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SOME OSTROWSKI-LIKE TYPE INEQUALITIES FOR DIFFERENTIABLE REAL (α, m)-CONVEX MAPPINGS

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Abstract

In this article, the author establishes some basic inequalities for (α, m)-convex mappings, and by using the properties of these basic inequalities, obtains several inequalities of Ostrowski-like type via differentiable real (α, m)-convex mappings.

1. Introduction

In [11, 12], Toader defined the m-convexity and the (α, m)-convexity.

Definition 1. The mapping $f : \mathbb{I} \subseteq [0, b^*] \to \mathbb{R}$ is said to be *m-convex* on $\mathbb{I}$, where $m \in [0, 1]$, $a, b \in \mathbb{I}$ with $a < b$ and $b^* > 0$, if the inequality

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

holds for $x, y \in \mathbb{I}$ and $t \in [0, 1]$.

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Definition 2. The mapping $f : \mathbb{I} \subseteq [0, b^*] \to \mathbb{R}$ is said to be (α, m)-*convex* on $\mathbb{I}$, where $m \in [0, 1]$, $a, b \in \mathbb{I}$ with $a < b$ and $b^* > 0$, if the inequality

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1 - t^{\alpha}) f(y)$$

holds for $x, y \in \mathbb{I}$ and $t \in [0, 1]$.

Denote by $K_m^{\alpha}(b)$ the set of all (α, m)-convex mappings on $[0, b]$ for which $f(0) \leq 0$. For recent results and generalizations concerning *m*-convex, (s, m)-convex and (α, m)-convex mappings, see [2-7].

Note that for $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$, one obtains the following classes of functions: increasing, α-star-shaped, star-shaped, *m*-convex, convex and α-convex [2-5,7, 8].

In [2, 3], the authors proved the following Hadamard's inequalities for *m*-convex mappings:

Theorem 1.1. *Let* $f : [a, b] \subseteq [0, b^*] \to \mathbb{R}$ *be an m-convex mapping with* $m \in (0, 1]$, $a < b$ *and* $b^* > 0$. *If* $f \in L_1([a, b])$, *then one has the inequality*:

(a) $$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{2}\min\left\{f(a) + mf\left(\frac{b}{m}\right),\ f(b) + mf\left(\frac{a}{m}\right)\right\},$$

(b) $$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)}\int_a^b \left\{f(x) + mf\left(\frac{x}{m}\right)dx\right\}$$

$$\leq \frac{m+1}{8}\left[\{f(a) + f(b)\} + m\left\{f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right\}\right].$$

Theorem 1.2. *Let* $f : [a, b] \subseteq [0, b^*] \to \mathbb{R}$ *be an m-convex mapping with* $m \in (0, 1]$, $0 \leq a < b < \infty$ *and* $b^* > 0$. *If* $f \in L_1([am, b])$, *then one has the inequality*:

$$\frac{1}{m+1}\left[\int_a^{mb} f(x)dx + \frac{mb-a}{b-ma}\int_{ma}^{b} f(x)dx\right] \le (mb-a)\frac{f(a)+f(b)}{2}.$$

In [8], the author established the following theorems for (α, m)-convex mapping:

Theorem 1.3. *Let* $f : [a, b] \subseteq [0, b^*] \to \mathbb{R}$ *be an* (α, m)*-convex mapping with* $\alpha, m \in [0, 1]$, $a < b$ *and* $b^* > 0$. *If* $f \in L_1([a, b])$, *then one has the inequality*:

(a) $\frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{\alpha+1}\min\left\{f(a)+\alpha mf\left(\frac{b}{m}\right),\ f(b)+\alpha mf\left(\frac{a}{m}\right)\right\},$

(b) $\frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{2(\alpha+1)}\left[\{f(a)+f(b)\}+\alpha m\left\{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right\}\right].$

Theorem 1.4. *Let* $f : [a, b] \subseteq [0, b^*] \to \mathbb{R}$ *be an* (α, m)*-convex mapping with* $\alpha, m \in [0, 1]$, $a < b$ *and* $b^* > 0$. *If* $f \in L_1([a, b])$, *then one has the inequality*:

(a) $$\frac{1}{x-a}\int_a^x f(x)dx + \frac{1}{b-x}\int_x^b f(x)dx$$

$$\le \frac{1}{\alpha+1}\left[2f(x)+\alpha m\left\{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right\}\right],$$

(b) $$f\left(\frac{a+b}{2}\right)$$

$$\le \frac{1}{b-a}\int_a^b\left\{\frac{1}{2^\alpha}f(x)+m\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{x}{m}\right)\right\}dx$$

$$\le \frac{1}{2^{\alpha+1}}\{f(a)+f(b)\}+\left(1-\frac{1}{2^\alpha}\right)\frac{m}{2(\alpha+1)}\left\{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right\}$$

$$+\left(1-\frac{1}{2^\alpha}\right)\frac{\alpha m^2}{2(\alpha+1)}\left\{f\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m^2}\right)\right\}.$$

In this article, the author establishes some basic inequalities for (α, m)-convex mappings in Section 2, and, by using these properties, obtains the inequalities of Hermite-Hadamard type via differentiable (α, m)-convex mappings.

2. Inequalities of Hermite-Hadamard Type via (α, m)-convex Mappings

To prove our new results, we need the following lemma [1]:

Lemma 1. *Let* $f : \mathbb{I} \subset [0, b^*] \to \mathbb{R}$ *be a differentiable mapping on the interior* $\mathbb{I}^0$ *of an interval* $\mathbb{I}$, *where* $a, b \in \mathbb{I}$ *with* $a < b$ *and* $b^* > 0$. *If* $f' \in L_1[a, b]$, *then the following inequality holds*:

$$(b - x) f(b) + (x - a) f(a) - \int_a^b f(x) dx$$

$$= (x - a)^2 \int_0^1 (t - 1) f'(tx + (1 - t)a) dt + (b - x)^2 \int_0^1 (1 - t) f'(tx + (1 - t)b) dt.$$

Theorem 2.1. *Let* $f : \mathbb{I} \subset [0, b^*] \to \mathbb{R}$ *be a differentiable mapping on the interior* $\mathbb{I}^0$ *of an interval* $\mathbb{I}$ *such that* $f' \in L([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$ *and* $b^* > 0$. *If* $|f'|$ *is* (α, m)-*convex on* $[a, b]$ *for* $(\alpha, m) \in [0, 1]^2$, *then the following inequality holds*:

$$(\alpha + 1)(\alpha + 2)\left| (b - x) f(b) + (x - a) f(a) - \int_a^b f(x) dx \right|$$

$$\leq (x - a)^2 M_{11} + (b - x)^2 M_{12}, \tag{1}$$

where

$$M_{11} = \min\left\{ \begin{array}{l} | f'(x) | + \dfrac{\alpha(\alpha + 3)}{2} m \left| f'\left(\dfrac{a}{m}\right) \right|, \\ m \left| f'\left(\dfrac{x}{m}\right) \right| + \dfrac{\alpha(\alpha + 3)}{2} | f'(a) | \end{array} \right\},$$

$$M_{12} = \min\left\{\begin{array}{l} |f'(x)| + \frac{\alpha(\alpha+3)}{2} m \left| f'\left(\frac{b}{m}\right)\right|, \\ m\left| f'\left(\frac{x}{m}\right)\right| + \frac{\alpha(\alpha+3)}{2} |f'(b)| \end{array}\right\}.$$

Proof. From Lemma 1, using the (α, m)-convexity of *f*, we get

$$\left| (b-x) f(b) + (x-a) f(a) - \int_a^b f(x) dx \right|$$

$$\leq (x-a)^2 \int_0^1 |(t-1) f'(tx + (1-t)a)| dt$$

$$+ (b-x)^2 \int_0^1 |(1-t) f'(tx + (1-t)b)| dt. \tag{2}$$

Since $|f'|$ is (α, m)-convex,

(i) $$\int_0^1 |(t-1) f'(tx + (1-t)a)| dt$$

$$\leq \int_0^1 (1-t) \left\{ t^{\alpha} |f'(x)| + m(1-t^{\alpha}) \left| f'\left(\frac{a}{m}\right)\right| \right\} dt$$

$$= \frac{1}{(\alpha+1)(\alpha+2)} |f'(x)| + \frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)} m \left| f'\left(\frac{a}{m}\right)\right| \tag{3}$$

and

(ii) $$\int_0^1 |(t-1) f'(tx + (1-t)b)| dt$$

$$\leq \int_0^1 (1-t) \left\{ t^{\alpha} |f'(x)| + m(1-t^{\alpha}) \left| f'\left(\frac{b}{m}\right)\right| \right\} dt$$

$$= \frac{1}{(\alpha+1)(\alpha+2)} |f'(x)| + \frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)} m \left| f'\left(\frac{b}{m}\right)\right|. \tag{4}$$

