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SOME FIXED POINT THEOREMS USING COMPATIBLE MAPS OF TYPE(α) IN $\mathcal{M}$-FUZZY METRIC SPACES

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Abstract

In this paper, we prove some fixed point theorems and present the example for six mappings satisfying the some conditions in $\mathcal{M}$-fuzzy metric space. We generalize and extend for the results of [3], [7] and [9].

1. Introduction

Kramosil and Michalek [4] introduced the concepts of fuzzy metric spaces, and George and Veeramani [2] modified the concept of fuzzy metric spaces due to Kramosil and Michalek. Jungck et al. [3] have studied the fixed point theory in these spaces. Also, many authors have defined the intuitionistic fuzzy metric space and proved the several fixed point theorems in intuitionistic fuzzy metric space ([5, 7-9]). Park et al. [6] and Sedghi et al. [11] introduced the concept of $\mathcal{M}$-fuzzy metric space which is

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generalization of fuzzy metric space and proved common fixed point theorems in complete $\mathcal{M}$-fuzzy metric space.

In this paper, we prove some fixed point theorems and give the example for six mappings satisfying the some conditions in $\mathcal{M}$-fuzzy metric space.

2. Preliminaries and Properties

Now, we give some definitions, properties on $\mathcal{M}$-fuzzy metric space as following:

Definition 2.1 [1]**.** Let X be a nonempty set. Then a generalized metric (or D-metric) on X is a function $D : X^3 \to \mathbf{R}^+$ satisfying the following conditions:

(a) $D(x, y, z) \geq 0$,

(b) $D(x, y, z) = 0$ if and only if $x = y = z$,

(c) $D(x, y, z) = D(p\{x, y, z\})$, where p is a permutation function,

(d) $D(x, y, z) \leq D(x, y, a) + D(a, z, z)$ for all $x, y, z, a \in X$.

The pair (X, D) is called a *generalized metric* (or *D-metric*) *space*.

Let us recall (see [10]) that a continuous t-norm is a binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions: (a) $*$ is commutative and associative, (b) $*$ is continuous, (c) $a * 1 = a$ for all $a \in [0, 1]$, (d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0, 1])$.

Definition 2.2 ([11])**.** The 3-tuple $(X, \mathcal{M}, *)$ is said to be an $\mathcal{M}$*-fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z, a \in X$ and $t, s > 0$,

(a) $\mathcal{M}(x, y, z, t) > 0$,

(b) $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,

(c) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function,

(d) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,

(e) $\mathcal{M}(x, y, z, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

We know that both *D*-metric and fuzzy metric induce an $\mathcal{M}$-fuzzy metric as following.

Example 2.3. Let (X, D) be a *D*-metric space. Denote $a * b = ab$ for all $a, b \in [0, 1]$ and for all $x, y, z \in X$ and $t > 0$,

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D(x, y, z)}.$$

Then $(X, \mathcal{M}, *)$ is an $\mathcal{M}$-fuzzy metric space.

Lemma 2.4 [11]**.** *Let* $(X, \mathcal{M}, *)$ *be an* $\mathcal{M}$*-fuzzy metric space. Then for any* $x, y, z \in X$ *and* $t > 0$, *we have*

(a) $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$, (b) $\mathcal{M}(x, y, z, \cdot)$ *is nondecreasing.*

Definition 2.5 [6]**.** Let X be an $\mathcal{M}$-fuzzy metric space and a sequence $\{x_n\} \subset X$. Then

(a) $\{x_n\}$ is convergent to an $x \in X$ if $\lim_{n\to\infty} \mathcal{M}(x, x, x_n, t) = 1$ for all $t > 0$;

(b) $\{x_n\}$ is called *Cauchy sequence* if $\lim_{n\to\infty} \mathcal{M}(x_{n+p}, x_{n+p}, x_n, t)$ $= 1$ for all $t > 0$ and $p > 0$;

(c) an $\mathcal{M}$-fuzzy metric in which every Cauchy sequence is convergent is said to be *complete.*

Let $(X, \mathcal{M}, *)$ be an $\mathcal{M}$-fuzzy metric space with the following condition:

$$\lim_{t\to\infty} \mathcal{M}(x, y, z, t) = 1 \text{ for all } x, y, z \in X, t > 0. \tag{2.1}$$

Lemma 2.6 [6]**.** *Let* $\{x_n\}$ *be a sequence in* $(X, \mathcal{M}, *)$ *with condition* (2.1). *If there exists a* $k \in (0, 1)$ *such that* $\mathcal{M}(x_{n+2}, x_{n+1}, x_{n+1}, kt) \geq \mathcal{M}(x_{n+1}, x_n, x_n, t)$ *for all* $t > 0$ *and* $n = 1, 2, ...,$ *then* $\{x_n\}$ *is a Cauchy sequence in X.*

Lemma 2.7 [6]**.** *Let* $\{x_n\}$ *be a sequence in an* $\mathcal{M}$*-fuzzy metric space* $(X, \mathcal{M}, *)$ *with condition* (2.1). *If for all* $x, y \in X$ *and for a number* $k \in (0, 1)$, $\mathcal{M}(x, y, z, kt) \geq \mathcal{M}(x, y, z, t)$, *then* $x = y = z$.

