JP Journal of Fixed Point Theory and Applications
Volume 6, Number 3, 2011, Pages 137-145
Published Online: February 2012
Available online at http://pphmj.com/journals/jpfpta.htm
Published by Pushpa Publishing House, Allahabad, INDIA

SOME NEW SECTION THEOREMS IN TOPOLOGICAL ORDERED SPACES

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Abstract

In this paper, in topological ordered spaces, by the generalized Fan-Browder fixed theorem, we obtain some new section theorems in different spaces.

1. Preliminaries

A semilattice is a partially ordered set X, with the partial ordering denoted by $\leq$, for which any pair (x, x') of elements has a least upper bound, denoted by $x \vee x'$. It is easy to see that any nonempty finite subset A of X has a least upper bound, denoted by $\sup A$. In a partially ordered set $(X, \leq)$, two arbitrary elements x and x' do not have to be comparable but, in the case where $x \leq x'$, the set $[x, x'] = \{y \in X : x \leq y \leq x'\}$ is called an *order interval*. Now assume that $(X, \leq)$ is a semilattice and $A \subseteq X$ is a nonempty finite subset, then the set $\Delta(A) = \bigcup_{a \in A}[a, \sup A]$ is well defined and it has

2010 Mathematics Subject Classification: 47H10, 54C60.

Keywords and phrases: fixed point theorem, section theorem, generalized order KKM mapping (GOKKM).

Received May 14, 2011

the following properties:

(a) $A \subseteq \Delta(A)$,

(b) if $A \subseteq A'$, then $\Delta(A) \subseteq \Delta(A')$.

We shall say that a subset $E \subseteq X$ is Δ-*convex* if, for any nonempty finite subset $A \subseteq E$, we have $\Delta(A) \subseteq E$ (see [1]).

Example 1.1 [2]**.** Let

$$X = \{(x, 1) : 0 \leq x < 1\} \cup \{(x, y) : 0 \leq y \leq 1, x \geq 1, y \geq x - 1\} \subset R^2,$$

the partial ordering of R^2 be defined by

$$(a, b), (c, d) \in R^2,$$

$$(a, b) \leq (c, d) \Leftrightarrow c - a \geq 0, d - b \geq 0 \text{ and } d - b \leq c - a.$$

Then X is Δ-convex.

For any $D \subset X$, $\mathcal{F}(D)$ denotes the family of all finite subsets of D, $\Delta(D) = \bigcup_{A \in \mathcal{F}(D)} \Delta(A)$.

Lemma 1.1 (Order KKM lemma [1])**.** *Let X be a topological semilattice with path-connected intervals and* $\{R_i : i = 0, 1, 2, ..., n\}$ *be a family of closed* (*open*) *subsets of X*. *If there exist points* $x_0, x_1, ..., x_n$ *of X such that*, *for any family* $\{i_0, ..., i_k\}$ *of indices* $\Delta(\{x_{i_0}, ..., x_{i_k}\}) \subseteq \bigcup_{j=0}^{k} R_{i_j}$, *then the set* $\bigcap_{i=0}^{n} R_i$ *is not empty*.

Definition 1.1. Let X be a nonempty set, M be a topological semilattice with path-connected intervals. A set-valued mapping $G : X \to 2^M \setminus \{\varnothing\}$ is said to be a *generalized order KKM mapping* (*GOKKM*) if for any finite subset $\{x_1, x_2, ..., x_n\} \subset X$, there exist points $y_1, ..., y_n$ of M, such that for each subset $\{y_{i_1}, ..., y_{i_k}\}$ of $\{y_1, ..., y_n\}$, we have

$$\Delta(\{y_{i_1}, y_{i_2}, ..., y_{i_k}\}) \subset \bigcup_{j=1}^{k} G(x_{i_j}).$$

Remark 1.1. If $G : X \to 2^M \backslash \{\varnothing\}$ is GOKKM, then G has the finite intersection property.