Analogously, also we get

(i)′ $$\int_0^1 |(t-1) f'(tx+(1-t)a)| dt$$

$$\leq \frac{1}{(\alpha+1)(\alpha+2)} m \left| f'\left(\frac{x}{m}\right) \right| + \frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)} | f'(a) |, \tag{5}$$

(ii)′ $$\int_0^1 |(t-1) f'(tx+(1-t)b)| dt$$

$$\leq \frac{1}{(\alpha+1)(\alpha+2)} m \left| f'\left(\frac{x}{m}\right) \right| + \frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)} | f'(b) |. \tag{6}$$

Hence, by (3)-(6), we have

$$\int_0^1 |(t-1) f'(tx+(1-t)a)| dt \leq \frac{1}{(\alpha+1)(\alpha+2)} M_{11}, \tag{7}$$

$$\int_0^1 |(t-1) f'(tx+(1-t)b)| dt \leq \frac{1}{(\alpha+1)(\alpha+2)} M_{12}. \tag{8}$$

By (2) and (7)-(8), the inequality (1) holds.

Theorem 2.2. *Let* $f : \mathbb{I} \subset [0, b^*] \to \mathbb{R}$ *be a differentiable mapping on the interior* $\mathbb{I}^0$ *of an interval* $\mathbb{I}$ *such that* $f' \in L([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$ *and* $b^* > 0$. *If* $|f'|^q$ *is* (α, m)*-convex on* $[a, b]$ *for* $(\alpha, m) \in [0, 1]^2$ *and* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequality holds*:

$$\left| (b-x) f(b) + (x-a) f(a) - \int_a^b f(x) dx \right|$$

$$\leq \left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}} \left\{ (x-a)^2 M_{21}^{\frac{1}{q}} + (b-x)^2 M_{22}^{\frac{1}{q}} \right\}, \tag{9}$$

where

$$M_{21} = \min\left\{ |f'(a)|^q + \alpha m\left|f'\left(\frac{x}{m}\right)\right|^q, \ |f'(x)|^q + \alpha m\left|f'\left(\frac{a}{m}\right)\right|^q \right\},$$

$$M_{22} = \min\left\{ |f'(x)|^q + \alpha m\left|f'\left(\frac{b}{m}\right)\right|^q, \ |f'(b)|^q + \alpha m\left|f'\left(\frac{x}{m}\right)\right|^q \right\}.$$

Proof. From Lemma 1, using the (α, m)-convexity of *f*, we get

$$\left|(b-x)f(b) + (x-a)f(a) - \int_a^b f(x)dx\right|$$

$$\leq (x-a)^2\left\{\int_0^1 (1-t)^p dt\right\}^{\frac{1}{p}}\left\{\int_0^1 |f'(tx + (1-t)a)|^q dt\right\}^{\frac{1}{q}}$$

$$+ (b-x)^2\left\{\int_0^1 (1-t)^p dt\right\}^{\frac{1}{p}}\left\{\int_0^1 |f'(tx + (1-t)b)|^q dt\right\}^{\frac{1}{q}}$$

$$\leq (x-a)^2\left\{\frac{1}{x-a}\int_a^x |f'(u)|^q du\right\}^{\frac{1}{q}}$$

$$+ (b-x)^2\left\{\frac{1}{b-x}\int_x^b |f'(u)|^q du\right\}^{\frac{1}{q}}, \tag{10}$$

where we have used the fact that $\frac{1}{2} \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \leq 1$ for $p > 1$.

