) &= \bigvee_{yz=x}[\mu_{Ra}(y)\wedge\mu_{Rb}(z)] = \bigvee_{yz=x}[\mu_R(a,\ y)\wedge\mu_R(b,\ z)]\\ &\leq \bigvee_{yz=x}[\mu_R(ab,\ yz)] \text{ (since } R \text{ is intuitionistic fuzzy compatible)}\\ &= \mu_R(ab,\ x) = \mu_{Rab}(x)\end{aligned}$$

and

$$\begin{aligned}\nu_{Ra\circ Rb}(x) &= \bigwedge_{yz=x}[\nu_{Ra}(y)\vee\nu_{Rb}(z)] = \bigwedge_{yz=x}[\nu_R(a,\ y)\vee\nu_R(b,\ z)]\\ &\geq \bigwedge_{yz=x}[\nu_R(ab,\ yz)] = \nu_R(ab,\ x) = \nu_{Rab}(x).\end{aligned}$$

Thus $Ra \circ Rb \subset Rab$. Hence, in all, $Ra \circ Rb \subset Rab$.

Proposition 2.18. *Let G be a group and let $R \in \mathrm{IFC}(G)$. We define the complex mapping $A_R = (\mu_{A_R}, \nu_{A_R}) : G \to I \times I$ as follows: for each $a \in G$,*

$$A_R(a) = R(a, e) = Re(a).$$

Then $A_R = Re \in \mathrm{IFN}(G)$.

Proof. From the definition of A_R, it is clear that $A_R \in \mathrm{IFS}(G)$. Let $a, b \in G$. Then

$$\begin{aligned}\mu_{A_R}(ab) &= \mu_R(ab, e) = \mu_R(a, b^{-1}) \text{ (by Lemma 2.9)}\\ &\geq \mu_{R \circ R}(a, b^{-1}) \text{ (since } R \text{ is intuitionistic fuzzy transitive)}\\ &= \bigvee_{t \in G}[\mu_R(a, t) \wedge \mu_R(t, b^{-1})] \geq \mu_R(a, e) \wedge \mu_R(e, b^{-1})\\ &= \mu_R(a, e) \wedge \mu_R(b, e) \text{ (by Lemma 2.9)}\\ &= \mu_{A_R}(a) \wedge \mu_{A_R}(b)\end{aligned}$$

and

$$\begin{aligned}\nu_{A_R}(a, b) &= \nu_R(ab, e) = \nu_R(a, b^{-1}) \leq \nu_{R \circ R}(a, b^{-1})\\ &= \bigwedge_{t \in G}[\nu_R(a, t) \vee \nu_R(t, b^{-1})] \leq \nu_R(a, e) \vee \nu_R(e, b^{-1})\\ &= \nu_R(a, e) \vee \nu_R(b, e) = \nu_{A_R}(a) \vee \nu_{A_R}(b).\end{aligned}$$

On the other hand

$$\begin{aligned}A_R(a^{-1}) &= (\mu_{A_R}(a^{-1}), \nu_{A_R}(a^{-1})) = (\mu_R(a^{-1}, e), \nu_R(a^{-1}, e))\\ &= (\mu_R(e, a), \nu_R(e, a)) \text{ (by Lemma 2.9)}\\ &= (\mu_R(a, e), \nu_R(a, e)) \text{ (since } R \text{ is intuitionistic fuzzy symmetric)}\\ &= (\mu_{A_R}(a), \nu_{A_R}(a)) = A_R(a).\end{aligned}$$

Moreover

$$A_R(e) = (\mu_{A_R}(e), \nu_{A_R}(e)) = (\mu_R(e, e), \nu_R(e, e)) = (1, 0).$$

So $A_R \in \mathrm{IFG}(G)$ such that $A_R(e) = (1, 0)$. On the other hand

$$A_R(ab) = (\mu_{A_R}(ab), \nu_{A_R}(ab)) = (\mu_R(ab, e), \nu_R(ab, e))$$

$$= (\mu_R(b(ab)b^{-1}, beb^{-1}), \nu_R(b(ab)b^{-1}, beb^{-1})) \text{ (by Lemma 2.9)}$$

$$= (\mu_R(ba, e), \nu_R(ba, e)) = (\mu_{A_R}(ba), \nu_{A_R}(ba)) = A_R(ba).$$

Hence $A_R \in \mathrm{IFN}(G)$. This completes the proof.