Example 1.2. The partial ordering of R^2 is defined by

$$(a, b), (c, d) \in R^2, (a, b) \leq (c, d) \Leftrightarrow a \leq c \text{ and } b \leq d.$$

Then $M = \{(x, 1) : 0 \leq x \leq 1\} \bigcup \{(1, y) : 0 \leq y \leq 1\} \subset R^2$ is Δ-convex. Take $X = [0, 1]$, $G : X \to 2^M \backslash \{\varnothing\}$, $G(x) = \{(z, 1) \in M : 0 \leq z \leq x\} \subset M$. For any finite subset $\{x_1, x_2, ..., x_n\} \subset X$, one takes $\{M_1(x_1, 1), M_2(x_2, 1), ..., M_n(x_n, 1)\} \subset M$, for each subset $\{M_{i_1}, ..., M_{i_k}\} \subset \{M_1(x_1, 1), M_2(x_2, 1), ..., M_n(x_n, 1)\}$, we have

$$\Delta(\{M_{i_1}, ..., M_{i_k}\}) = [(\min\{x_{i_1}, ..., x_{i_k}\}, 1), (\max\{x_{i_1}, ..., x_{i_k}\}, 1)]$$

$$\subset \bigcup_{j=1}^{k} G(x_{i_j}) = \{(z, 1) : 0 \leq z \leq \max\{x_{i_1}, ..., x_{i_k}\}\}.$$

Then G is GOKKM.

Definition 1.2 [3]**.** Let X, Y be two topological spaces. Then $T : X \to 2^Y$ is said to have *the local intersection property* if for each $x \in X$ with $T(x) \neq \varnothing$, there exists an open neighborhood $N(x)$ of x such that $\bigcap_{z \in N(x)} T(z) \neq \varnothing$.

Definition 1.3. Let Y be a nonempty set, X be a Δ-convex subset in a topological semilattice with path-connected intervals M and $\gamma \in (-\infty, +\infty)$. Suppose $f : X \times Y \to (-\infty, +\infty)$ is a function. Then f is said to be $\Delta - \gamma$

generalized quasi-convex (resp., *concave*) in Y if for each non-empty finite subset $\{y_1, y_2, ..., y_n\} \subset Y$, there exists a sequence $x_1, x_2, ..., x_n$ in X such that for each subsequence $x_{i_1}, x_{i_2}, ..., x_{i_k}$ and any $x_0 \in \Delta(\{x_{i_1}, x_{i_2}, ..., x_{i_k}\})$, we have

$$\gamma \leq \max_{1 \leq j \leq k} f(x_0, y_{i_j}) \quad (\text{resp., } \gamma \geq \min_{1 \leq j \leq k} f(x_0, y_{i_j})).$$

Lemma 1.2 (Generalized Fan-Browder Fixed Theorem [2]). *Let X be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals M and $F : X \to 2^X$ have the local intersection property with nonempty Δ-convex valued. Then F has a fixed point*, *i.e.*, *there exists $x^* \in X$ such that $x^* \in F(x^*)$.*

Lemma 1.3. *Let X be a nonempty set of a topological semilattice with path-connected intervals H, and $\{A_i\}_{i=1}^n$ be a finite family of n closed* (*open*) *subsets of X such that $\bigcup_{i=1}^n A_i = X$. Then for any n points $x_1, ..., x_n$ of X, there exist k indices $i_1 < \cdots < i_k$ such that*

$$\Delta(\{x_{i_1}, ..., x_{i_k}\}) \cap \left(\bigcap_{j=1}^{k} A_{i_j} \right) \neq \varnothing.$$

Proof. Let $X_0 = \{x_1, ..., x_n\} \subset X$. Define a set-valued mapping $F : X_0 \to 2^X$ by $F(x_i) = X \setminus A_i$, for each $i = 1, 2, ..., n$.

Suppose that the conclusion is false. Then for any $i_1 < \cdots < i_m$ between 1 and n,

$$\Delta(\{x_{i_1}, ..., x_{i_m}\}) \cap \left(\bigcap_{j=1}^{m} A_{i_j} \right) = \varnothing,$$

i.e.,

$$\Delta(\{x_{i_1}, ..., x_{i_m}\}) \subset X \setminus \left(\bigcap_{j=1}^{m} A_{i_j} \right) = \bigcup_{j=1}^{m} (X \setminus A_{i_j}) = \bigcup_{j=1}^{m} F(x_{i_j}).$$

By Lemma 1.1, $\bigcap_{i=1}^{n} F(x_i) \neq \varnothing,$ then $\bigcup_{i=1}^{n} A_i \neq X,$ which contradicts our hypothesis and this contradiction completes the proof.

