Advances in Differential Equations and Control Processes
Volume 8, Number 2, 2011, Pages 83-90
Published Online: January 2012
Available online at http://pphmj.com/journals/adecp.htm
Published by Pushpa Publishing House, Allahabad, INDIA

DELAY-DEPENDENT STABILITY CONDITION FOR NEURAL NETWORKS WITH DISCRETE AND UNBOUNDED DISTRIBUTED DELAYS

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Abstract

This paper is concerned with the problem of stability analysis for neural networks with discrete and unbounded distributed delays. In terms of linear matrix inequalities, a new delay-dependent condition is proposed, which ensures the existence of a unique equilibrium point and its global asymptotic stability of the delayed neural networks. A numerical example is given to demonstrate the reduced conservatism of the condition.

1. Introduction

In the past two decades, neural networks have found many applications in pattern classification, associative memory and combinatorial optimization. It is well known that the stability of neural networks plays an important role in such applications. In hardware implementation of neural networks, time delays will unavoidable occur due to the finite switching speed of amplifiers.

2010 Mathematics Subject Classification: 34K20, 92B20.

Keywords and phrases: neural networks, distributed delays, stability.

Received September 26, 2011

The existence of time delays in a neural network may be a source of oscillation and instability [1]. Therefore, the problem of stability analysis of delayed neural networks has attracted extensive attention. Until now, however, most studies are based on the assumption that the time delays are discrete. Although this assumption is not unreasonable, as noted in [2], it is more appropriate to introducing continuously distributed delays such that the distant past has less influence compared to the recent behavior of the state. For previous stability results of neural networks with distributed delays, please refer to [3-5] and the references cited therein.

The purpose of this paper is to study further the global asymptotic stability of neural networks with both discrete and unbounded distributed delays given in [5]. A new delay-dependent condition for the existence of a unique equilibrium point and its global asymptotic stability of the networks is developed by using the Lyapunov-Krasovskii functional method. The condition is expressed in terms of linear matrix inequalities (LMIs), and hence can be easily verified with the help of the Matlab LMI Toolbox. A numerical example is provided to demonstrate the less conservatism of the obtained result by comparing with those reported recently in the literature.

2. Problem Formulation

Consider the following neural network with discrete and unbounded distributed delays [5]:

$$\dot{u}_i(t) = -c_i u_i(t) + \sum_{j=1}^{n} w_{ij} g_j(u_j(t)) + \sum_{j=1}^{n} a_{ij} g_j(u_j(t-\tau))$$

$$+ \sum_{j=1}^{n} b_{ij} \int_{-\infty}^{t} k_j(t-s) g_j(u_j(s))ds + I_i, \tag{1}$$

for $i = 1, 2, \ldots, n,$ where n denotes the number of neurons in the system, $u_i(t)$ is the state of the ith neuron at time t, $c_i > 0$ is the passive decay rate,

w_{ij}, a_{ij} and b_{ij} are the synaptic connection strengths, I_i is the external constant input, $\tau > 0$ is the discrete transmission delay from one neuron to another, k_j is a real value non-negative continuous function defined on $[0, \infty)$ satisfying $\int_0^\infty k_j(s)ds = 1,$ and g_j is the neuron activation which is assumed to be bounded and to satisfy $0 \le (g_j(\xi_1) - g_j(\xi_2))/(\xi_1 - \xi_2) \le l_j$ for any $\xi_1, \xi_2 \in R,$ $\xi_1 \ne \xi_2,$ where $l_j,$ $j = 1, 2, ..., n,$ are positive constants.

With the boundedness of functions $g_j,$ $j = 1, 2, ..., n,$ system (1) has at least one equilibrium point [1]. Suppose that $\breve{u} = [\breve{u}_1\ \breve{u}_2 \cdots \breve{u}_n]^T$ is an equilibrium point of the system. Then, we can shift $\breve{u}$ to the origin by taking the transformation $x(\cdot) = u(\cdot) - \breve{u},$ which puts system (1) into the following form:

$$\dot{x}(t) = -Cx(t) + Wf(x(t)) + Af(x(t - \tau)) + B\int_{-\infty}^{t} K(t - s)f(x(s))ds, \quad (2)$$

where

$K(t - s) = diag[k_1(t - s), k_2(t - s), ..., k_n(t - s)],$ $C = diag(c_1, c_2, ..., c_n),$

$W = [w_{ij}]_{n\times n},$ $A = [a_{ij}]_{n\times n},$ $B = [b_{ij}]_{n\times n},$ $x = [x_1\ x_2 \cdots x_n]^T,$

$f(x) = [f_1(x_1)\ f_2(x_2) \cdots f_n(x_n)]^T$

with $f_j(x_j) = g_j(x_j + \breve{u}_j) - g_j(\breve{u}_j),$ $j = 1, 2, ..., n.$ Note that $f_1, f_2, ..., f_n$ satisfy

$$0 \le f_j(x_j)/x_j \le l_j \text{ and } f_j(0) = 0. \quad (3)$$

The following fact will be used in the proof of our main result.

