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TWO FORMULAS OF THETA FUNCTION AND THEIR APPLICATIONS IN MODULAR EQUATIONS

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Abstract

By using well-known Jacobian theta function additive identities, a formula of Jacobian theta function is obtained. A product formula of theta functions is also established. To illustrate the application of these two formulas, we derive some interesting modular equations of degree three, five, seven and nine.

1. Introduction

We suppose, throughout this paper, q denotes $\exp(\pi i\tau)$, where τ has positive imaginary part. To carry out study, we need some basic facts about the Jacobian theta functions $\theta_1(z|\tau)$, $\theta_2(z|\tau)$, $\theta_3(z|\tau)$, and $\theta_4(z|\tau)$ which are defined as

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$$\theta_1(z\,|\,\tau) = -iq^{1/4}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1)}e^{(2n+1)iz}$$

$$= 2q^{1/4}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\sin(2n+1)z, \tag{1}$$

$$\theta_2(z\,|\,\tau) = q^{1/4}\sum_{n=-\infty}^{\infty}q^{n(n+1)}e^{(2n+1)iz}$$

$$= 2q^{1/4}\sum_{n=0}^{\infty}q^{n(n+1)}\cos(2n+1)z, \tag{2}$$

$$\theta_3(z\,|\,\tau) = \sum_{n=-\infty}^{\infty}q^{n^2}e^{2niz} = 1 + 2\sum_{n=0}^{\infty}q^{n^2}\cos 2nz, \tag{3}$$

$$\theta_4(z\,|\,\tau) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}e^{2niz} = 1 + 2\sum_{n=0}^{\infty}(-1)^n q^{n^2}\cos 2nz. \tag{4}$$

From this, we readily find that $\theta_1(z\,|\,\tau)$ is an odd function and others are even functions and that

$$\theta_1(z+\pi\,|\,\tau) = -\theta_1(z\,|\,\tau), \quad \theta_1(z+\pi\tau\,|\,\tau) = -q^{-1}e^{-2iz}\theta_1(z\,|\,\tau), \tag{5}$$

$$\theta_2(z+\pi\,|\,\tau) = -\theta_2(z\,|\,\tau), \quad \theta_2(z+\pi\tau\,|\,\tau) = q^{-1}e^{-2iz}\theta_2(z\,|\,\tau), \tag{6}$$

$$\theta_3(z+\pi\,|\,\tau) = \theta_3(z\,|\,\tau), \quad \theta_3(z+\pi\tau\,|\,\tau) = -q^{-1}e^{-2iz}\theta_3(z\,|\,\tau), \tag{7}$$

$$\theta_4(z+\pi\,|\,\tau) = \theta_4(z\,|\,\tau), \quad \theta_4(z+\pi\tau\,|\,\tau) = -q^{-1}e^{-2iz}\theta_4(z\,|\,\tau). \tag{8}$$

Using the well-known Jacobi triple product identity [2, pp. 21, 22], we have

$$\theta_1(z\,|\,\tau) = 2q^{1/4}(\sin z)(q^2,\, q^2e^{2iz},\, q^2e^{-2iz};\, q^2)_\infty, \tag{9}$$

$$\theta_2(z\,|\,\tau) = 2q^{1/4}(\cos z)(q^2,\, -q^2e^{2iz},\, -q^2e^{-2iz};\, q^2)_\infty, \tag{10}$$

$$\theta_3(z|\tau) = (q^2, -qe^{2iz}, -qe^{-2iz}; q^2)_\infty, \tag{11}$$

$$\theta_4(z|\tau) = (q^2, qe^{2iz}, qe^{-2iz}; q^2)_\infty, \tag{12}$$

see for example, [7, p. 469]. For brevity, by $\theta_2(\tau)$, $\theta_3(\tau)$ and $\theta_4(\tau)$, we will represent $\theta_2(0|\tau)$, $\theta_3(0|\tau)$ and $\theta_4(0|\tau)$, respectively.

In the course of searching, we require the following Lemmas 1 and 2 can be found in [1] or [7].

Lemma 1. *If the elliptic function f has no poles, then it is constant.*

Lemma 2. *An elliptic function has two poles at least.*

Now we recall the following additive formulas (see for example [3, 4, 6]).

