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ON A GENERALIZATION OF INJECTIVE MODULES WITH *IN*-CONDITIONS

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Abstract

A right R-module M is called PQ-injective if every right R-homomorphism from a cyclic module A into M extends from M to M. It is shown that M is IP-quasi-injective (i.e., if every right R-homomorphism f from a module A into M with cyclic image $f(A)$ in M extends from M to M, where A is a submodule of M) if and only if M is PQ-injective and GIN-module (i.e., $l_S(A \cap B) = l_S(A) + l_S(B)$ for any submodules A and B of M). We prove that M is quasi HN-injective (i.e., if every right R-homomorphism f from A to M with finitely generated image $f(A)$ in M extends from M to M) if and only if M is PQ-injective and $l_S(N \cap K) = l_S(N) + l_S(K)$ for any finitely generated submodule N and submodule K. We also show that, for a right R-module M; the idempotents of $End(M)$ are central if and only if every direct summand of M is fully invariant. Two examples are given to show that a commutative IN-ring R need not be $CSSES$-ring and the idempotents of $End(M)$ are central for a right R-module M is not necessarily M is duo respectively.

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1. Introduction

Throughout the paper all rings have unity and all modules are unitary. The right (resp. left) annihilator of a subset N of a module is denoted by $r(N)$ (resp. $l(N)$). If M is an R-module, then we write $Soc(M)$ for the socle of M. If R is a ring, then we denote by $Soc(R_R)$, $Soc({}_RR)$ and $J(R)$, for the right socle, the left socle, and the Jacobson radical of R, respectively.

The module M is called *CS-module* if for any submodule of M is essential in a direct summand of M. CS-module is also said to be C_1 or *extending module* in the context. Every injective module is CS-module. A ring R is called *right-CS ring* if the right R-module R_R is CS-module. A module M is said to *satisfy* C_2 *condition* if every submodule, that is, isomorphic to a direct summand of M is itself a direct summand, and is said to *satisfy* C_3 *condition* if for any direct summands M_1 and M_2 of M with $M_1 \cap M_2 = 0$, $M_1 \oplus M_2$ is also a direct summand of M. A module M is called *continuous* if it is CS and (C_2); M is called *quasi-continuous* if it is CS and (C_3); and M is called $(GC2)$ if, for any submodule N of M with $N \cong M$, N is a summand of M. A ring R is called *right* (resp. *left*) *Kasch* if every simple right (resp. left) R-module embeds in R_R, or equivalently if $l(I) \neq 0$ (resp. $r(I) \neq 0$) for any maximal right (resp. left) ideal I of R. Recall that a right ideal A is called *complement* of a right ideal B if A is maximal such that $A \cap B = 0$, in which case $A \oplus B$ is essential in R_R.

Let N be any submodule of the module M. N is said to be *small* in M if $N + K \neq M$ for any proper submodule K of M. Let M be any module. If there exists an epimorphism $p : P \to M$ such that P is projective and $Ker(p)$ is small in P, then it is said that P is a *projective cover* of M and M is said to be *semiperfect* if any homomorphic image of M has a projective cover. We call a module M *CSSES-module* if M is a CS, then semiperfect module with essential socle. $CSSES$-modules generalize semisimple modules, projective uniform modules, and any domain considered as a module over itself. We call the ring R *right CSSES-ring* if the right

R-module R_R is $CSSES$-module over R. We say that a submodule K of M is *fully invariant* in M if $\lambda(K) \subseteq K$ for every $\lambda \in End(M)$ and M is called *duo module* if every submodule of M is fully invariant. The ring R is called a *duo ring* if every one-sided ideal is two-sided (equivalently $aR = Ra$ for all $a \in R$). For the unexplained terminology, the reader is referred to ([1], [5]), ([6] or [7]).

We consider the following condition (1) for rings

$$l(T \cap T') = l(T) + l(T') \text{ for all right ideals } T \text{ and } T' \text{ of } R. \tag{1}$$

The ring R is called *right Ikeda-Nakayama ring* (*right IN-ring*, for short) (see namely [2]), if (1) holds for all right ideals T and T' of R, while R is called *right Generalized Ikeda-Nakayama rings* (*right GIN-rings*, for short), if (1) holds for all right ideals T and T' with T principal.