Note that

$$\frac{1}{x-a}\int_a^x |f'(u)|^q du \leq \frac{1}{\alpha+1} M_{21}, \tag{11}$$

$$\frac{1}{b-x}\int_x^b |f'(u)|^q du \leq \frac{1}{\alpha+1} M_{22}. \tag{12}$$

By the inequalities (10)-(12), the inequality (9) is proved.

Theorem 2.3. *Let* $f : \mathbb{I} \subset [0, b^*] \to \mathbb{R}$ *be a differentiable mapping on the interior* $\mathbb{I}^0$ *of an interval* $\mathbb{I}$ *such that* $f' \in L([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$ *and* $b^* > 0$. *If* $|f'|^q$ *is* (α, m)*-convex on* $[a, b]$ *for* $(\alpha, m) \in [0, 1]^2$ *and* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequality holds*:

$$2^{\frac{1}{p}}\{(\alpha+1)(\alpha+2)\}^{\frac{1}{q}}\left|(b-x)f(b)+(x-a)f(a)-\int_a^b f(x)dx\right|$$

$$\leq (x-a)^2 M_{31}^{\frac{1}{q}} + (b-x)^2 M_{32}^{\frac{1}{q}}, \tag{13}$$

where

$$M_{31} = \min\left\{\begin{matrix} |f'(x)|^q + \left(\frac{\alpha(\alpha+3)}{2}\right)m\left|f'\left(\frac{a}{m}\right)\right|^q, \\ m\left|f'\left(\frac{x}{m}\right)\right|^q + \left(\frac{\alpha(\alpha+3)}{2}\right)m|f'(a)|^q \end{matrix}\right\},$$

$$M_{32} = \min\left\{\begin{matrix} |f'(x)|^q + \left(\frac{\alpha(\alpha+3)}{2}\right)m\left|f'\left(\frac{b}{m}\right)\right|^q, \\ m\left|f'\left(\frac{x}{m}\right)\right|^q + \left(\frac{\alpha(\alpha+3)}{2}\right)m|f'(b)|^q \end{matrix}\right\}.$$

Proof. From Lemma 1 and by using the well-known power mean inequality, we get

$$\left|(b-x)f(b)+(x-a)f(a)-\int_a^b f(x)dx\right|$$

$$\leq (x-a)^2\left\{\int_0^1(1-t)dt\right\}^{\frac{1}{p}}\left\{\int_0^1(1-t)|f'(tx+(1-t)a)|^q dt\right\}^{\frac{1}{q}}$$

$$+ (b-x)^2\left\{\int_0^1(1-t)dt\right\}^{\frac{1}{p}}\left\{\int_0^1(1-t)|f'(tx+(1-t)b)|^q dt\right\}^{\frac{1}{q}}$$

$$= \frac{(x-a)^2}{2^{\frac{1}{p}}} \left\{ \int_0^1 (1-t) | f'(tx + (1-t)a) |^q dt \right\}^{\frac{1}{q}}$$

$$+ \frac{(b-x)^2}{2^{\frac{1}{p}}} \left\{ \int_0^1 (1-t) | f'(tx + (1-t)b) |^q dt \right\}^{\frac{1}{q}}. \tag{14}$$

Note that

(i) $\int_0^1 (1-t) | f'(tx + (1-t)a) |^q dt$

$$\le \frac{| f'(x) |^q}{(\alpha+1)(\alpha+2)} + \left(\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)} \right) m \left| f'\left(\frac{a}{m} \right) \right|^q, \tag{15}$$

(ii) $\int_0^1 (1-t) | f'(tx + (1-t)b) |^q dt$

$$\le \frac{| f'(x) |^q}{(\alpha+1)(\alpha+2)} + \left(\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)} \right) m \left| f'\left(\frac{b}{m} \right) \right|^q. \tag{16}$$

Analogously, also we have

(i)′ $\int_0^1 (1-t) | f'(tx + (1-t)a) |^q dt$

$$\le \frac{m \left| f'\left(\frac{x}{m} \right) \right|^q}{(\alpha+1)(\alpha+2)} + \left(\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)} \right) | f'(a) |^q, \tag{17}$$

(ii)′ $\int_0^1 (1-t) | f'(tx + (1-t)b) |^q dt$

$$\le \frac{m \left| f'\left(\frac{x}{m} \right) \right|^q}{(\alpha+1)(\alpha+2)} + \left(\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)} \right) | f'(b) |^q. \tag{18}$$