Fact 1. For any real matrices $M_i,$ $i = 1, 2, ..., 5$ with appropriate dimensions, we have

$$2\Bigg[\dot{x}^T(t)M_1 + x^T(t)M_2 + f^T(x(t))M_3 + f^T(x(t-\tau))M_4$$

$$+\left(\int_{-\infty}^{t} K(t-s)f(x(s))ds\right)^T M_5\Bigg]$$

$$\times\left[\dot{x}(t) + Cx(t) - Wf(x(t)) - Af(x(t-\tau)) - B\int_{-\infty}^{t} K(t-s)f(x(s))ds\right] = 0.$$

3. Stability Condition

Theorem 1. *The origin of system* (2) *is the unique equilibrium point and it is globally asymptotically stable if there exist matrices* $M_1, M_2, ..., M_5$, $P_1 > 0,\ P_3 > 0,\ P_6 > 0$ *and diagonal matrices* $P_2 > 0,\ P_4 > 0,\ P_5 > 0,$ $P_7 > 0,\ P_8 > 0$ *such that the following LMI holds*:

$$\Gamma = \begin{bmatrix} Z_{11} & Z_{12} & -M_1W + M_3^T & -M_1A + M_4^T & -M_1B + M_5^T & 0 \\ * & Z_{22} & Z_{23} & Z_{24} & -M_2B + CM_5^T & 0 \\ * & * & Z_{33} & -M_3A - W^TM_4^T & -M_3B - W^TM_5^T & 0 \\ * & * & * & Z_{44} & -M_4B - A^TM_5^T & Z_{46} \\ * & * & * & * & Z_{55} & 0 \\ * & * & * & * & * & Z_{66} \end{bmatrix} < 0, \tag{4}$$

where

$Z_{11} = M_1 + M_1^T + \tau P_6,\ Z_{12} = M_1C + M_2^T + P_1,\ Z_{22} = M_2C + CM_2^T + P_2,$

$Z_{23} = -M_2W + CM_3^T + P_5 + LP_7,\ Z_{24} = -M_2A + CM_4^T - P_5 + LP_8,$

$Z_{33} = -M_3W - W^TM_3^T + P_3 + P_4 - 2P_7,$

$Z_{44} = -M_4A - A^TM_4^T - L^{-1}P_2L_2^{-1} - P_3 - 2P_8,\ Z_{46} = -\tau(P_5 - LP_8),$

$Z_{55} = -M_5B - B^TM_5^T - P_4,\ Z_{66} = -\tau P_6,\ L = diag[l_1, l_2, ..., l_n].$

Proof. First, we will show the origin of system (2) is the unique equilibrium point. Let $\breve{x}$ be the equilibrium point of system (2). Then, we have

$$-C\breve{x} + (W + A + B)f(\breve{x}) = 0. \tag{5}$$

Multiplying both sides of (5) by

$$2[\breve{x}^T M_2 + f^T(\breve{x})(M_3 + M_4 + M_5)]$$

gives

$$2[\breve{x}^T M_2 + f^T(\breve{x})(M_3 + M_4 + M_5)][-C\breve{x} + (W + A + B)f(\breve{x})] = 0. \tag{6}$$

Noting that (6) can be rewritten as

$$\begin{aligned}
&[\breve{x}^T \quad f^T(\breve{x}) \quad f^T(\breve{x}) \quad f^T(\breve{x})]\Omega[\breve{x}^T \quad f^T(\breve{x}) \quad f^T(\breve{x}) \quad f^T(\breve{x})]^T \\
&= [\breve{x}^T P_2 \breve{x} - f^T(\breve{x})L^{-1}P_2L^{-1}f(\breve{x})] \\
&\quad + 2[\breve{x}^T L(P_7 + P_8)f(\breve{x}) - f^T(\breve{x})(P_7 + P_8)f(\breve{x})],
\end{aligned} \tag{7}$$

where

$$\Omega = \begin{bmatrix} Z_{22} & Z_{23} & Z_{24} & -M_2B + CM_5^T \\ * & Z_{33} & -M_3A - W^TM_4^T & -M_3B - W^TM_5^T \\ * & * & Z_{44} & -M_4B - A^TM_5^T \\ * & * & * & Z_{55} \end{bmatrix}.$$

By (3), it follows from (7) that

$$[\breve{x}^T \quad f^T(\breve{x}) \quad f^T(\breve{x}) \quad f^T(\breve{x})]\Omega[\breve{x}^T \quad f^T(\breve{x}) \quad f^T(\breve{x}) \quad f^T(\breve{x})]^T \geq 0.$$

On the other hand, (4) gives $\Omega < 0.$ Thus we have $\breve{x} = 0,$ which means the origin of system (2) is the unique equilibrium point.