Let M be a right R-module and $S = End(M)$. Then M is S-R-bimodule. For any $X \subseteq M$ and any $T \subseteq S$, consider

$$l_S(X) = \{s \in S : sX = 0\} \text{ and } r_M(T) = \{m \in M : Tm = 0\},$$

where $l_S(X)$ denotes the annihilator of X in S and $r_M(T)$ denotes the annihilator of T in M.

We consider the following condition (2) for modules M

$$l_S(A \cap B) = l_S(A) + l_S(B) \text{ for any submodules } A \text{ and } B \text{ of } M. \tag{2}$$

By extending "IN-rings" notion studied in [8] to modules, M is called *Ikeda-Nakayama Module* (*IN-module*, for short), if (1) holds for all submodules A and B of M. A module M is called *Generalized Ikeda-Nakayama Module* (*GIN-module*, for short) if it satisfies (2) for each pair of submodules A and B with A cyclic. GIN-modules generalizes right GIN-rings. In [3] it is proved that R is right IP-injective if and only if R is right P-injective and right GIN-ring.

2. *IN*-modules with some Injectivities

Definition 2.1. Let M be a right R-module. We call M *P-injective* (or *Principally Quasi injective*, *PQ-injective*, for short) if every right

R-homomorphism from a cyclic module A into M extends from B to M, where A and B are modules with exact row $0 \to A \to B$ (or where A is a submodule of $B = M$).

Definition 2.2. Let M be a right R-module. We call M *IP-injective module* (or *IP-quasi-injective*) if every right R-homomorphism f from a module A into M with cyclic image $f(A)$ in M extends from B (or M) to M, where A and B are modules with exact row $0 \to A \to B$ (or where A is a submodule of M).

Definition 2.3. We call a right R-module M *HN-injective* (or *simple-injective*) if every right R-homomorphism f from A to M with finitely generated (or simple) image $f(A)$ in M extends from B to M, where A and B are modules with exact row $0 \to A \to B$. If $B = M$, then HN-injective (or simple-injective) module is called *quasi HN-injective* (or *quasi simple-injective*) module.

It is obvious that every HN-injective module is IP-injective, and any IP-injective module is simple-injective and PQ-injective module.

Definition 2.4. We call a right R-module M *f-injective* if every right R-homomorphism from a finitely generated module A into M extends from B to M, where A and B are modules with exact diagram $0 \to A \to B$.

Lemma 2.5. *Let M be a right R-module with $S = End(M_R)$. Then the following are equivalent*:

(1) ${}_SM$ *is PQ-injective.*

(2) $r_M(l_S(m)) \leq Sm$ *for all* $m \in M$.

(3) $l_S(m) \leq l_S(m_1)$, *where* $m, m_1 \in M$ *implies that* $Sm_1 \leq Sm$.

(4) $Sr_M(Sf \cap l_S(m)) = Sr_M(Sf) + Sm$ *for all* $f \in S$.

(5) *If* $Sm \overset{\alpha}{\to} M$ *is S-linear, then* $(m)\alpha \in Sm$.

Proof. (1) $\Rightarrow$ (2) Let M be any PQ-injective as left S-module and $x \in r_M(l_S(m))$ for any $m \in M$. Define $Sm \overset{\varphi}{\to} M$ with $\varphi(fm) = fx$, where

$f \in S$. Then φ is well defined S-homomorphism, by assumption φ extends to a g on M. Hence $x = 1_M x = \varphi(1_M m) = gm \in Sm$. Thus $r_M(l_S(m)) \leq Sm$.

(2) ⇒ (3) Let $m, m_1 \in M$ and $l_S(m) \leq l_S(m_1)$. Then $m_1 \in r_M(l_S(m_1)) \leq r_M(l_S(m))$. By (2), $Sm_1 \leq Sr_M(l_S(m)) \leq Sm$.

(3) ⇒ (4) Let $x \in r_M(l_S(m))$. Then $l_S(m) \leq l_S(x)$. By (3), $Sx \leq Sm$, and so $Sr_M(l_S(m)) \leq Sm$, in particular, $r_M(l_S(m)) \leq Sm$. Since $f \in Sf$, it follows that $r_M(Sf) \leq r_M(f)$ and $r_M(Sf \cap l_S(m)) \leq r_M(Sf) + r_M(l_S(m)) \leq r_M(f) + Sm$, and so $Sr_M(Sf \cap l_S(m)) \leq Sr_M(f) + Sm$. As for the reverse inclusion, since $(Sf \cap l_S(m))m = 0$, $m \in r_M(Sf \cap l_S(m))$. Hence $Sm \leq Sr_M(Sf \cap l_S(m))$. On the other hand, $Sf \cap l_S(m) \leq Sf$ implies $r_M(Sf) \leq r_M(Sf \cap l_S(m))$. Hence $r_M(Sf) + Sm \leq r_M(Sf \cap l_S(m))$, and so $Sr_M(Sf) + Sm \leq Sr_M(Sf \cap l_S(m))$ which is what we aimed at proving.