Definition 2.8 [11]**.** Let A, B be mappings from $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ into itself. Then the mappings are said to be *compatible* if

$$\lim_{n\to\infty} \mathcal{M}(ABx_n, BAx_n, BAx_n, t) = 1$$

for all $t > 0$, whenever $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 2.9 [6]**.** Let A, B be mappings from $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ into itself. Then the mappings are called *compatible* of type(α) if

$$\lim_{n\to\infty} \mathcal{M}(ABx_n, BBx_n, BBx_n, t) = 1$$

and

$$\lim_{n\to\infty} \mathcal{M}(BAx_n, AAx_n, AAx_n, t) = 1$$

for all $t > 0$, whenever $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Proposition 2.10 [9]**.** *Let X be an* $\mathcal{M}$*-fuzzy metric space and A, B be mappings from X into itself. If A, B are compatible of type*(α) *and* $\{x_n\} \subset X$ *such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ *for some* $z \in X$, *then*

(a) $\lim_{n\to\infty} BAx_n = Az$ *if A is continuous at* $z \in X$,

(b) $ABz = BAz$ *and* $Az = Bz$ *if A and B are continuous at* $z \in X$.

3. Some Common Fixed Point using Compatible Maps of Type(α)

Theorem 3.1. *Let X be a complete* $\mathcal{M}$*-fuzzy metric space with* $t * t \geq t$ *for all* $t \in [0, 1]$ *and satisfy the condition* (2.1). *Let A, B, S, T, P and Q be mappings from X into itself such that*

(a) $P(X) \subset AB(X), Q(X) \subset ST(X)$,

(b) $AB = BA, ST = TS, PB = BP, QS = SQ$ *and* $QT = TQ$,

(c) *A, B, S and T are continuous*,

(d) (P, AB) *and* (Q, ST) *are compatible of type*(α),

(e) *There exists a* $k \in (0, 1)$ *such that for all* $x, y \in X$, $\beta \in (0, 2)$ *and* $t > 0$,

$$\begin{aligned}&\mathcal{M}(Px, Qy, Qy, kt)\\ &\geq \mathcal{M}(ABx, Px, Px, t) * \mathcal{M}(STy, Qy, Qy, t) * \mathcal{M}(STy, Px, Px, \beta t)\\ &\quad * \mathcal{M}(ABx, Qy, Qy, (2-\beta)t) * \mathcal{M}(ABx, STy, STy, t).\end{aligned}$$

Then A, B, S, T, P and Q have a common fixed point in X.

Proof. Since $P(X) \subset AB(X)$ from (a), we can choose a point $x_1 \in X$ for any $x_0 \in X$ such that $Px_0 = ABx_1$. Also, since $Q(X) \subset ST(X)$, we can choose $x_2 \in X$ for this point x_1 such that $Qx_1 = STx_2$. Inductively, construct sequence $\{y_n\} \subset X$ such that $y_{2n} = Px_{2n} = ABx_{2n+1}$, $y_{2n+1} = Qx_{2n+1} = STx_{2n+2}$ for $n = 1, 2, \ldots$. By (b), we have for all $t > 0$ and $\beta = 1 - q$ with $q \in (0, 1)$,

$$\begin{aligned}&\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt)\\ &= \mathcal{M}(Px_{2n+1}, Qx_{2n+2}, Qx_{2n+2}, kt)\\ &\geq \mathcal{M}(ABx_{2n+1}, Px_{2n+1}, Px_{2n+1}, t) * \mathcal{M}(STx_{2n+2}, Qx_{2n+2}, Qx_{2n+2}, t)\\ &\quad * \mathcal{M}(STx_{2n+2}, Px_{2n+1}, Px_{2n+1}, \beta t)\end{aligned}$$

$$* \mathcal{M}(ABx_{2n+1}, Qx_{2n+2}, Qx_{2n+2}, (2-\beta)t)$$

$$* \mathcal{M}(ABx_{2n+1}, STx_{2n+2}, STx_{2n+2}, t)$$

$$\geq \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t) * \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t)$$

$$* \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, qt).$$

Letting $q \to 1$, we get

$$\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt)$$

$$\geq \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t) * \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t). \tag{3.1}$$

