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# PARAMETER ESTIMATION FOR GARCH(1, 1) MODELS BASED ON KALMAN FILTER

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## Abstract

In this work, we propose a new estimate algorithm for the parameters of a GARCH(1, 1) model. This algorithm turns out to be very reliable in estimating the true parameter values of a given model. It combines quasi-maximum likelihood method, Kalman filter algorithm and the SPSA method (Simultaneous Perturbation Stochastic Approximation). Simulation results demonstrate that the algorithm is viable and promising.

## 1. Introduction

State-space models and Kalman filtering have become important and powerful tools for the statistician and the econometrician. Together they provide the researcher with a modeling framework and a computationally efficient way to compute parameter estimates over a wide range of situations. Problems involving stationary and nonstationary stochastic processes, systematically or stochastically varying parameters, and unobserved or latent



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variables (as signal extraction problems) all have been fruitfully approached with these tools. In addition, smoothing problems and time series with missing observations have been studied with methodologies based on this combination. The state-space model and the Kalman filter recursions were first introduced in linear time series models, especially for estimation and prediction of autoregressive moving average (ARMA) processes (Harvey and Phillips [19]; Pearlman [40]; Gardner et al. [15]; Jones [22]). In each of these instances the state-space formulation and the Kalman filter have yielded a modeling and estimation methodology that is much less cumbersome than more traditional regression-based approaches.

In this paper, we focus our interest for estimating parameters of GARCH(1, 1) models, which are mainly used in finance, speech signals, daily and monthly temperature measurements, wind speeds and atmospheric $CO_2$ concentrations, etc.

Autoregressive conditionally heteroscedastic (ARCH) models were introduced by Engle [10] and their GARCH (generalized ARCH) extension is due to Bollerslev [5]. In these models, the key concept is the conditional variance, that is, the variance conditional on the past. In the classical GARCH models, the conditional variance is expressed as a linear function of the squared past values of the series.

For more details, see Francq and Zakoïan [13], GARCH Models - Structure, statistical inference and financial applications.

**Definition 1.1** (Strong GARCH($p$, $q$) process)**.** Let $(\eta_t)$ be sequence of independent and identically distributed (i.i.d.) random variables $(E(\eta_t) = 0,$ $E(\eta_t^2) = 1)$. Then the process $(\varepsilon_t)$ is called a *strong GARCH*($p$, $q$) (*with respect to the sequence* $(\eta_t)$) if

$$\begin{cases} \varepsilon_t = \sigma_t \eta_t, \\ \sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{i-1}^2 + \sum_{j=1}^{p} \beta_j \sigma_{j-1}^2, \end{cases} \tag{1}$$

where $\alpha_i$ and $\beta_j$ are nonnegative constants and $\omega$ is a (strictly) positive constant.

Weiss [46, 47] established the asymptotic properties of the ordinary least squares (OLS) estimator, in the ARMA-GARCH framework, under eighth-order moments assumptions. For the same models, asymptotic results of the quasi-maximum likelihood estimator (QMLE) have been established by Ling and Li [28, 29], Ling and McAleer [31, 33] and Francq and Zakoïan [12]. In the GARCH(1, 1) case, the asymptotic properties have been established by Lumsdaine [34] (see also Lee and Hansen, [27]) for the local QMLE under the strict stationarity assumption. Berkes et al. [2] was the first paper to give a rigorous proof of the asymptotic properties of the QMLE in the GARCH($p, q$) case under very weak assumptions; see also Berkes and Horváth [3, 4], together with Boussama [8, 9]. The assumptions given in Berkes et al. [2] were weakened slightly in Francq and Zakoïan [12]. Recently Francq and Zakoïan [14] considered the test of the strict stationarity of GARCH(1, 1) models and studied the asymptotic properties of the quasi-maximum likelihood estimator without strict stationarity constraints.

The purpose of this work, is to investigate a new approach for estimating the parameters of GARCH(1, 1) model.

It deals with the quasi-maximum likelihood method, and the Kalman filter algorithm. Indeed, the main idea is to express the concerned model in state-space form, and then deduce the log-likelihood function, which can be computed with Kalman filter algorithm (see Kalman [23, 24]). To obtain the maximum of the log-likelihood function, we used the SPSA method, which provides a global optimum regardless of the initial values. We have used two examples of GARCH(1, 1) models to examine the performance of the proposed method.