2. Main Results

Theorem 2.1. *Let X be a nonempty Δ-convex subset of a topological semilattice with path-connected intervals M, Y be a nonempty set, $A \subset X \times Y,$ for any $y \in Y,$ the mapping $y \to \{x \in X : (x, y) \in A\}$ nonempty closed and GOKKM, and there exists a $y_0 \in Y$ such that $\{x \in X : (x, y_0) \in A\}$ be compact. Then there exists $x_0 \in X$ such that $\{x_0\} \times Y \subset A.$*

Proof. Let $G : Y \to 2^X,$ where $G(y) = \{x \in X : (x, y) \in A\},$ for any $y \in Y.$

Suppose that there exist $y_1, ..., y_n \in Y$ such that $\bigcap_{i=1}^{n} G(y_i) = \varnothing.$ Then $X \setminus \bigcap_{i=1}^{n} G(y_i) = X,$ take $A_i = X \setminus G(y_i).$ Since $G(y_i)$ closed, A_i open and $\bigcup_{i=1}^{n} A_i = X,$ by Lemma 1.3, for any n points $x_1, ..., x_n$ of X, there exist k indices $i_1 < \cdots < i_k$ such that

$$\Delta(\{x_{i_1}, ..., x_{i_k}\}) \cap \left(\bigcap_{j=1}^{k} A_{i_j} \right) \neq \varnothing.$$

Take $x^* \in \Delta(\{x_{i_1}, ..., x_{i_k}\}) \cap \left(\bigcap_{j=1}^{k} A_{i_j} \right),$ then for each $j = 1, 2, ..., k,$ $x^* \in A_{i_j} = X \setminus G(y_{i_j}),$ $x^* \notin G(y_{i_j}),$ then $\Delta(\{x_{i_1}, ..., x_{i_k}\}) \not\subset \bigcup_{j=1}^{k} G(y_{i_j}),$ G is not GOKKM, which contradicts the hypothesis, hence G has the finite intersection property, and $G(y_0)$ is compact, $\bigcap_{y \in Y} G(y) \neq \varnothing.$ Take $x_0 \in \bigcap_{y \in Y} G(y),$ then $\{x_0\} \times Y \subset A.$

Corollary 2.1. *Let X be a nonempty compact* Δ*-convex subset of a topological semilattice with path-connected intervals M, Y be a nonempty Hausdorff topological space and* $\gamma \in (-\infty, +\infty)$, $f, g : X \times Y \to (-\infty, +\infty)$ *be two functions. Suppose that*:

(1) *g is* $\Delta - \gamma$ *generalized quasi-concave in Y*;

(2) $s : Y \to X$ *is continuous such that* $g(s(y), y) \leq \gamma$ *for any* $y \in Y$;

(3) *For any* $y \in Y$, $f(\cdot, y) : X \to (-\infty, +\infty)$ *is lower semi-continuous*;

(4) *For any* $x \in X$, $y \in Y$, $f(x, y) \leq g(x, y)$.

Then there exists $x_0 \in X$ *such that*

$$f(x_0, y) \leq \gamma \text{ for any } y \in Y.$$

Proof. We define two mappings $F, G : Y \to 2^X$ by

$$F(y) = \{x \in X : f(x, y) \leq \gamma\}, \quad G(y) = \{x \in X : g(x, y) \leq \gamma\}$$

for each $y \in Y$. By (2) and (4), $G(y)$ is nonempty and $G(y) \subset F(y)$ for any $y \in Y$, and hence $F(y)$ is nonempty. By (3), $F(y)$ is closed and hence compact for any $y \in Y$. By (1), G is GOKKM and hence F is so, then $\bigcap_{y \in Y} F(y) \neq \emptyset$. Take $x_0 \in \bigcap_{y \in Y} F(y)$, we have that

$$f(x_0, y) \leq \gamma \text{ for any } y \in Y.$$

Theorem 2.2. *Let X and Y be two nonempty compact* Δ*-convex subsets of two topological semilattice with path-connected intervals M and E,* $A \subset X \times Y$.

(1) *For any* $y \in Y$, *the mapping* $G(y) = \{x \in X : (x, y) \in A\}$ *nonempty closed and* $\Delta(\{x_1, ..., x_n\}) \subset \bigcup_{i=1}^n G(y_i)$, *for any nonempty finite subset* $\{x_1, ..., x_n\} \subset X$ *with* $x_i \in G(y_i)$, $i = 1, ..., n (y_1, ..., y_n \in Y)$;

(2) *For any* $x \in X$, $\{y \in Y : (x, y) \notin A\}$ *is empty or* Δ*-convex.*

Then there exists $x^* \in X$ *such that* $\{x^*\} \times Y \subset A$.