Theorem 1. *Let x, y be any complex numbers. Then*

$$\theta_1(x|\tau)\theta_1(y|\tau) = \theta_2(x-y|2\tau)\theta_3(x+y|2\tau) - \theta_2(x+y|2\tau)\theta_3(x-y|2\tau), \tag{13}$$

$$\theta_2(x|\tau)\theta_2(y|\tau) = \theta_2(x-y|2\tau)\theta_3(x+y|2\tau) + \theta_2(x+y|2\tau)\theta_3(x-y|2\tau), \tag{14}$$

$$\theta_3(x|\tau)\theta_3(y|\tau) = \theta_2(x-y|2\tau)\theta_2(x+y|2\tau) + \theta_3(x+y|2\tau)\theta_3(x-y|2\tau), \tag{15}$$

$$\theta_4(x|\tau)\theta_4(y|\tau) = \theta_3(x-y|2\tau)\theta_3(x+y|2\tau) - \theta_2(x+y|2\tau)\theta_2(x-y|2\tau), \tag{16}$$

from the above equations, we are able to deduce that

Theorem 2. *Let x, y be any complex numbers. Then*

$$\theta_4(x|\tau)\theta_4(y|\tau) + \theta_3(x|\tau)\theta_3(y|\tau) = 2\theta_3(x+y|2\tau)\theta_3(x-y|2\tau), \tag{17}$$

$$\theta_3(x|\tau)\theta_3(y|\tau) - \theta_4(x|\tau)\theta_4(y|\tau) = 2\theta_2(x+y|2\tau)\theta_2(x-y|2\tau), \tag{18}$$

$$\theta_2(x|2\tau)\theta_2(y|2\tau)$$

$$= \frac{1}{2}\left[\theta_2\left(\frac{x+y}{2}\Big|\tau\right)\theta_2\left(\frac{x-y}{2}\Big|\tau\right) - \theta_1\left(\frac{x+y}{2}\Big|\tau\right)\theta_1\left(\frac{x-y}{2}\Big|\tau\right)\right]. \tag{19}$$

2. Two Formulas of the Theta Functions and their Proofs

Theorem 3. *Let x, y and a be any complex numbers. Then*

$$\begin{aligned}&\theta_3(x|\tau)\theta_3(y|\tau)\theta_3(x+a|\tau)\theta_3(y+a|\tau)\\&-\theta_4(x|\tau)\theta_4(y|\tau)\theta_4(x+a|\tau)\theta_4(y+a|\tau)\\=\ &2\theta_3(x-y|2\tau)\theta_2(x-y|2\tau)\theta_2(x+y+a|\tau)\theta_2(a|\tau).\end{aligned}\tag{20}$$

Proof. For brevity, we set

$$A=\theta_3(x|\tau)\theta_3(y|\tau),\quad A'=\theta_4(x|\tau)\theta_4(y|\tau),$$

$$B=\theta_3(x+a|\tau)\theta_3(y+a|\tau),\quad B'=\theta_4(x+a|\tau)\theta_4(y+a|\tau),$$

$$W=AB-A'B'.$$

By (21) and (22), we readily find that

$$A+A'=2\theta_3(x+y|2\tau)\theta_3(x-y|2\tau),$$

$$A-A'=2\theta_2(x+y|2\tau)\theta_2(x-y|2\tau),$$

$$B+B'=2\theta_3(x+y+a|2\tau)\theta_3(x-y|2\tau),$$

$$B-B'=2\theta_2(x+y+a|2\tau)\theta_2(x-y|2\tau).$$

Substituting these in the identity

$$2(AB-A'B')=(A+A')(B-B')+(A-A')(B+B')$$

and using (23), we obtain that

$$W=2\theta_3(x-y|2\tau)\theta_2(x-y|2\tau)\theta_2(x+y+a|\tau)\theta_2(a|\tau)$$

which is what we intend to prove.

Theorem 4. *Let x be any complex number and k be any natural number. Then*

$$\begin{aligned}&\prod_{r=0}^{2k}\theta_3\left(x+\frac{r}{2k+1}\pi|\tau\right)\\&=\frac{(q;q)_\infty^{2k+1}}{(q^{2k+1};q^{2k+1})_\infty}\theta_3((2k+1)x|(2k+1)\tau).\end{aligned}\tag{21}$$

Proof. Let

$$f(x) = \prod_{r=0}^{2k} \theta_3\left(x + \frac{r}{2k+1}\pi \mid \tau\right),$$

noting that $1 + 2 + \cdots + 2k = \frac{2k(2k+1)}{2}$ and $e^{-2ki\pi} = 1.$ From (5)-(8), we are able to find that

$$f(x + \pi) = f(x),$$

$$f(x + \pi\tau) = q^{-(2k+1)} e^{-2(2k+1)ix} f(x).$$

From above two identities, we deduce that $f(x)/\theta_3((2k+1)x \mid (2k+1)\tau)$ is an elliptic function with periods π and $\pi\tau$. It is clear that $\theta_3((2k+1)x \mid (2k+1)\tau)$ has only a simple zero at $x = \frac{\pi + (2k+1)x\tau}{2(2k+1)}$ in the period parallelogram. By Lemmas 1 and 2, we know that $f(x)/\theta_3((2k+1)x \mid (2k+1)\tau)$ is an independent x. To determine the constant $C = f(x)/\theta_3((2k+1)x \mid (2k+1)\tau)$, we set $x = 0,$ using the product representations of theta functions, after simple reduction, we obtain $C = \frac{(q;\, q)_\infty^{2k+1}}{(q^{2k+1};\, q^{2k+1})_\infty}$. Thus, we finish the proof of Theorem 4.