The remainder of the paper proceeds as follows. Section 2 lays out the GARCH(1, 1) models and its main properties. Section 3 deals with the estimation algorithm of the parameters of the model of interest. In this section, we state the definition of the state-space form of the GARCH(1, 1) models, and the expression of the log-likelihood function obtained by applying the Kalman filter to the state space, and then maximize the likelihood using SPSA method. In Section 4, we use two numerical examples to illustrate the estimation technique discussed in Section 3. The conclusion is provided in Section 5.

## 2. Stationarity Study and Moment Properties

This section is concerned with the existence of stationary solutions (in the strict and second-order senses) and some moment properties.

When $p = q = 1$, model (1) has the form

$$\begin{cases} \varepsilon_t = \sigma_t \eta_t, & (\eta_t)\, iid\, (0, 1), \\ \sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \end{cases} \tag{2}$$

with $\omega \geq 0$, $\alpha > 0$, $\beta > 0$. Let $a(z) = \alpha z^2 + \beta$ and $\log^+ x = \max\{\log x, 0\}$.

The strict stationarity of the GARCH(1, 1) model was first studied by Nelson [39] under the assumption $E \log^+ \eta_t^2 < \infty$. His results were extended by Klüppelberg et al. [26] to the case of $E \log^+ \eta_t^2 = +\infty$. For GARCH($p$, $q$) models, the strict stationarity conditions were established by Bougerol and Picard [7]. The second-order stationarity condition for the GARCH($p$, $q$) model was obtained by Bollerslev [5].

The fourth-order moment structure and the autocovariances of the squares of GARCH processes were analyzed by Milhøj [38], Karanasos [25] and He and Teräsvirta [20]. The necessary and sufficient condition for the existence of even-order moments was established by Ling and McAleer [30], the sufficient part having been obtained by Chen and An [11]. Ling and McAleer [32] derived an existence condition for the moment of order $s$, with $s > 0$.

**Theorem 2.1** (Strict stationarity of the strong GARCH(1, 1) process, Bougerol and Picard [7])**.** *If*

$$-\infty \leq \gamma = E \log\{\alpha \eta_t^2 + \beta\} < 0, \tag{3}$$

*then the infinite sum*

$$h_t = \left\{1 + \sum_{i=1}^{\infty} a(\eta_{t-1}) \cdots a(\eta_{t-i})\right\} \omega, \tag{4}$$

*converges almost surely* (*a.s.*) *and the process* $(\varepsilon_t)$ *defined by*

$$\varepsilon_t = \sqrt{h_t}\eta_t = \left\{\omega + \sum_{i=1}^{\infty} a(\eta_{t-1})\cdots a(\eta_{t-i})\omega\right\}^{\frac{1}{2}} \eta_t, \tag{5}$$

*is the unique strictly stationary solution of model* (2). *This solution is nonanticipative and ergodic*. *If* $\gamma \geq 0$ *and* $\omega > 0$, *then there exists no strictly stationary solution*.

**Remark 2.1.** A nonanticipative solution is a process $(\varepsilon_t)$ such that $\varepsilon_t$ is a measurable function of the variables $\eta_{t-s},\ s \geq 0.$ For such processes, $\sigma_t$ is independent of the $\sigma$-field generated by $\{\eta_{t+h},\ h \geq 0\}$ and $\varepsilon_t$ is independent of the $\sigma$-field generated by $\{\eta_{t+h},\ h > 0\}$.

Condition (3) implies $\beta < 1.$ Now, if $\alpha + \beta < 1,$ then (3) is satisfied.

**Theorem 2.2** (Second-order stationarity of the GARCH(1, 1) process, Bollerslev [5])**.** *Let* $\omega > 0.$ *If* $\alpha + \beta \geq 1,$ *then a nonanticipative and second-order stationary solution to the GARCH*(1, 1) *model does not exist*. *If* $\alpha + \beta < 1,$ *then the process* $(\varepsilon_t)$ *defined by* (5) *is second-order stationary*. *More precisely*, $(\varepsilon_t)$ *is a weak white noise*. *Moreover*, *there exists no other second-order stationary and nonanticipative solution*.

**Theorem 2.3.** *We suppose that* $\alpha + \beta < 1,$ *then*

$$E(\varepsilon_t) = 0, \tag{6}$$

$$E(\varepsilon_t^2) = \frac{\omega}{1 - \alpha - \beta}, \tag{7}$$

$$E(\varepsilon_t^4) = \frac{\omega^2(1 + \alpha + \beta)}{(1 - \alpha - \beta)(1 - \mu_4\alpha^2 - \beta^2 - 2\alpha\beta)}\mu_4;\ \mu_4 = E(\eta_t^4), \tag{8}$$

$$E(\sigma_t^4) = \frac{E(\varepsilon_t^4)}{\mu_4}. \tag{9}$$

*If* $\nu_t = \varepsilon_t^2 - \sigma_t^2$ *is a martingale difference, then*

$$E(\nu_t) = 0, \tag{10}$$

$$E(\nu_t^2) = \frac{\omega^2(1+\alpha+\beta)(\mu_4 - 1)}{(1-\alpha-\beta)(1-\mu_4\alpha^2 - \beta^2 - 2\alpha\beta)} \tag{11}$$

*and the process* $(\nu_t)$ *is weak white noise*.