(4) ⇒ (5) Let $Sm \overset{\alpha}{\to} M$ be a left S-module homomorphism with $(m)\alpha = m_1$. Then $l_S(m) \leq l_S(m_1)$, and so $r_M(l_S(m_1)) \leq r_M(l_S(m))$, therefore $Sr_M(l_S(m_1)) \leq Sr_M(l_S(m))$. By taking $f = 1_M$ in (4), $Sr_M(l_S(m)) = Sm$ holds for all $m \in M$. Hence $Sm_1 = Sr_M(l_S(m_1)) \leq Sr_M(l_S(m)) = Sm$. Since $m_1 \in Sm_1$, $m_1 \in Sm$ and then (5) follows.

(5) ⇒ (1) Let $Sm \overset{\alpha}{\to} M$ be any left S-module homomorphism. By (5), $(m)\alpha = fm$ for some $f \in S$. So the left S-homomorphism α is a left multiplication by f. Let $g \in S$ be any. Then $(gf)m = g(fm) = g(m\alpha) = (gm)\alpha = f(gm) = (fg)m$. Hence $fg = gf$ for all $g \in S$. Define $M \overset{\beta}{\to} M$ by $(m')\beta = f(m')$, where $m' \in M$. It is clear that β is a left S-homomorphism of M and $\beta_{|_{Sm}} = \alpha$. Thus (1) holds.

It is well known that a ring R is right IP-injective if and only if R is right P-injective and right GIN, that is, $l(K \cap L) = l(K) + l(L)$ for each pair of right ideals K and L of R with K principal (see [3]). It is clear that every f-injective module is PQ-injective. Also in [3] it is proved that a ring R is right f-injective if and only if R is right P-injective and $l(K \cap L) =$

$l(K)+l(L)$ for each pair of finitely generated right ideals K and L of R (see [3]). We generalize these results to module cases.

Theorem 2.6. *Let M be right R-module and $S = End(M)$. Then the following are equivalent*:

(1) *M is IP-quasi-injective*.

(2) *M is PQ-injective and GIN-module*.

Proof. We use the Hajarnavis-Norton technique (*HN* for short) (see the proof of [4, Proposition 5.2]) as it is used in the proof of [3, Theorem 2.2] in ring case.

(2) $\Rightarrow$ (1) First we suppose that $f : A_1 + A_2 \to M$ is a right R-homomorphism such that $f|_{A_1} : A_1 \to M$ extends to a $g \in S$ and $f|_{A_2} : A_2 \to M$ extends to an $h \in S$ with A_1 cyclic. Let $x \in A_1 \cap A_2$. Then $g(x) = h(x) = f(x)$ and so $(g-h)x = 0$. Then $g - h \in l_S(A_1 \cap A_2)$. By (2), there exist $g_1 \in l_S(A_1)$ and $g_2 \in l_S(A_2)$ such that $g - h = g_1 + g_2$. Let $a_1 \in A_1$ and $a_2 \in A_2$. Then $g_1(a_1) = 0$ and $g_2(a_2) = 0$, and $f(a_1 + a_2)$ $= g(a_1) + h(a_2) = (g - g_1)(a_1) + (h + g_2)(a_2) = (h + a_2)(a_1) + (h + g_2)(a_2)$ $= (h + g_2)(a_1 + a_2)$. It follows that f extends to $h + g_2$ on M.

Now let N be a submodule of M and $f \in Hom(N, M)$ with $f(N)$ cyclic. So $f(N) = f(n)R$ for some $n \in N$. Hence $N = nR + Ker(f)$. Since M is PQ-injective, $f|_{nR}$ extends on M and $f|_{Ker(f)}$ extends on M to zero homomorphism. By the preceding paragraph f extends on M.