3. Special Cases and Applications of Theorems

In this section, we find some well-known formulas of theta functions which happen to be special cases of Theorems 3 and 4. At the same time, we find some interesting results of theta functions.

In (24), setting $x = y = a = 0,$ we get that

$$\theta_3^4(\tau) - \theta_4^4(\tau) = 2\theta_3(2\tau)\theta_2(2\tau)\theta_2^2(\tau). \tag{22}$$

Using (14) and (15), we are able to know that

$$2\theta_3(2\tau)\theta_2(2\tau) = \theta_2^2(\tau). \tag{23}$$

From above two identities, we get following famous Jacobi identity:

$$\theta_3^4(\tau) - \theta_4^4(\tau) = \theta_2^4(\tau). \tag{24}$$

In (24), letting $x = y$ and $a = 0$, we are able to find that

$$\theta_3^4(x|\tau) - \theta_4^4(x|\tau) = \theta_2^3(\tau)\theta_2(2x|\tau). \tag{25}$$

In (24), letting $x = y = a$, we are able to find that

$$\theta_3^2(x|\tau)\theta_3^2(2x|\tau) - \theta_4^2(x|\tau)\theta_4^2(2x|\tau) = \theta_2^2(\tau)\theta_2(3x|\tau)\theta_2(x|\tau). \tag{26}$$

By the way, in (23), letting $x = y$, we have

$$2\theta_2(x|2\tau)\theta_3(x|2\tau) = \theta_2(\tau)\theta_2(x|\tau). \tag{27}$$

Now we study (25). We know easily that (21) holds too for $\theta_2(x|\tau)$ and $\theta_4(x|\tau)$. By (7), we note that, when $x = 0$,

$$\begin{aligned}\theta_j\left(\frac{r}{2k+1}\pi|\tau\right) &= \theta_j\left(\pi - \frac{2k+1-r}{2k+1}\pi|\tau\right) \\ &= \theta_j\left(-\frac{2k+1-r}{2k+1}\pi|\tau\right) = \theta_j\left(\frac{2k+1-r}{2k+1}\pi|\tau\right)\end{aligned} \tag{28}$$

in which $j = 2, 3, 4$. Combining with (21), we have

$$\prod_{r=1}^{k}\theta_j(\tau)\theta_j^2\left(\frac{r}{2k+1}\pi|\tau\right) = \frac{(q;\, q)_\infty^{2k+1}}{(q^{2k+1};\, q^{2k+1})_\infty}\theta_j((2k+1)\tau). \tag{29}$$

In the identity above, letting $k = 1$ and $k = 2$, we get that [6, p. 108]

$$\theta_j(\tau)\theta_j^2\left(\frac{1}{3}\pi|\tau\right) = \frac{(q;\, q)_\infty^3}{(q^3;\, q^3)_\infty}\theta_j(3\tau) \text{ for } j = 2, 3, 4, \tag{30}$$

$$\theta_j(\tau)\theta_j^2\left(\frac{1}{5}\pi|\tau\right)\theta_j^2\left(\frac{2}{5}\pi|\tau\right) = \frac{(q;\, q)_\infty^5}{(q^5;\, q^5)_\infty}\theta_j(5\tau) \text{ for } j = 2, 3, 4. \tag{31}$$

Similarly, we could get that

$$\theta_j(\tau)\theta_j^2\left(\frac{1}{7}\pi\,|\,\tau\right)\theta_j^2\left(\frac{2}{7}\pi\,|\,\tau\right)\theta_j^2\left(\frac{3}{7}\pi\,|\,\tau\right) = \frac{(q;\,q)_\infty^7}{(q^7;\,q^7)_\infty}\theta_j(7\tau)$$

$$\text{for } j = 2, 3, 4, \tag{32}$$

$$\theta_j(\tau)\theta_j^2\left(\frac{1}{9}\pi\,|\,\tau\right)\theta_j^2\left(\frac{2}{9}\pi\,|\,\tau\right)\theta_j^2\left(\frac{1}{3}\pi\,|\,\tau\right)\theta_j^2\left(\frac{4}{9}\pi\,|\,\tau\right) = \frac{(q;\,q)_\infty^9}{(q^9;\,q^9)_\infty}\theta_j(9\tau)$$

$$\text{for } j = 2, 3, 4, \tag{33}$$

and so on. From the above conclusion, we get that

Theorem 5. *We have*

$$\frac{\theta_4^2(3\tau)}{\theta_4^2(\tau)} - \frac{\theta_3^2(3\tau)}{\theta_3^2(\tau)} = 8q\frac{(q^3;\,q^3)_\infty^2}{(q;\,q)_\infty^6}(q^2;\,q^2)_\infty^3(-q^2;\,q^2)_\infty^4(-q^6;\,q^6)_\infty. \tag{34}$$