By the inequalities (15)-(18), we have

$$\text{(a)}\ \int_0^1 (1-t)|\, f'(tx+(1-t)a)|^q dt \le \frac{1}{(\alpha+1)(\alpha+2)} M_{31}, \tag{19}$$

$$\text{(b)}\ \int_0^1 (1-t)|\, f'(tx+(1-t)b)|^q dt \le \frac{1}{(\alpha+1)(\alpha+2)} M_{32}, \tag{20}$$

which imply that the assertion (13) holds by (14), (19)-(20) and Theorem 2.1.

Lemma 2. *Let* $f : \mathbb{I} \subset [0, b^*] \to \mathbb{R}$ *be a differentiable mapping on the interior* $\mathbb{I}^0$ *of an interval* $\mathbb{I}$, *where* $a, b \in \mathbb{I}$ *with* $a < b$ *and* $b^* > 0$. *If* $f' \in L_1[a, b]$, *then the following inequality holds*:

$$\left| f(x) + \frac{1}{b-a}\int_a^b f(u)du - \{f(a) + f(b)\} \right|$$

$$= (b-a)\int_0^1 p(t)\, f'(ta+(1-t)b)dt,$$

where

$$p(t) = \begin{cases} t-1 & t \in \left[0, \dfrac{b-x}{b-a}\right) \\ t & t \in \left[\dfrac{b-x}{b-a}, 1\right]. \end{cases}$$

Proof. By the integration by parts, this identity is proved.

Theorem 2.4. *Let* $f : \mathbb{I} \subset [0, b^*] \to \mathbb{R}$ *be a differentiable mapping on the interior* $\mathbb{I}^0$ *of an interval* $\mathbb{I}$ *such that* $f' \in L([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$ *and* $b^* > 0$. *If* $|\, f'|$ *is* (α, m)*-convex on* $[a, b]$ *for* $(\alpha, m) \in [0, 1]^2$, *then the following inequality holds*:

$$\left| f(x) + \frac{1}{b-a}\int_a^b f(u)du - \{f(a) + f(b)\} \right|$$

$$\leq (b-a)\begin{Bmatrix} (\mu_{11}+\mu_{12})|\, f'(a)\,| + (\nu_{11}+\nu_{12})m\left|\, f'\left(\frac{b}{m}\right)\right|, \\ (\mu_{11}+\mu_{12})m\left|\, f'\left(\frac{a}{m}\right)\right| + (\nu_{11}+\nu_{12})|\, f'(b)\,| \end{Bmatrix},$$

where

$$\mu_{11} = \frac{2b-a-x}{(\alpha+1)(\alpha+2)(b-a)}\left(\frac{b-x}{b-a}\right)^{\alpha+1},$$

$$\mu_{12} = \frac{1}{\alpha+2}\left\{1-\left(\frac{b-x}{b-a}\right)^{\alpha+2}\right\},$$

$$\nu_{11} = \left(\frac{x-a}{b-a}\right) - \frac{1}{2}\left(\frac{x-a}{b-a}\right)^2 - \mu_{11},$$

$$\nu_{12} = \frac{1}{2}\left\{1-\left(\frac{b-x}{b-a}\right)^2\right\} - \mu_{12}.$$

Proof. From Lemma 2, using the $(\alpha,\ m)$-convexity of *f*, we get

$$\frac{1}{b-a}\left|\, f(x) + \frac{1}{b-a}\int_a^b f(u)du - \{f(a) + f(b)\}\right|$$

$$\leq \int_0^{\frac{b-x}{b-a}} (1-t)|\, f'(ta+(1-t)b)\,|dt + \int_{\frac{b-x}{b-a}}^1 t|\, f'(ta+(1-t)b)\,|dt$$

$$= \left\{\int_0^{\frac{b-x}{b-a}} (1-t)t^{\alpha}dt + \int_{\frac{b-x}{b-a}}^1 t^{\alpha+1}dt\right\}|\, f'(a)\,|$$

$$+ \left\{\int_0^{\frac{b-x}{b-a}} (1-t)(1-t^{\alpha})dt + \int_{\frac{b-x}{b-a}}^1 t(1-t^{\alpha})dt\right\}m\left|\, f'\left(\frac{b}{m}\right)\right|$$

$$= (\mu_{11}+\mu_{12})|\, f'(a)\,| + (\nu_{11}+\nu_{12})m\left|\, f'\left(\frac{b}{m}\right)\right|.$$