(1) $\Rightarrow$ (2) Let N be any cyclic submodule of M. The image of N under any R-homomorphism is cyclic. By (1), any homomorphism from N to M extends on M. Hence M is PQ-injective. To show M is GIN-module, let N be a cyclic submodule of M and K be any submodule of M. Since $N \cap K$ $\leq N$ and $N \cap K \leq K$, so $l_S(N) + l_S(K) \leq l_S(N \cap K)$. Let $g \in l_S(N \cap K)$. For $n + k \in N + K$ with $n \in N$ and $k \in K$, let $f(n+k) = g(n)$. Then f is a well-defined R-homomorphism from $N + K$ to M with $f(N + K) =$ $g(N)$. Since $g(N)$ is cyclic submodule, by (1), f extends to $h \in S$ and so $g(n) = f(n+k) = h(n+k)$ for all $n \in N$ and $k \in K$. Let $k = 0$. Then $g(n)$

$= h(n)$ for all $n \in N$ and so $g - h \in l_S(N)$. Let $n = 0$. Then $h(k) = 0$ for all $k \in K$ and so $h \in l_S(K)$. Hence $g = (g - h) + h \in l_S(N) + l_S(K)$. Thus $l_S(N \cap K) = l_S(N) + l_S(K)$.

Theorem 2.7. *Let M be a right R-module and* $S = End(M)$. *Then the following are equivalent*:

(1) *M is quasi HN-injective*.

(2) *M is PQ-injective and* $l_S(N \cap K) = l_S(N) + l_S(K)$ *for any finitely generated submodule N and submodule K*.

Proof. (2) $\Rightarrow$ (1) First we suppose that $f : A_1 + A_2 \to M$ is an R-homomorphism such that $f|_{A_1} : A_1 \to M$ extends to a $g \in S$ and $f|_{A_2} : A_2 \to M$ extends to an $h \in S$ with $f(A_1)$ finitely generated. We prove that f extends to an element of S. Let $x \in A_1 \cap A_2$. Then $g(x) = h(x) = f(x)$ and so $(g - h)x = 0$. Then $g - h \in l_S(A_1 \cap A_2)$. By (2), there exist $g_1 \in l(A_1)$ and $g_2 \in l(A_2)$ such that $g - h = g_1 + g_2$. Let $a_1 \in A_1$ and $a_2 \in A_2$. Then $g_1(a_1) = 0$, $g_2(a_2) = 0$, and $f(a_1 + a_2) = g(a_1) + h(a_2) = (g - g_1)(a_1) + (h + g_2)(a_2) = (h + a_2)(a_1) + (h + g_2)(a_2) = (h + g_2)(a_1 + a_2)$. It follows that f extends to $h + g_2$ on M.

Now let N be a submodule of M and $f \in Hom(N, M)$ with $f(N)$ finitely generated. So $f(N) = f(n_1)R + f(n_2)R + \cdots + f(n_t)R$ for some $n_1, n_2, \ldots, n_t \in N$. Let $K = n_1R + n_2R + \cdots + n_tR$. Then $N = K + Ker(f)$. Since M is PQ-injective. If K is cyclic, then $f|_K$ extends to $f_1 \in S$. Assume that $K = n_1R + n_2R$ is 2-generated. Then by the preceding paragraph $f|_K$ extends to an $f_1 \in S$. By induction on the generators of K, $f|_K$ extends to an element of S. Clearly $f|_{Ker(f)}$ extends to an element of S. As in the first paragraph f extends to an element of S.

(1) $\Rightarrow$ (2) Let N be any cyclic submodule of M. The image of N under any R-homomorphism is cyclic. By (1) any homomorphism from N to M extends on M. Hence M is PQ-injective. To show M is GIN-module, let N be a finitely generated submodule of M and K be any submodule of M. Since $N \cap K \leq N$ and $N \cap K \leq K$, and so $l_S(N) + l_S(K) \leq l_S(N \cap K)$.

Let $g \in l_S(N \cap K)$. For $n + k \in N + K$ with $n \in N$ and $k \in K$, let $f(n + k) = g(n)$. Then f is a well defined R-homomorphism from $N + K$ to M. Since $g(N)$ is finitely generated, so is $f(N + K)$. By (1), f extends to an $h \in S$ and so $g(n) = f(n + k) = h(n + k)$ for all $n \in N$ and $k \in K$. Let $k = 0$. Then $g(n) = h(n)$ for all $n \in N$ and so $g - h \in l_S(N)$. Let $n = 0$. Then $h(k) = 0$ for all $k \in K$ and so $h \in l_S(K)$. Hence $g = (g - h) + h \in l_S(N) + l_S(K)$. Thus $l_S(N \cap K) = l_S(N) + l_S(K)$.