## 3. Estimating Algorithm of Parameters of the GARCH(1, 1) Model

Let $(\varepsilon_t)$ be a GARCH(1, 1) model defined by (2).

We suppose that $\alpha + \beta < 1$.

**Remark 3.1.** Denote by $\theta = (\omega, \alpha, \beta)'$ the GARCH(1, 1) parameter and define the QMLE by minimizing:

$$\tilde{\ell}_n(\theta) = n^{-1}\sum_{t=1}^{n}\left\{\frac{\varepsilon_t^2}{\tilde{\sigma}_t^2(\theta)} + \log\tilde{\sigma}_t^2(\theta)\right\}, \tag{12}$$

where

$$\tilde{\sigma}_t^2(\theta) = \omega + \alpha\varepsilon_{t-1}^2 + \beta\tilde{\sigma}_{t-1}^2(\theta) \text{ for } t = 1, ..., n. \tag{13}$$

With initial values for $\varepsilon_0^2$ and $\tilde{\sigma}_0^2(\theta)$ (in practice the choice of the initial values may be important in QMLE method).

Our aim is to generate $(\tilde{\sigma}_t^2(\theta))$ without any assumptions about initial values $(\varepsilon_0^2$ and $\tilde{\sigma}_0^2(\theta))$ witch are not known in practice.

Let $\theta = (\theta_1, \theta_2, \theta_3)$, where $\theta_1 = \omega, \theta_2 = \alpha, \theta_3 = \beta$, denote the vector of unknown parameters, $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ the observed data, and $F_t = (\varepsilon_1, ..., \varepsilon_t)$ be the set of observations available at time $t = 1, ..., n$. In this study, we propose estimating $\theta$ by using quasi-maximum likelihood, given by minimizing:

$$L(\varepsilon_1, \ldots, \varepsilon_n; \theta) = n^{-1}\sum_{t=1}^{n}\left\{\frac{\varepsilon_t^2}{\hat{\sigma}_{t|t-1}^2} + \log \hat{\sigma}_{t|t-1}^2\right\}, \tag{14}$$

where the $\hat{\sigma}_{t|t-1}^2$ are defined recursively, for $t \geq 1$, by the Kalman Filter, without any assumptions about initial values which is essential in other methods of estimating the likelihood function.

The key step of our algorithm is to construct a convenient state-space representation of our model.

This representation is given by:

$$\begin{cases} Z_t = AZ_{t-1} + GX_{t-1} + \omega & : \text{state equation} \\ X_t = Z_t + \nu_t & : \text{observation equation.} \end{cases} \tag{15}$$

Here the state vector is $Z_t = \sigma_t^2$, $X_t = \varepsilon_t^2$ and $\nu_t = \varepsilon_t^2 - \sigma_t^2$.

$A = \beta$ and $G = \alpha$.

The Kalman filter recursively generates an optimal forecast $\hat{Z}_{t+1|t} = E[Z_{t+1} | F_t] = \hat{\sigma}_{t+1|t}^2$ of the state vector $Z_{t+1}$, with associated mean square error $P_{t+1|t} = V[Z_{t+1} - \hat{Z}_{t+1|t}]$, $t = 1, \ldots, n$.

Given starting values $\hat{Z}_{1|0}$ and $P_{1|0}$ of Kalman filter which are derived from Theorem 2.3, the next step in Kalman filter algorithm is to calculate $\hat{Z}_{2|1}$ and $P_{2|1}$.

The calculations for $t = 2, 3, \ldots, n$ all have the same basic form, so we will describe them in general terms for step $t$.

(i) Updating the state vector $\hat{Z}_{t|t}$. Compute $P_{t|t}$ the MSE of this updated projection.

(ii) Calculate the forecast $\hat{Z}_{t+1|t}$, and the MSE $P_{t+1|t}$ of this forecast.