The following is the immediate result of Proposition 2.18 and Result 1.G.

Proposition 2.19. *Let G be a group and let e be the identity element of G. If* $P, Q \in \mathrm{IFC}(G)$, *then* $Pe \circ Qe = Qe \circ Pe$.

Proposition 2.20. *Let G be a group. If* $R \in \mathrm{IFC}(G)$, *then any intuitionistic fuzzy congruence class Rx of* $x \in G$ *by R is an intuitionistic fuzzy coset of Re. Conversely, each intuitionistic fuzzy coset of Re is an intuitionistic fuzzy congruence class by R.*

Proof. Suppose $R \in \mathrm{IFC}(G)$ and let $x, g \in G$. Then $Rx(g) = R(x, g)$. Since R is intuitionistic fuzzy left compatible, by Lemma 2.9, $R(x, g) = R(e, x^{-1}g)$. Thus $Rx(g) = R(e, x^{-1}g) = Re(x^{-1}g) = (xRe)(g)$. So $Rx = xRe$. Hence Rx is an intuitionistic fuzzy coset of Re.

Conversely, let A be any intuitionistic fuzzy coset of Re. Then there exists an $x \in G$ such that $A = xRe$. Let $g \in G$. Then $A(g) = (xRe)(g) = Re(x^{-1}g) = R(e, x^{-1}g)$. Since R is left compatible, $R(e, x^{-1}g) = R(x, g) = Rx(g)$. Thus $A(g) = Rx(g)$. So $A = Rx$. Hence A is an intuitionistic fuzzy congruence class of x by R.

Proposition 2.21. *Let R be an intuitionistic fuzzy congruence on a groupoid S. We define the binary operation* $*$ *on* S/R *as follows: for any* $a, b \in S$,

$$Ra * Rb = Rab.$$

Then $*$ *is well-defined.*

Proof. Suppose $Ra = Rx$ and $Rb = Ry$, where $a, b, x, y \in S$. Then, by Result 1.D, $R(a, x) = R(b, y) = (1, 0)$. Thus

$$\begin{aligned}\mu_R(ab, xy) &\geq \bigvee_{z \in S}[\mu_R(ab, z) \wedge \mu_R(z, xy)] \\ &\quad \text{(since } R \text{ is intuitionistic fuzzy transitive)} \\ &\geq \mu_R(ab, xb) \wedge \mu_R(xb, xy) \geq \mu_R(a, x) \wedge \mu_R(b, y) \\ &\quad \text{(since } R \text{ is intuitionistic fuzzy right and left compatible)} \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\nu_R(ab, xy) &\leq \bigwedge_{z \in S}[\nu_R(ab, z) \vee \nu_R(z, xy)] \leq \nu_R(ab, xb) \vee \nu_R(xb, xy) \\ &\leq \nu_R(a, x) \vee \nu_R(b, y) = 0.\end{aligned}$$

Thus $\mu_R(ab, xy) = 1$ and $\nu_R(ab, xy) = 0$, i.e., $R(ab, xy) = (1, 0)$. By Result 1.D, $Rab = Rxy$. So $Ra * Rb = Rx * Ry$. Hence $*$ is well-defined.

From Proposition 2.21 and the condition of semigroup, we obtain the following result.

Theorem 2.22. *Let R be an intuitionistic fuzzy congruence on a semigroup S. Then $(S/R, *)$ is a semigroup.*

A semigroup S is called an *inverse semigroup* [10] if every $a \in S$ possesses a unique inverse, i.e., there exists a unique $a^{-1} \in S$ such that $aa^{-1}a = a$ and $a^{-1} = a^{-1}aa^{-1}$.