Theorem 2.8. *Let M be a right R-module and* $S = End(M)$. *Consider the following conditions*:

(1) *M is quasi simple-injective.*

(2) (a) $l_S(N \cap K) = l_S(N) + l_S(K)$ *for any submodules N and K with N simple.*

(b) *Every homomorphism from a cyclic submodule of M to M with simple image extends to an endomorphism of M.*

Then (1) $\Rightarrow$ (2).

Proof. (1) $\Rightarrow$ (2) (a) To prove $l_S(N \cap K) = l_S(N) + l_S(K)$ for any submodule N and K with N simple, we may assume that $N \cap K = 0$, otherwise that equality is obvious. Then $l_S(N \cap K) = S$. Let $g \in S$. For $n + k \in N + K$ with $n \in N$ and $k \in K$, let $f(n + k) = g(n)$. Then f is a well defined R-homomorphism from $N + K$ to M with $f(N + K) = g(N)$ simple. By (1), f extends to an $h \in S$ and so $g(n) = f(n + k) = h(n + k)$ for all $n \in N$ and $k \in K$. Let $k = 0$. Then $g(n) = h(n)$ for all $n \in N$ and so $g - h \in l_S(N)$. Let $n = 0$. Then $h(k) = 0$ for all $k \in K$ and so $h \in l_S(K)$. Hence $g = (g - h) + h \in l_S(N) + l_S(K)$. Thus $l_S(N \cap K) = l_S(N) + l_S(K)$ and so (2) (a) holds.

(1) $\Rightarrow$ (2) (b) Clear by definitions.

Theorem 2.9. *Let M be a right R-module and* $S = End(M)$. *Consider the following*:

(1) *M is quasi simple-injective.*

(2) (a) $l_S(N \cap K) = l_S(N) + l_S(K)$ *for any submodules N and K with N cyclic*.

(b) *Every homomorphism from a cyclic submodule of M to M with simple image extends to an endomorphism of M.*

Then (2) $\Rightarrow$ (1).

Proof. (2) $\Rightarrow$ (1) Let N be a submodule of M and $0 \neq f \in Hom(N, M)$ with $f(N)$ simple. So $f(N) = f(n)R$ for some $0 \neq n \in N$. Hence $N = nR + Ker(f)$ and f is defined on the cyclic submodule nR with $f(N)$ simple. By (2)(b), the restriction, say f_1 of f on nR extends to a g since $f_1(N) = f(N)$ is simple, and the restriction of f on $Ker(f)$ extends to an $h = 0$. By HN technique and the condition 2(a), we show that f extends on M as was done in the previous results:

Suppose that $f : A_1 + A_2 \to M$ is an R-homomorphism such that $f|_{A_1}: A_1 \to M$ extends to a $g \in S$ and $f|_{A_2} : A_2 \to M$ extends to an $h \in S$ with A_1 cyclic. Let $x \in A_1 \cap A_2$. Then $g(x) = h(x) = f(x)$ and so $(g - h)x = 0$. Then $g - h \in l_S(A_1 \cap A_2)$. By (2)(a) there exist $g_1 \in l_S(A_1)$ and $g_2 \in l_S(A_2)$ such that $g - h = g_1 + g_2$. Let $a_1 \in A_1$ and $a_2 \in A_2$. Then $g_1(a_1) = 0$, $g_2(a_2) = 0$, and $f(a_1 + a_2) = g(a_1) + h(a_2) = (g - g_1)(a_1) + (h + g_2)(a_2) = (h + g_2)(a_1) + (h + g_2)(a_2) = (h + g_2)(a_1 + a_2)$. It follows that f extends to $h + g_2$ on M.

Now let N be a submodule of M and $f \in Hom(N, M)$ with $f(N)$ cyclic. So $f(N) = f(n)R$ for some $n \in N$. Hence $N = nR + Ker(f)$. Since M is PQ-injective, $f|_{nR}$ extends on M to f_1 and $f|_{Ker(f)}$ extends on M to zero homomorphism. By the preceding paragraph f extends on M and (1) follows.

3. *CS*-modules with *IN*-conditions

In the rest of the paper, we discuss the implication between *CSSES*-rings and *IN*-rings and give a proof to generalize Proposition 14 of [8].