Proof. Suppose that the conclusion is false, i.e., for any $x \in X$, there exists $y \in Y$ such that $(x, y) \notin A$, then $F(x) = \{y \in Y : (x, y) \notin A\}$ is nonempty Δ-convex.

Defined $M = G \circ F : X \to 2^X$, where $M(x) = G(F(x))$, then $M(x)$ is nonempty.

Suppose that there exists $\bar{x} \in X$ such that $M(\bar{x})$ is not Δ-convex. Then there exist $x_1, ..., x_n \in M(\bar{x})$ such that $\Delta(\{x_1, ..., x_n\}) \not\subset M(\bar{x})$. Take $x_0 \in \Delta(\{x_1, ..., x_n\})$ and $x_0 \notin M(\bar{x}) = G(F(\bar{x})) = \bigcup_{y \in F(\bar{x})} G(y)$. Since $x_i \in F(\bar{x})$, there exists $y_i \in F(\bar{x})$ such that $x_i \in G(y_i)$, $i = 1, ..., n$. By $F(\bar{x})$ is Δ-convex, $\Delta(\{y_1, ..., y_n\}) \subset F(\bar{x})$, then

$$\Delta(\{x_1, ..., x_n\}) \subset \bigcup_{i=1}^{n} G(y_i) \subset G(\Delta(\{y_1, ..., y_n\})) \subset G(F(\bar{x})) = M(\bar{x})$$

and hence $x_0 \in M(\bar{x})$, which is a contradiction. Hence, $M(x)$ is nonempty Δ-convex for any $x \in X$.

For any $z \in X$ and

$$x_0 \in M^{-1}(z),\ z \in M(x_0) = G(F(x_0)) = \bigcup_{y \in F(x_0)} G(y),$$

there exists $y_0 \in F(x_0)$ such that $z \in G(y_0)$, then $x_0 \in F^{-1}(y_0)$. Since $F^{-1}(y_0) = X \backslash G(y_0)$ is open, there exists an open neighborhood $N(x_0)$ of x_0 such that $N(x_0) \subset F^{-1}(y_0)$, i.e., for any $x \in N(x_0)$, $y_0 \in F(x)$, then $z \in G(y_0) \subset GF(x) = M(x)$, $x \in M^{-1}(z)$, hence $M^{-1}(z)$ is open. By Lemma 1.2, there exists $x^* \in X$ such that $x^* \in M(x^*) = GF(x^*)$, hence there exists $y^* \in F(x^*)$ such that $x^* \in G(y^*)$, hence $(x^*, y^*) \notin A$ and $(x^*, y^*) \in A$, which is a contradiction. Then there exists $x^* \in X$ such that $\{x^*\} \times Y \subset A$.

Theorem 2.3. *Let X and Y be two nonempty compact* Δ*-convex subsets of two topological semilattice with path-connected intervals M and E*, $A \subset X \times Y$.

(1) *The mapping* $G : Y \to 2^X$ *is transfer closed valued with nonempty* Δ*-convex values and has the local intersection property*, *where* $G(y) = \{x \in X : (x, y) \in A\}$, *for any* $y \in Y$;

(2) *For any* $x \in X$, $\{y \in Y : (x, y) \notin A\}$ *is empty or* Δ*-convex.*

Then there exists $x^* \in X$ *such that* $\{x^*\} \times Y \subset A$.

Proof. Suppose that the conclusion is false, i.e., for any $x \in X$, there exists $y \in Y$ such that $(x, y) \notin A$, then $F(x) = \{y \in Y : (x, y) \notin A\}$ is nonempty Δ-convex. And $G^{-1}(x) = Y \backslash F(x)$, $F^{-1}(y) = X \backslash G(y)$. Since G be transfer closed valued, F has the local intersection property.

Define $K = G \times F : X \times Y \to 2^{X \times Y}$, where $K(x, y) = G(y) \times F(x)$, then $K(x, y)$ is nonempty Δ-convex of $X \times Y$, for any $(x, y) \in X \times Y$, and K has local intersection property. By Lemma 1.2, there exists $(x^*, y^*) \in X \times Y$ such that $(x^*, y^*) \in K(x^*, y^*) = G(y^*) \times F(x^*)$, hence $y^* \in F(x^*)$ and $x^* \in G(y^*)$, which is a contradiction. Then there exists $x^* \in X$ such that $\{x^*\} \times Y \subset A$.

Acknowledgment

This research was supported by the Natural Science Foundation of Guangdong Province (9251064101000015), China.

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