Corollary 2.22-1. *Let R be an intuitionistic fuzzy congruence on an inverse semigroup S. Then $(S/R, *)$ is an inverse semigroup.*

Proof. By Theorem 2.22, $(S/R, *)$ is a semigroup. Let $a \in S$. Since S is an inverse semigroup, there exists a unique inverse $a^{-1} \in S$ such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. Let $(Ra)^{-1} = Ra^{-1}$. Then

$$(Ra)^{-1} * Ra * (Ra)^{-1} = Ra^{-1} * Ra * Ra^{-1} = Ra^{-1}aa^{-1} = Ra^{-1} = (Ra)^{-1}$$

and

$$Ra * (Ra)^{-1} * Ra = Ra * Ra^{-1} * Ra = Raa^{-1}a = Ra.$$

Hence Ra^{-1} is an inverse of Ra for each $a \in S$.

An element a of a semigroup S is said to be *regular* if $a \in aSa$, i.e., there exists an $x \in S$ such that $a = axa$. The semigroup S is said to be *regular* if for each $a \in S$, a is a regular element. Corresponding to a regular element a, there exists at least one $a' \in S$ such that $a = aa'a$ and $a' = a'aa'$. Such an element a' is called an *inverse* of a.

Corollary 2.22-2. *Let R be an intuitionistic fuzzy congruence on a regular semigroup S. Then $(S/R, *)$ is a regular semigroup.*

Proof. By Theorem 2.22, $(S/R, *)$ is a semigroup. Let $a \in S$. Since S is a regular semigroup, there exists an $x \in S$ such that $a = axa$. Then clearly $Rx \in S/R$. Moreover, $Ra * Rx * Ra = Raxa = Ra$. So Ra is a regular element of S/R. Hence S/R is a regular semigroup.

Corollary 2.22-3. *Let R be an intuitionistic fuzzy congruence on a group G. Then $(G/R, *)$ is a group.*

Proof. By Theorem 2.22, $(G/R, *)$ is a semigroup. Let $x \in G$. Then $Rx * Re = Rxe = Rx = Rex = Re * Rx$. Thus Re is the identity in G/R with respect to $*$. Moreover, $Rx * Rx^{-1} = Rxx^{-1} = Re = Rx^{-1}x = Rx^{-1} * Rx$. So Rx^{-1} is the inverse of Rx with respect to $*$. Hence $(G/R, *)$ is a group.

Proposition 2.23. *Let G be a group and let $R \in \mathrm{IFC}(G)$. We define a complex mapping $\pi = (\mu_\pi, \nu_\pi) : G/R \to I \times I$ as follows: for each $x \in G$,*

$$\pi(Rx) = (\mu_{Rx}(e), \nu_{Rx}(e)).$$

Then $\pi \in \mathrm{IFG}(G/R)$.

Proof. From the definition of π, it is clear that $\pi = (\mu_\pi, \nu_\pi) \in \mathrm{IFS}(G/R)$. Let $x, y \in G$. Then

$$\mu_\pi(Rx * Ry) = \mu_\pi(Rxy) = \mu_{Rxy}(e) = \mu_R(xy, e)$$

$$\geq \mu_R(x, e) \wedge \mu_R(y, e)$$

(since R is intuitionistic fuzzy compatible)

$$= \mu_{Rx}(e) \wedge \mu_{Ry}(e) = \mu_\pi(Rx) \wedge \mu_\pi(Ry)$$

and

$$\nu_\pi(Rx * Ry) = \nu_\pi(Rxy) = \nu_{Rxy}(e) = \nu_R(xy, e) \leq \nu_R(x, e) \vee \nu_R(y, e)$$

$$= \nu_{Rx}(e) \vee \nu_{Ry}(e) = \nu_\pi(Rx) \vee \nu_\pi(Ry).$$

By the process of the proof of Corollary 2.22-1, $(R_x)^{-1} = R_{x^{-1}}$. Thus $\pi((R_x)^{-1}) = \pi(R_x^{-1}) = R(x^{-1}, e) = R(e, x) = \pi(Rx)$. So $\pi((R_x)^{-1}) = \pi(Rx)$ for each $x \in G$. Hence $\pi \in \mathrm{IFG}(G/R)$.