Proof. In (24), setting $x = y = a = \frac{\pi}{3}$, we easily know that

$$\theta_3^2\left(\frac{1}{3}\pi\,|\,\tau\right)\theta_3^2\left(\frac{2}{3}\pi\,|\,\tau\right) - \theta_4^2\left(\frac{1}{3}\pi\,|\,\tau\right)\theta_4^2\left(\frac{2}{3}\pi\,|\,\tau\right)$$

$$= 2\theta_3(2\tau)\theta_2(2\tau)\theta_2(\tau)\theta_2\left(\frac{1}{3}\pi\,|\,\tau\right).$$

Noting that $\theta_j\left(\frac{1}{3}\pi\,|\,\tau\right) = \theta_j\left(\frac{2}{3}\pi\,|\,\tau\right)$ for $j = 2, 3, 4,$ we obtain

$$\theta_4^4\left(\frac{1}{3}\pi\,|\,\tau\right) - \theta_3^4\left(\frac{1}{3}\pi\,|\,\tau\right) = 2\theta_3(2\tau)\theta_2(2\tau)\theta_2(\tau)\theta_2\left(\frac{1}{3}\pi\,|\,\tau\right). \tag{35}$$

From (31), we get

$$2\theta_2(2\tau)\theta_3(2\tau) = \theta_2^2(\tau).$$

Substituting (34) into the identity (39) and simplifying the resulting equation, we arrive at Theorem 5.

Theorem 6. *We have*

$$\frac{\theta_4(5\tau)}{\theta_4(\tau)} - \frac{\theta_3(5\tau)}{\theta_3(\tau)} = 4q\frac{(q^5;\, q^5)_\infty}{(q;\, q)_\infty^5}(q^2;\, q^2)_\infty^4(-q^2;\, q^2)_\infty^3(-q^{10};\, q^{10})_\infty. \quad (36)$$

Proof. In (24), letting $x = y = \frac{1}{5}\pi = a,$ we get that

$$\theta_3^2\left(\frac{1}{5}\pi \,|\, \tau\right)\theta_3^2\left(\frac{2}{5}\pi \,|\, \tau\right) - \theta_4^2\left(\frac{1}{5}\pi \,|\, \tau\right)\theta_4^2\left(\frac{2}{5}\pi \,|\, \tau\right)$$

$$= 2\theta_3(2\tau)\theta_2(2\tau)\theta_2\left(\frac{1}{5}\pi \,|\, \tau\right)\theta_2\left(\frac{2}{5}\pi \,|\, \tau\right).$$

Substituting (35) into the identity above and after simplification gives

$$\theta_3(5\tau)\theta_4(\tau) - \theta_4(5\tau)\theta_3(\tau)$$

$$= 4q\frac{(q^5;\, q^5)_\infty}{(q;\, q)_\infty^5}\theta_3(\tau)\theta_4(\tau)(q^2;\, q^2)_\infty^4(-q^2;\, q^2)_\infty^3(-q^{10};\, q^{10})_\infty.$$

This completes the proofs of the theorem.

Using the same method, we derive following two theorems which are about modular equations of degree seven and nine. The proofs are omitted.

Theorem 7. *We have* [5, p. 138]

$$\sqrt{\theta_3(7\tau)\theta_3(\tau)} - \sqrt{\theta_4(7\tau)\theta_4(\tau)} = \sqrt{\theta_2(7\tau)\theta_2(\tau)}. \quad (37)$$

Theorem 8. *We have that*

$$\sqrt{\frac{\theta_3(9\tau)}{\theta_3(\tau)}} - \sqrt{\frac{\theta_4(9\tau)}{\theta_4(\tau)}}$$

$$= 4q\sqrt{\frac{(q^9;\, q^9)_\infty}{(q;\, q)_\infty^9}\frac{(q^2;\, q^2)_\infty^4(-q^2;\, q^2)_\infty(-q^{18};\, q^{18})_\infty}{(-q^6;\, q^6)_\infty}}. \quad (38)$$

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