Next, we prove that the origin of (2) is globally asymptotically stable. To the end, the following Lyapunov-Krasovskii functional is constructed:

$$V(x(t)) = x^T(t)P_1x(t) + \int_{t-\tau}^{t}(x^T(s)P_2x(s) + f^T(x(s))P_3f(x(s))$$

$$+ 2x^T(s)P_5f(x(s)))ds + \sum_{j=1}^{n} p_{4j}\int_0^\infty k_j(s)\int_{t-s}^{t} f_j^2(x_j(r))drds$$

$$+ \int_{-\tau}^{0}\int_{t+s}^{t}\dot{x}^T(r)P_6\dot{x}(r)drds.$$

Calculating the time derivative of $V(x(t))$ yields

$$\begin{aligned}\dot{V}(x(t)) = {} & 2x^T(t)P_1\dot{x}(t) + x^T(t)P_2x(t) + 2x^T(t)P_5f(x(t)) \\ & + f^T(x(t))P_3f(x(t)) + \tau\dot{x}^T(t)P_6\dot{x}(t) \\ & - x^T(t-\tau)P_2x(t-\tau) - 2x^T(t-\tau)P_5f(x(t-\tau)) \\ & - f^T(x(t-\tau))P_3f(x(t-\tau)) + f^T(x(t))P_4f(x(t)) \\ & - \sum_{j=1}^{n} p_{4j}\int_0^\infty k_j(s)ds\int_0^\infty k_j(s)f_j^2(x_j(t-s))ds \\ & - \int_{t-\tau}^{t}\dot{x}^T(s)P_6\dot{x}(s)ds. \end{aligned} \tag{8}$$

By the Cauchy inequality (see [4]), we obtain

$$\begin{aligned} & -\sum_{j=1}^{n} p_{4j}\int_0^\infty k_j(s)ds\int_0^\infty k_j(s)f_j^2(x_j(t-s))ds \\ & \le -\left(\int_{-\infty}^{t} k(t-s)f(x(s))ds\right)^T P_4\left(\int_{-\infty}^{t} k(t-s)f(x(s))ds\right). \end{aligned} \tag{9}$$

In the light of the well-known Jensen inequality, we have

$$-\int_{t-\tau}^{t}\dot{x}^T(s)P_6\dot{x}(s)ds \le -\frac{1}{\tau}[x(t) - x(t-\tau)]^T P_6[x(t) - x(t-\tau)]. \tag{10}$$

In view of (3), we obtain

$$-x^T(t-\tau)P_2x(t-\tau) \le -f^T(x(t-\tau))L^{-1}P_2L^{-1}f(x(t-\tau)), \tag{11}$$

$$-2x^T(t)LP_7f(x(t)) \le -2f^T(x(t))P_7f(x(t)), \tag{12}$$

$$-2x^T(t-\tau)LP_8f(x(t-\tau)) \le -2f^T(x(t-\tau))P_8f(x(t-\tau)). \tag{13}$$

By Fact 1, it follows from (8)-(13) that

$$\dot{V}(x(t)) \le \eta(t)\psi\eta^T(t),$$

where

$$\eta(t) = \Bigg[\dot{x}^T(t)\ x^T(t)\ f^T(x(t))\ f^T(x(t-\tau))$$

$$\cdot\left(\int_{-\infty}^{t} K(t-s)f(x(s))ds\right)^T x^T(t-\tau)/\tau\Bigg],$$

$$\Psi = \begin{bmatrix} Z_{11} & Z_{12} & -M_1W + M_3^T & -M_1A + M_4^T & -M_1B + M_5^T & 0 \\ * & Z_{22} - P_6/\tau & Z_{23} & Z_{24} + P_5 - LP_8 & -M_2B + CM_5^T & P_6 \\ * & * & Z_{33} & -M_3A - W^TM_4^T & -M_3B - W^TM_5^T & 0 \\ * & * & * & Z_{44} & -M_4B - A^TM_5^T & Z_{46} \\ * & * & * & * & Z_{55} & 0 \\ * & * & * & * & * & Z_{66} \end{bmatrix}.$$

If

$$\Psi < 0, \tag{14}$$

holds, then $\dot{V}(x(t)) < 0$ for all $x(t) \ne 0$. That is, the origin of system (2) is globally stable under condition (14). The proof of Theorem 1 is completed due to the fact that (14) is equivalent to (4).

4. Numerical Example

Example 1. Consider a second-order neural network (2) with

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; W = 0, \; A = \begin{bmatrix} 0.78 & -0.07 \\ 0.15 & 0.63 \end{bmatrix}, \; B = \begin{bmatrix} 0.52 & 0.02 \\ -0.12 & 0.37 \end{bmatrix},$$

$$f_j(x_j) = \tanh(x_j), \; j = 1, 2.$$

Note that (3) holds with $l_1 = l_2 = 1$.

For this delayed neural network, it can be verified that the stability conditions in [4] and [5] are not satisfied. Thus the conditions in [4] and [5] fail to check whether the system is globally asymptotically stable or not. However, by resorting to the Matlab LMI Toolbox, we find the LMI (4) is feasible. Hence, by Theorem 1, we can conclude that the system is globally asymptotically stable.

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