Proposition 2.24. *If R is an intuitionistic fuzzy congruence on an inverse semigroup S, then $R(x^{-1}, y^{-1}) = R(x, y)$ for any $x, y \in S$.*

Proof. By Corollary 2.22-1, $(S/R, *)$ is an inverse semigroup with $(Rx)^{-1} = Rx^{-1}$ for each $x \in S$. Let $x, y \in S$. Then $R(x^{-1}, y^{-1}) = Rx^{-1}(y^{-1})$ $= [Rx(y^{-1})]^{-1} = [Ry^{-1}(x)]^{-1} = [[Ry(x)]^{-1}]^{-1} = Ry(x) = R(y, x) = R(x, y)$. Hence $R(x^{-1}, y^{-1}) = R(x, y)$.

The following is the immediate result of Proposition 2.24.

Corollary 2.24. *Let R be an IFC on a group G. Then*

$$R(x^{-1}, y^{-1}) = R(x, y)$$

for any $x, y \in G$.

Proposition 2.25. *Let R be an intuitionistic fuzzy congruence on a semigroup S. Then $R^{-1}((1, 0)) = \{(a, b) \in S \times S : R(a, b) = (1, 0)\}$ is a congruence on S.*

Proof. It is clear that $R^{-1}((1, 0))$ is reflexive and symmetric. Let $(a, b), (b, c) \in R^{-1}((1, 0))$. Then $R(a, b) = R(b, c) = (1, 0)$. Thus

$$\mu_R(a, c) \geq \bigvee_{x \in S} [\mu_R(a, x) \wedge \mu_R(x, c)]$$

(since R is intuitionistic fuzzy transitive)

$$\geq \mu_R(a, b) \wedge \mu_R(b, c) = 1$$

and

$$\nu_R(a, c) \leq \bigwedge_{x \in S} [\nu_R(a, x) \vee \nu_R(x, c)] \leq \nu_R(a, b) \vee \nu_R(b, a) = 0.$$

So $R(a, c) = (1, 0)$, i.e., $(a, c) \in R^{-1}((1, 0))$. Hence $R^{-1}((1, 0))$ is an equivalence relation on S. Now let $(a, b) \in R^{-1}((1, 0))$ and let $x \in S$. Since R is an intuitionistic fuzzy congruence on S, $\mu_R(ax, bx) \geq \mu_R(a, b) = 1$ and $\nu_R(ax, bx) \leq \nu_R(a, b) = 0$. Then $R(ax, bx) = (1, 0)$. So $(ax, bx) \in R^{-1}((1, 0))$. Similarly, we have $(xa, xb) \in R^{-1}((1, 0))$. Thus $R^{-1}((1, 0))$ is compatible. Hence $R^{-1}((1, 0))$ is a congruence on S.

Let S be a semigroup. Then S^1 denotes the monoid defined as follows:

$$S^1 = \begin{cases} S & \text{if } S \text{ has the identity } 1, \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

Definition 2.26. Let S be a semigroup and let $R \in \mathrm{IFR}(S)$. Then we define a complex mapping $R^* = (\mu_{R^*}, \nu_{R^*}) : S \times S \to I \times I$ as follows: for any $c, d \in S$,

$$\mu_{R^*}(c, d) = \bigvee_{\substack{xay = c,\, xby = d \\ x,\, y \in S^1}} \mu_R(a, b)$$

and

$$\nu_{R^*}(c, d) = \bigwedge_{\substack{xay = c,\, xby = d \\ x,\, y \in S^1}} \nu_R(a, b).$$

It is clear that $R^* \in \mathrm{IFR}(S)$.

Proposition 2.27. *Let S be a semigroup and let $R, P, Q \in \mathrm{IFR}(S)$. Then*

(1) $R \subset R^*$.

(2) $(R^*)^{-1} = (R^{-1})^*$.

(3) *If* $P \subset Q$, *then* $P^* \subset Q^*$.

(4) $(R^*)^* = R^*$.

(5) $(P \cup Q)^* = P^* \cup Q^*$.

(6) $R = R^*$ *if and only if R is left and right compatible.*

Proof. The proofs of (1), (2) and (3) are clear from Definition 2.26.