Similarly, we have

$$\mathcal{M}(y_{2n+2}, y_{2n+3}, y_{2n+3}, kt)$$

$$\geq \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t) * \mathcal{M}(y_{2n+2}, y_{2n+3}, y_{2n+3}, t). \tag{3.2}$$

From (3.1) and (3.2),

$$\mathcal{M}(y_{n+1}, y_{n+2}, y_{n+2}, kt) \geq \mathcal{M}(y_n, y_{n+1}, y_{n+1}, t) * \mathcal{M}(y_{n+1}, y_{n+2}, y_{n+2}, t)$$

for $n = 1, 2, \ldots.$ And for positive integers n, p,

$$\mathcal{M}(y_{n+1}, y_{n+2}, y_{n+2}, kt)$$

$$\geq \mathcal{M}(y_n, y_{n+1}, y_{n+1}, t) * \mathcal{M}\left(y_{n+1}, y_{n+2}, y_{n+2}, \frac{t}{k^p}\right).$$

Thus since $\lim_{n\to\infty} \mathcal{M}\left(y_{n+1}, y_{n+2}, y_{n+2}, \frac{t}{k^p}\right) = 1,$ we have

$$\mathcal{M}(y_{n+1}, y_{n+2}, y_{n+2}, kt) \geq \mathcal{M}(y_n, y_{n+1}, y_{n+1}, t).$$

By Lemma 2.6, $\{y_n\}$ is a Cauchy sequence in X and since X is complete, $\{y_n\}$ converges to $z \in X$. Also, since $\{Px_{2n}\}$, $\{Qx_{2n+1}\}$, $\{ABx_{2n+1}\}$ and $\{STx_{2n+2}\}$ are subsequences of $\{y_n\}$, hence they converge to the point

$z \in X$. Since A, B are continuous and (P, AB) is compatible maps of type(α), by Proposition 2.10(a), we have

$$\lim_{n\to\infty} P(AB)x_{2n+1} = ABz \text{ and } \lim_{n\to\infty} (AB)^2 x_{2n+1} = ABz.$$

Also, since S, T are continuous and (Q, ST) is compatible maps of type(α), by Proposition 2.10(b), we have $\lim_{n\to\infty} Q(ST)x_{2n+2} = STz$ and $\lim_{n\to\infty}(ST)^2 x_{2n+2} = STz$.

First, let $x = (AB)x_{2n+1}$ and $y = x_{2n+2}$ with $\beta = 1$ and $n \to \infty$ in (e), we obtain

$$\begin{aligned}\mathcal{M}(ABz, z, z, kt) &\geq 1 * 1 * 1 * \mathcal{M}(ABz, z, z, t) * \mathcal{M}(ABz, z, z, t)\\ &\geq \mathcal{M}(ABz, z, z, t).\end{aligned}$$

Hence, by Lemma 2.7, $ABz = z$.

Second, let $x = Px_{2n}$ and $y = x_{2n+1}$ with $\beta = 1$ and $n \to \infty$ in (e), we have

$$\begin{aligned}&\mathcal{M}(Pz, z, z, kt)\\ &\geq \mathcal{M}(Pz, z, z, t) * \mathcal{M}(z, z, z, t) * \mathcal{M}(z, Pz, Pz, t)\\ &\quad * \mathcal{M}(z, z, z, t) * \mathcal{M}(z, z, z, t)\\ &\geq \mathcal{M}(Pz, z, z, t).\end{aligned}$$

Therefore, $Pz = z$. Hence $Pz = z = ABz$.

Third, let $x = Bz$ and $y = x_{2n+1}$ with $\beta = 1$ and $n \to \infty$ in (e), we obtain

$$\begin{aligned}\mathcal{M}(Bz, z, z, kt) \geq\ &\mathcal{M}(Bz, Bz, Bz, t) * \mathcal{M}(z, z, z, t) * \mathcal{M}(z, Bz, Bz, t)\\ &* \mathcal{M}(Bz, z, z, t) * \mathcal{M}(Bz, z, z, t).\end{aligned}$$

Therefore, $Bz = z$. Also, since $ABz = z$, hence $Az = z$.

Fourth, let $x = z$ and $y = STx_{2n+2}$ with $\beta = 1$ and $n \to \infty$ in (e), we have

$$\begin{aligned}
&\mathcal{M}(z, STz, STz, kt) \\
&\geq \mathcal{M}(z, z, z, t) * \mathcal{M}(STz, STz, STz, t) * \mathcal{M}(STz, z, z, t) \\
&\quad * \mathcal{M}(z, STz, STz, t) * \mathcal{M}(z, STz, STz, t) \\
&\geq \mathcal{M}(z, STz, STz, t).
\end{aligned}$$

Thus $STz = z$.