Lemma 3.1 [8, Corollary 12]**.** *If R satisfies the condition that, for any set* $\{A_i : i \in I\}$ *of right ideals such that* $\bigcap_{i\in I} A_i = 0$, $R = \sum_{i\in I} l_R(A_i)$ *and* R_R *satisfies* $(GC2)$, *then R is a semiperfect right continuous ring with a finitely generated essential right socle. In particular, R is left and right Kasch.*

Motivated by preceding Lemma 3.1 we prove the following:

Theorem 3.2. *Let R be a semiperfect ring. If R satisfies the condition that, for any set* $\{A_i : i \in I\}$ *of right ideals such that* $\bigcap_{i\in I} A_i = 0$, $R = \sum_{i\in I} l_R(A_i)$, *then R is a right CSSES-ring.*

Proof. Let R be a ring satisfying the condition that, for any set $\{A_i : i \in I\}$ of right ideals such that $\bigcap_{i\in I} A_i = 0$, $R = \sum_{i\in I} l_R(A_i)$, by [8, Proposition 11(2)] R_R is finitely cogenerated. In particular, $Soc(R_R)$ is essential in R_R. Also by [8, Theorem 8], R_R is π-injective (that is, quasi-continuous). Hence R is right *CSSES*-ring.

Definition 3.3. A right R-module M is called *strongly Ikeda-Nakayama module* if, for any set $\{A_i : i \in I\}$ of submodules such that $l_S(\bigcap_{i\in I} A_i) = \sum_{i\in I} l_S(A_i)$. M is called *dual module* if every submodule N of M is a right annihilator of a subset of $S = End_R(M)$. A ring R is called *strongly right-IN* if, for any set $\{A_i : i \in I\}$ of right ideals such that $l_R(\bigcap_{i\in I} A_i) = \sum_{i\in I} l_R(A_i)$. The ring R is called *right dual* if every right ideal of R is a right annihilator.

The following example shows that there is no implication between right *CSSES*-rings and right *IN*-rings. Notice that the following is an interesting example to be considered (see [6, Example 6.42]).

Example 3.4. There exists a commutative *IN*-ring R such that R is neither semiperfect nor $GC2$ nor Kasch nor dual. Hence R is not *CSSES*-ring.

Proof. Let R be the trivial extension of $\mathbb{Z}$ with the $\mathbb{Z}$-module $\mathbb{Z}_{2^\infty}$. Then R is also considered as the matrix ring with usual matrix operations

$$R = \left\{ \begin{bmatrix} n & m \\ 0 & n \end{bmatrix} : n \in \mathbb{Z},\ m \in \mathbb{Z}_{2^\infty} \right\}.$$

We will prefer to use matrix form for R

$$Soc(R) = \left\{ \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} : m \in (1/2 + \mathbb{Z})\mathbb{Z} \leq \mathbb{Z}_{2^\infty} \right\}$$

is essential minimal ideal, so R is finitely cogenerated.

Let $\{A_i : i \in I\}$ be any right ideals (in fact they are two sided ideals) of R such that $\bigcap_{i \in I} A_i = 0$. Then $R = \sum_{i \in I} l_R(A_i)$ since at least one of A_i is zero.

Moreover, R is IN-ring. Clear: If A is a nonzero ideal in R, then it is easily checked that $l(A) = Soc(R_R)$. If A_1 and A_2 are nonzero, then $A_1 \cap A_2$ is nonzero and so $l(A_1 \cap A_2) = Soc(R_R) = l(A_1) + l(A_2)$. Assume at least one of A_1 and A_2 is zero. Then $A_1 \cap A_2$ is zero and so $l(A_1 \cap A_2) = R = l(A_1) + l(A_2)$.

But R contains nonzero divisors which are not invertible, so R is not $(GC2)$. In fact let $a = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. Then any annihilator of a is zero. But a is not invertible, and so $aR \cong R$. Hence aR is not direct summand since R is uniform.

Let

$$I = \left\{ \begin{bmatrix} 3n & m \\ 0 & 3n \end{bmatrix} : n \in \mathbb{Z},\ m \in \mathbb{Z}_{2^\infty} \right\}.$$

Clearly R/I is a simple R-module and R/I is not isomorphic to the minimal ideal $Soc(R)$ of R, since R/I and $Soc(R)$ have distinct orders. Hence R is not Kasch. Since $Soc(R) = J(R)$ and $R/J(R)$ is not semisimple. Hence R is not semiperfect and so is not $CSSES$-ring. If R were right dual, then R would be Kasch. Hence R is not dual.