(4) It is clear that $R^* \subset (R^*)^*$ by (1) and (3). Let $c, d \in S$. Then, by the process of the proof of Proposition 3.5(iv) in [21], we have $\mu_{(R^*)^*}(c, d) \leq \mu_{R^*}(c, d)$. On the other hand

$$\begin{aligned}
\nu_{(R^*)^*}(c, d) &= \bigwedge_{\substack{xay=c,\, xby=d \\ x,\, y \in S^1}} \nu_{R^*}(a, b) \\
&= \bigwedge_{\substack{xay=c,\, xby=d \\ x,\, y \in S^1}} \bigwedge_{\substack{zpt=a,\, zqt=b \\ z,\, t \in S^1}} \nu_R(p, q) \\
&\geq \bigwedge_{\substack{xzpty=c,\, xzqty=d \\ xz,\, ty \in S^1}} \nu_R(p, q) = \nu_{R^*}(c, d).
\end{aligned}$$

Thus $(R^*)^* \subset R^*$. Hence $(R^*)^* = R^*$.

(5) By (3), $P^* \subset (P \cup Q)^*$ and $Q^* \subset (P \cup Q)^*$. Thus $P^* \cup Q^* \subset (P \cup Q)^*$. Let $c, d \in S$. Then, by the process of the proof of Proposition 3.5(v) in [21], we have $\mu_{(P \cup Q)^*}(c, d) \leq \mu_{P^*}(c, d) \vee \mu_{Q^*}(c, d)$. On the other hand

$$\nu_{(P \cup Q)^*}(c, d) = \bigwedge_{\substack{xay=c,\, xby=d \\ x,\, y \in S^1}} \nu_{P \cup Q}(a, b)$$

$$= \bigwedge_{\substack{xay=c,\, xby=d \\ x,\, y\in S^1}} [\nu_P(a, b) \vee \nu_Q(a, b)]$$

$$\geq \bigwedge_{\substack{xay=c,\, xby=d \\ x,\, y\in S^1}} \nu_P(a, b) \vee \bigwedge_{\substack{xay=c,\, xby=d \\ x,\, y\in S^1}} \nu_Q(a, b)$$

$$= \nu_{R^*}(c, d) \vee \nu_{QP^*}(c, d).$$

Thus $(P \cup Q)^* \subset P^* \cup Q^*$. Hence $(P \cup Q)^* = P^* \cup Q^*$.

(6) ($\Rightarrow$) Suppose $R = R^*$ and let $c, d, e \in S$. Then

$$\mu_R(ec, ed) = \mu_{R^*}(ec, ed) = \bigvee_{\substack{xay=ec,\, xby=ed \\ x,\, y\in S^1}} \mu_R(a, b) \geq \mu_R(c, d)$$

and

$$\nu_R(ec, ed) = \nu_{R^*}(ec, ed) = \bigwedge_{\substack{xay=ec,\, xby=ed \\ x,\, y\in S^1}} \nu_R(a, b) \leq \nu_R(c, d).$$

Similarly, we have $\mu_R(ce, de) \geq \mu_R(c, d)$ and $\nu_R(ce, de) \leq \nu_R(c, d)$. Hence R is intuitionistic fuzzy left and right compatible.

($\Leftarrow$) Suppose R is intuitionistic fuzzy left and right compatible. Let $c, d \in S$. Then

$$\mu_{R^*}(c, d) = \bigvee_{\substack{xay=c,\, xby=d \\ x,\, y\in S^1}} \mu_R(a, b) \leq \bigvee_{\substack{xay=c,\, xby=d \\ x,\, y\in S^1}} \mu_R(xay, xby) = \mu_R(c, d)$$

and

$$\nu_{R^*}(c, d) = \bigwedge_{\substack{xay=c,\, xby=d \\ x,\, y\in S^1}} \nu_R(a, b) \geq \bigwedge_{\substack{xay=c,\, xby=d \\ x,\, y\in S^1}} \nu_R(xay, xby) = \nu_R(c, d).$$

Thus $R^* \subset R$. Hence, $R^* = R$.