Fifth, let $x = z$ and $y = Qx_{2n+1}$ with $\beta = 1$ and $n \to \infty$ in (e), we have

$$\begin{aligned}
&\mathcal{M}(z, Qz, Qz, kt) \\
&\geq \mathcal{M}(z, z, z, t) * \mathcal{M}(z, Qz, Qz, t) * \mathcal{M}(z, z, z, t) \\
&\quad * \mathcal{M}(z, Qz, Qz, t) * \mathcal{M}(z, z, z, t) \\
&\geq \mathcal{M}(z, Qz, Qz, t).
\end{aligned}$$

Therefore, $Qz = z$ and hence $STz = z = Qz$.

Sixth, let $x = z$ and $y = Tz$ with $\beta = 1$ and $n \to \infty$ in (e), we have

$$\begin{aligned}
&\mathcal{M}(Pz, Q(Tz), Q(Tz), kt) \\
&\geq \mathcal{M}(ABz, Pz, Pz, t) * \mathcal{M}(ST(Tz), Q(Tz), Q(Tz), t) \\
&\quad * \mathcal{M}(ST(TZ), Pz, Pz, t) \\
&\quad * \mathcal{M}(ABz, Q(Tz), Q(Tz), t) * \mathcal{M}(ABz, ST(Tz), ST(Tz), t) \\
&\geq \mathcal{M}(Tz, z, z, t).
\end{aligned}$$

Thus, $Tz = z$. Since $STz = z$, hence $STz = z = Sz$. Therefore, z is a common fixed point of A, B, S, T, P and Q.

Finally, let $u(u \neq z)$ be another common fixed point of A, B, S, T, P and Q, and $\beta = 1.$ Then by (e),

$$\mathcal{M}(Pz, Qu, Qu, kt)$$

$$\geq \mathcal{M}(ABz, Pz, Pz, t) * M(STu, Qu, Qu, t) * \mathcal{M}(STu, Pz, Pz, t)$$

$$* \mathcal{M}(ABz, Qu, Qu, t) * \mathcal{M}(ABz, STu, STu, t).$$

It follows that

$$M(z, u, u, kt) \geq M(z, z, z, t) * M(u, u, u, t) * M(u, z, z, t)$$

$$* M(z, u, u, t) * M(z, u, u, t)$$

$$\geq M(z, u, u, t).$$

Therefore, z is a unique common fixed point of A, B, S, T, P and Q. □

Corollary 3.2. *Let X be a complete* $\mathcal{M}$ *-fuzzy metric space with* $t * t \geq t$ *for all* $t \in [0, 1]$ *and satisfy the condition* (2.1). *Let P, S be compatible maps of type*(α) *on X such that* $P(X) \subset S(X)$. *If S is continuous and there exists a constant* $k \in (0, 1)$ *such that*

$$\mathcal{M}(Px, Py, Py, kt)$$

$$\geq \mathcal{M}(Sx, Px, Px, t) * \mathcal{M}(Sy, Py, Py, t) * \mathcal{M}(Sy, Px, Px, \beta t)$$

$$* \mathcal{M}(Sx, Py, Py, (2 - \beta)t) * \mathcal{M}(Sx, Sy, Sy, t)$$

for all $x, y \in X$, $\beta \in (0, 2)$ *and* $t > 0$, *then P and S have a unique common fixed point in X.*

Proof. The proof follows from Theorem 3.1 with $P = Q$, $S = A$ and $B = T = I_X$. □

Example 3.3. Let $X = [0, 1]$ and (X, D) be a D-metric space. Define $D(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$ and $\mathcal{M}(x, y, z, t) =$

$\frac{t}{t + D(x, y, z)}$ for all $a, b \in [0, 1]$, $x, y, z \in X$ and $t > 0$. Then X is a complete $\mathcal{M}$-fuzzy metric space. Let A, B, S, T, P and Q be defined as $Ax = x$, $Bx = \frac{x}{2}$, $Sx = \frac{x}{5}$, $Tx = \frac{x}{3}$, $Px = \frac{x}{6}$ and $Qx = 0$ for all $x \in X$. Then $P(X) = \left[0, \frac{1}{6}\right] \subset \left[0, \frac{1}{2}\right] = AB(X)$. Also, since $Q(X) = \{0\}$ and $ST(X) = \left[0, \frac{1}{15}\right]$, hence $Q(X) \subset ST(X)$. If we take $k = \frac{1}{2}$, $t = 1$ and $\beta = 1$, then we see that Theorem 3.1(e) is satisfied. Furthermore, Theorems 3.1(b) and (c) are satisfied, and (P, AB) is compatible maps of type(α) if $\lim_{n\to\infty} x_n = 0$, where $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} ABx_n = 0$ for some $0 \in X$. Similarly, (Q, ST) is also compatible maps of type(α). Thus all conditions of Theorem 3.1 are satisfied and 0 is the unique common fixed point of A, B, S, T, P and Q.

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