Using the Kalman filter we have constructed the log-likelihood function (see Hamilton [18]). It is laborious to calculate the partial derivatives of $L(\varepsilon_1, ..., \varepsilon_n; \theta)$, so there is a definite need for minimization procedures which do not require them. Therefore, we used the SPSA method, which is a stochastic optimization algorithm that does not depend on direct gradient information or measurements, rather this method is based on an approximation to the gradient formed from measurements of the loss function (see Spall [42, 45]). It ensures convergence in a finite number of steps. Also, SPSA has recently attracted considerable international attention in areas such as statistical parameter estimation, feedback control, simulation-based optimization, signal and image processing, and experimental design.

Before describing our algorithm MLKF (quasi-maximum likelihood and Kalman filter estimation), it is worthwhile to provide a sub algorithm which tests if parameters fulfill the conditions of stationarity, we will denote it by Test.

The second sub algorithm, which we must provide, concerns the computation of $L(\varepsilon_1, ..., \varepsilon_n; \theta)$ by Kalman filter, we will denote it by *KF*. These two sub algorithms will be implemented in our global estimating algorithm.

**Sub algorithm Test** $(\theta)$

Step 1: If $(\theta_2 + \theta_3 < 1)$ Then go to next.

Step 2: Else return to the previous step and take the previous point as starting point

**End Sub**

**Sub algorithm *KF*** $(\theta)$

Step 1: Given the starting condition $\hat{Z}_{1|0}$ and $P_{1|0}$

Step 2: For $t = 1$ to $n$ Do

compute $K_t,\ \hat{Z}_{t|t},\ P_{t|t},\ \hat{Z}_{t+1|t},\ P_{t+1|t}$

End For

Step 3: $som = 0$

For $t = 1$ to $n$ Do

$$som = som + \frac{1}{n}\frac{\varepsilon_t^2}{\hat{Z}_{t|t-1}} + \frac{1}{n}\log \hat{Z}_{t|t-1}$$

End For

$L(\varepsilon_1, ..., \varepsilon_n; \theta) = som.$

**End Sub.**

Now, we propose the global algorithm for parameter estimation, where we integrate all the sub algorithms described above.

**MLKF Algorithm**

Step 1: **Initialization and coefficient selection.**

Select counter index $k = 0$;

Let $\theta_0$ be an initial point(check the test of stationarity) and let $a$, $C$, $A$, $\lambda$ and $\gamma$ non-negative coefficients and $p$ a number of parameters.

Step 2: **Compute:**

$$a_k = \frac{a}{(A + k + 1)^{\lambda}} \text{ and } c_k = \frac{C}{(k+1)^{\gamma}};$$

Step 3: **Generation of the simultaneous perturbation vector.**

Generate a $p$-dimensional random perturbation vector $\Delta_k$.

A simple (and theoretically valid) choice for each component of $\Delta_k$ is to use a Bernoulli ±1 distribution with probability of $\frac{1}{2}$;

Step 4: **Loss function evaluations.**

Call sub algorithm $KF$;

Set $y(\theta_k + c_k\Delta_k) \leftarrow KF(\theta_k + c_k\Delta_k)$;

Set $y(\theta_k - c_k\Delta_k) \leftarrow KF(\theta_k - c_k\Delta_k)$;

Step 5: **Gradient approximation.**

Generate the simultaneous perturbation approximation to the unknown gradient $g(\theta_k)$:

$$g_k(\theta_k) = \frac{y(\theta_k + c_k\Delta_k) - y(\theta_k - c_k\Delta_k)}{2c_k}\begin{pmatrix} \Delta_{k1}^{-1} \\ \Delta_{k2}^{-1} \\ \cdot \\ \cdot \\ \cdot \\ \Delta_{kp}^{-1} \end{pmatrix}; \tag{16}$$

where $\Delta_{ki}$ is the *i*th component of $\Delta_k$ vector.

Step 6: **Updating θ estimate.**

Call sub algorithm Test $(\theta_k - a_k g_k(\theta_k))$;

Set $\theta_{k+1} \leftarrow \theta_k - a_k g_k(\theta_k)$;

Step 7: **Iteration or termination.**

Return to Step 3;

Set $k \leftarrow k + 1$;

Terminate the algorithm if there is little change in several successive iterates or the maximum allowable number of iterations has been reached.

**End Algorithm**

**Remark 3.2.**

- A possible choice of $\lambda$ and $\gamma$ is: $\lambda = 0.602$, $\gamma = 0.101$.
- The parameter $A$ is equal to 10% (or less) of the number of iterations.

## 4. Simulation Study

To assess the performance of our estimate algorithm, we have conducted series of simulation experiments.