Proposition 2.28. *If R is an IFR on a semigroup S that is intuitionistic fuzzy left and right compatible, then so is R^∞.*

Proof. Let $a, b, c \in S$ and let $n \geq 1$. Then, by the process of the proof of Proposition 3.6 in [21], $\mu_{R^n}(a, b) \leq \mu_{R^n}(ac, bc)$. On the other hand

$$\nu_{R^n}(a, b) = \bigwedge_{z_1, \ldots, z_{n-1}} [\nu_R(a, z_1) \vee \nu_R(z_1, z_2) \vee \cdots \vee \nu_R(z_{n-1}, b)]$$

$$\geq \bigwedge_{z_1, \ldots, z_{n-1}} [\nu_R(ac, z_1c) \vee \nu_R(z_1c, z_2c) \vee \nu_R(z_{n-1}c, bc)]$$

$$= \nu_{R^n}(ac, bc).$$

Similarly, we have $\mu_{R^n}(a, b) \leq \mu_{R^n}(ca, cb)$ and $\nu_{R^n}(a, b) \geq \nu_{R^n}(ca, cb)$. So R^n is intuitionistic fuzzy left and right compatible for each $n \geq 1$. Hence R^∞ is intuitionistic fuzzy left and right compatible.

Let $R \in \mathrm{IFR}(S)$ and let $\{R_\alpha\}_{\alpha \in \Gamma}$ be the family of all IFCs on S containing R. Then the intuitionistic fuzzy relation $\hat{R}$ defined by $\hat{R} = \bigcap_{\alpha \in \Gamma} R_\alpha$ is clearly the smallest intuitionistic fuzzy congruences on S containing R. In this case, $\hat{R}$ is called the *intuitionistic fuzzy congruences on S generated by R.*

Theorem 2.29. *If R is an IFR on a semigroup S, then* $\hat{R} = (R^*)^e$.

Proof. By Definition 1.9, $(R^*)^e \in \mathrm{IFE}(S)$ such that $R^* \subset (R^*)^e$. Then $R \subset (R^*)^e$. By Proposition 2.27(2) and (5), $R^* \cup (R^*)^{-1} \cup \Delta = (R \cup R^{-1} \cup \Delta)^*$. Thus, by Proposition 2.27(6) and Result 1.F, $R^* \cup (R^*)^{-1} \cup \Delta$ is intuitionistic fuzzy left and right compatible. So, by Proposition 2.28, $(R^*)^e = [R^* \cup (R^*)^{-1} \cup \Delta]^\infty$ is intuitionistic fuzzy left and right compatible. Hence, by Proposition 2.7, $(R^*)^e \in \mathrm{IFC}(S)$. Suppose $Q \in \mathrm{IFC}(S)$ such that $R \subset Q$. Then, by Proposition 2.27(3) and (4), $R^* \subset Q^* = Q$. Thus $(R^*)^e \subset Q$. Therefore $\hat{R} = (R^*)^e$.

3. Homomorphisms

Let $f : S \to T$ be a semigroup homomorphism. Then it is well known that the relation

$$Ker(f) = \{(a, b) \in S \times S : f(a) = f(b)\}$$

is a congruence on S.

The following is the immediate result of Theorem 2.8.

Proposition 3.1. *Let $f : S \to T$ be a semigroup homomorphism. Then R is an intuitionistic fuzzy congruence on S, where $R = (\chi_{Ker(f)}, \chi_{[Ker(f)]^c})$.*

In this case, R is called the *intuitionistic fuzzy kernel* of f and denoted by $\mathrm{IFK}(f)$. In fact, for any $a, b \in S$,

$$\mu_{\mathrm{IFK}(f)}(a, b) = \begin{cases} 1 & \text{if } f(a) = f(b), \\ 0 & \text{if } f(a) \neq f(b), \end{cases}$$

and

$$\nu_{\mathrm{IFK}(f)}(a, b) = \begin{cases} 0 & \text{if } f(a) = f(b), \\ 1 & \text{if } f(a) \neq f(b). \end{cases}$$

Theorem 3.2. (1) *Let R be an intuitionistic fuzzy congruence on a semigroup S. Then, the mapping $p : S \to S/R$ defined in Result* 1.D, *is an epimorphism.*

(2) *If $f : S \to T$ is a semigroup homomorphism, then there is a monomorphism $g : S/\mathrm{IFK}(f) \to T$ such that the diagram*

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "{[\mathrm{IFK}(f)]^{\sharp}}"'] & & T \\ & S/\mathrm{IFK}(f) \arrow[ur, "g"'] & \end{tikzcd}$$

commutes, where $[\mathrm{IFK}(f)]^{\sharp}$ denotes the natural mapping.