The following lemma generalizes Proposition 14 of [8].

Lemma 3.5. *Let R be a ring. Consider the following*:

(1) *Every closed right ideal of R is a right annihilator of a finite subset of R.*

(2) *R is right CS-ring.*

(3) *R is right continuous.*

Then (1) $\Leftrightarrow$ (2), (3) $\Rightarrow$ (1). *Suppose further that every finitely generated left ideal of R is a left annihilator. Then* (1) $\Rightarrow$ (3).

Proof. Note that a ring R is right continuous if and only if $R = l(P) + l(Q)$ for any right ideals P and Q with $P \cap Q = 0$, (see namely [6, Theorem 6.31]).

(1) $\Rightarrow$ (2) and (3). Let I and K be right ideals of R that are complements of each other. Since they are closed, as in [8, Proof of Proposition 14, (1) $\Rightarrow$ (2)] $R = l_R(I) + l_R(K)$. Hence R_R is right quasi-continuous. In particular R is right CS-ring.

(3) and (2) $\Rightarrow$ (1). Clear from definitions.

Lemma 3.6. *Let M_R be a right R-module and $S = End_R(M)$. Consider the following*:

(1) $l_S(A \cap B) = l_S(A) + l_S(B)$ *for all submodules A and B of M_R.*

(2) *For any submodules A and B of M_R with $A \cap B = 0$, $S = l_S(A) + l_S(B)$.*

(3) ${}_S M$ *is a CS-module as a left S-module.*

Then (1) $\Rightarrow$ (2) *and* (2) $\Leftrightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Clear from [8, Corollary 4] since M is faithful left S-module.

(3) $\Rightarrow$ (2) Let A be any submodule of M. By Zorn's lemma there exists a direct summand K of M such that A is essential in K. Let $M = K \oplus L$. By (3) $S = l_S(K) + l_S(L)$. Since $A \leq K$, $l_S(K) \leq l_S(A)$ and so $S = l_S(A) + l_S(L)$.

Proposition 3.7. *Let* M_R *be a right R-module and consider the following*:

(1) M_R *is CS-module and every idempotent of* $End(M_R)$ *is central.*

(2) M_R *is CS-module and every direct summand of* M_R *is fully invariant.*

(3) M_R *is CS-module and duo module.*

Then (1) $\Leftrightarrow$ (2), *and* (3) $\Rightarrow$ (1), (2).

Proof. (1) $\Rightarrow$ (2) Let K be any direct summand of M, π_K be the idempotent corresponding to K in S and $f \in S$ any. By (1) $f\pi_K = \pi_K f$. Hence $f(K) = f\pi_K(M) = \pi_K f(K) \leq \pi_K(M) = K$ and (2) follows.

(2) $\Rightarrow$ (1) Let π be any idempotent in S and $f \in S$ any. Since $\pi(M)$ and $(1 - \pi)(M)$ are direct summands of M, by (2) $f\pi(M) \leq \pi(M)$ and $f(1 - \pi)(M) \leq (1 - \pi)(M)$. Left multiply $f\pi(M) \leq \pi(M)$ by $1 - \pi$ to obtain $(1 - \pi)f\pi = 0$. Then $f\pi = \pi f\pi$. Left multiply $f(1 - \pi)(M) \leq (1 - \pi)(M)$ by π to obtain $\pi f(1 - \pi) = 0$. Then $\pi f = \pi f\pi$. Hence $\pi f = f\pi$. Thus π is central idempotent of S.

(3) $\Rightarrow$ (1) and (2) Clear.

The converse to Proposition 3.7 of [(3) $\Rightarrow$ (1)] is false by Faith-Menal's example as following (see namely [6, Example 8.16]).

Example 3.8. Let D be any countable, existentially closed division ring over a field F, $R = D \otimes_F F(x)$, and $T(R, D) = \left\{\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| a \in R, b \in D\right\}$ denote the extension of D by R. Then the ring $T(R, D)$ is not a duo ring and every idempotent of $T(R, D)$ is central.

Proof. It is obvious that the ring $T(R, D)$ is not a duo, since it is not a commutative ring. It is easy to check that the only direct summands of $T(R, D)$ are itself and zero right ideal or it has the identity and zero as the only idempotents. Hence every idempotent of $T(R, D)$ is central.

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