In this study, we are interested to verify that our method improves the estimations obtained by quasi-maximum likelihood method (QMLE) considered in the literature see for example Francq and Zakoïan [12].

Consider two examples of model GARCH(1, 1):

1.

$$\begin{cases} \varepsilon_t = \sigma_t \eta_t, & (\eta_t) \; iid \; N(0, 1), \\ \sigma_t^2 = 1 + 0.2\varepsilon_{t-1}^2 + 0.6\sigma_{t-1}^2, \end{cases}$$

2.

$$\begin{cases} \varepsilon_t = \sigma_t \eta_t, & (\eta_t) \; iid \; N(0, 1), \\ \sigma_t^2 = 1 + 0.7\varepsilon_{t-1}^2 + 0.2\sigma_{t-1}^2. \end{cases}$$

For the models above, we generated 1000 replications of sample sizes $n = 50,$ 100 and 150.

The results of this experiment are displayed in Tables 1-2 where for each estimator we give the mean and MSE, where we used notation QMLE for the quasi-maximum likelihood estimators, MLKF for the estimation by our algorithm.

Note that the method we use to obtain the quasi-maximum likelihood estimator (QMLE) is the SPSA method.

**Table 1.** Mean and MSE of estimated parameters

| | | | MLKF | | QMLE | |
|---|---|---|---|---|---|---|
| | | True | Mean | MSE | Mean | MSE |
| $n = 50$ | $\omega$ | 1 | 1.0033 | 0.0044 | 0.9425 | 0.0109 |
| | $\alpha$ | 0.2 | 0.1970 | 0.0135 | 0.1837 | 0.0139 |
| | $\beta$ | 0.6 | 0.5953 | 0.0037 | 0.5995 | 0.0042 |
| $n = 100$ | $\omega$ | 1 | 1.0022 | 0.0036 | 0.9554 | 0.0098 |
| | $\alpha$ | 0.2 | 0.1975 | 0.0130 | 0.2110 | 0.0130 |
| | $\beta$ | 0.6 | 0.6003 | 0.0033 | 0.6035 | 0.0034 |
| $n = 150$ | $\omega$ | 1 | 0.9934 | 0.0034 | 0.9934 | 0.0033 |
| | $\alpha$ | 0.2 | 0.1998 | 0.0127 | 0.2049 | 0.0126 |
| | $\beta$ | 0.6 | 0.6002 | 0.0028 | 0.5995 | 0.0032 |

**Table 2.** Mean and MSE of estimated parameters

| | | | MLKF | | QMLE | |
|---|---|---|---|---|---|---|
| | | True | Mean | MSE | Mean | MSE |
| $n = 50$ | $\omega$ | 1 | 0.9817 | 0.0056 | 0.9662 | 0.0117 |
| | $\alpha$ | 0.7 | 0.6687 | 0.0191 | 0.6360 | 0.0178 |
| | $\beta$ | 0.2 | 0.1758 | 0.0100 | 0.1726 | 0.0260 |
| $n = 100$ | $\omega$ | 1 | 1.0036 | 0.0036 | 0.9988 | 0.0076 |
| | $\alpha$ | 0.7 | 0.6838 | 0.0058 | 0.6777 | 0.0054 |
| | $\beta$ | 0.2 | 0.1775 | 0.0088 | 0.1705 | 0.0247 |
| $n = 150$ | $\omega$ | 1 | 1.0016 | 0.0029 | 1.0044 | 0.0036 |
| | $\alpha$ | 0.7 | 0.7027 | 0.0036 | 0.6890 | 0.0111 |
| | $\beta$ | 0.2 | 0.1816 | 0.0086 | 0.1819 | 0.0110 |

The numerical results presented in the table above, showed that our algorithm succeeds, as is seen from the fact that the sample mean square errors are generally smaller than for the quasi-maximum likelihood estimators (QMLE). Hence, we can conclude that the performance of our estimation procedure is promising.

## 5. Conclusion

In this paper, we consider the quasi-maximum likelihood estimation of the parameters of GARCH(1, 1) model. The log-likelihood function constructed using the Kalman filter and is numerically maximized applying the SPSA method. The results of our simulation study show that our estimation approach succeeds and it performs better than the competitor.

The quasi-likelihood procedure (QMLE) in the nonstationary case has recently been studied by Francq and Zakoïan [14].

In the work that follows, we will see if our estimate algorithm based on Kalman filter also continue to hold irrespective of the stationarity of the underlying process. We first generate the Kalman filter in this case, then to assess the performance of our estimate algorithm we will use examples like those in the paper cited above.

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