Proof. (1) Let $a, b \in S$. Then, by the definition of p and Theorem 2.22,

$$p(ab) = Rab = Ra * Rb = p(a) * p(b).$$

Thus p is a homomorphism. By Result 1.D(4), p is surjective. Hence p is an epimorphism.

(2) We define $g : S/\mathrm{IFK}(f) \to T$ by $g([\mathrm{IFK}(f)]_a) = f(a)$ for each $a \in S$.

Suppose $[\mathrm{IFK}(f)]_a = [\mathrm{IFK}(f)]_b$ for any $a, b \in S$. Then $\mathrm{IFK}(f)(a, b) = (1, 0)$, i.e., $\chi_{\mathrm{IFK}(f)}(a, b) = 1$ and $\chi_{[\mathrm{IFK}_{(f)}]^c}(a, b) = 0$. Thus $(a, b) \in Ker(f)$. So $g([\mathrm{IFK}(f)]_a) = f(a) = f(b) = g([\mathrm{IFK}(f)]_b)$. Hence g is well-defined. For any $a, b \in S$, suppose $g([\mathrm{IFK}(f)]_a) = g([\mathrm{IFK}(f)]_b)$. Then $f(a) = f(b)$. Thus $\mathrm{IFK}(f)(a, b) = (1, 0)$. By Result 1.D(1), $[\mathrm{IFK}(f)]_a = [\mathrm{IFK}(f)]_b$. So g is injective. Now let $a, b \in S$. Then

$$\begin{aligned} g([\mathrm{IFK}(f)]_a * [\mathrm{IFK}(f)]_b) &= g([\mathrm{IFK}(f)]_{ab}) = f(ab) = f(a)f(b) \\ &= g([\mathrm{IFK}(f)]_a)g([\mathrm{IFK}(f)]_b). \end{aligned}$$

So g is a homomorphism. Let $a \in S$. Then $(g([\mathrm{IFK}(f)]^\sharp))(a) = g \circ [\mathrm{IFK}(f)]^\sharp$ $= f(a)$. Hence $g \circ [\mathrm{IFK}(f)]^\sharp = f$. This completes the proof.

Theorem 3.3. *Let R and Q be intuitionistic fuzzy congruence on a semigroup such that $R \subset Q$. Then there exists a unique semigroup homomorphism $g : S/R \to S/Q$ such that the diagram*

$$\begin{tikzcd} S \arrow[rr, "Q^\sharp"] \arrow[dr, "R^\sharp"'] & & S/Q \\ & S/R \arrow[ur] & \end{tikzcd}$$

commutes and $(S/R)/\mathrm{IFK}(g)$ *is isomorphic to* S/Q, *where* $R^\sharp$ *and* $Q^\sharp$ *denote the natural mappings*, *respectively*.

Proof. Define $g : S/R \to S/Q$ by $g(Ra) = Qa$ for each $a \in S$. Suppose $Ra = Rb$. Then, by Result 1.D, $R(a, b) = (1, 0)$. Since $R \subset Q$, $1 = \mu_R(a, b) \leq \mu_Q(a, b)$ and $0 = \nu_R(a, b) \geq \nu_Q(a, b)$. Then $Q(a, b) = (1, 0)$. So $Qa = Qb$, i.e., $g(Ra) = g(Rb)$. Hence g is well-defined. Let $a, b \in S$. Then

$$g(Ra * Rb) = g(Rab) = Qab = Qa * Qb = g(Ra) * g(Rb).$$

So g is a semigroup homomorphism. The remainder of the proofs is easy. This completes the proof.

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Kul Hur and Su Youn Jang
Division of Mathematics and Informational Statistic and
Institute of Basic Natural Science
Wonkwang University
Iksan, Chonbuk, Korea 570-749
e-mail: kulhur@wonkwang.ac.kr
suyoun123@yahoo.co.kr

Young Bae Jun
Department of Mathematics Education
Gyeongsang National University
Chinju, Korea 660-701
e-mail: ybjun@gsnu.ac.kr