Analogously, also we have

$$\frac{1}{b-a}\left| f(x)+\frac{1}{b-a}\int_a^b f(u)du-\{f(a)+f(b)\}\right|$$

$$\leq (\mu_{11}+\mu_{12})m\left| f'\left(\frac{a}{m}\right)\right| + (\nu_{11}+\nu_{12})|f'(b)|,$$

which completes the proof by the simple calculations.

Theorem 2.5. *Let* $f : \mathbb{I} \subset [0, b^*] \to \mathbb{R}$ *be a differentiable mapping on the interior* $\mathbb{I}^0$ *of an interval* $\mathbb{I}$ *such that* $f' \in L([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$ *and* $b^* > 0$. *If* $|f'|^q$ *is* (α, m)*-convex on* $[a, b]$ *for* $(\alpha, m) \in [0, 1]^2$ *and* $q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds*:

$$\frac{1}{b-a}\left| f(x)+\frac{1}{b-a}\int_a^b f(u)du-\{f(a)+f(b)\}\right|$$

$$\leq \left\{\left(\frac{x-a}{b-a}\right)^{p+1}\right\}^{\frac{1}{p}}\left\{\left(\frac{b-x}{b-a}\right)\frac{1}{\alpha+1}M_{41}\right\}^{\frac{1}{q}}$$

$$+\left\{1-\left(\frac{b-x}{b-a}\right)^{p+1}\right\}^{\frac{1}{p}}\left\{\left(\frac{x-a}{b-a}\right)\frac{1}{\alpha+1}M_{42}\right\}^{\frac{1}{q}}, \tag{21}$$

where

$$M_{41} = \min\left\{|f'(a)|^q+\alpha m\left| f'\left(\frac{x}{m}\right)\right|^q, |f'(x)|^q+\alpha m\left| f'\left(\frac{a}{m}\right)\right|^q\right\},$$

$$M_{42} = \min\left\{|f'(x)|^q+\alpha m\left| f'\left(\frac{b}{m}\right)\right|^q, |f'(b)|^q+\alpha m\left| f'\left(\frac{x}{m}\right)\right|^q\right\}.$$

Proof. From Lemma 2 and using the (α, m)-convexity of *f*, we get

$$\frac{1}{b-a}\left| f(x) + \frac{1}{b-a}\int_a^b f(u)du - \{f(a) + f(b)\}\right|$$

$$\leq \int_0^{\frac{b-x}{b-a}} (1-t)|\, f'(ta + (1-t)b)\,|dt + \int_{\frac{b-x}{b-a}}^1 t|\, f'(ta + (1-t)b)\,|dt$$

$$\leq \left\{\int_0^{\frac{b-x}{b-a}} (1-t)^p\, dt\right\}^{\frac{1}{p}} \left\{\int_0^{\frac{b-x}{b-a}} |\, f'(ta + (1-t)b)\,|^q\, dt\right\}^{\frac{1}{q}}$$

$$+ \left\{\int_{\frac{b-x}{b-a}}^1 t^p dt\right\}^{\frac{1}{p}} \left\{\int_{\frac{b-x}{b-a}}^1 |\, f'(ta + (1-t)b)\,|^q\, dt\right\}^{\frac{1}{q}}$$

$$\leq \left\{1 - \left(\frac{x-a}{b-a}\right)^{p+1}\right\}^{\frac{1}{p}} \left\{\int_0^{\frac{b-x}{b-a}} |\, f'(ta + (1-t)b)\,|^q\, dt\right\}^{\frac{1}{q}}$$

$$+ \left\{1 - \left(\frac{b-x}{b-a}\right)^{p+1}\right\}^{\frac{1}{p}} \left\{\int_{\frac{b-x}{b-a}}^1 |\, f'(ta + (1-t)b)\,|^q\, dt\right\}^{\frac{1}{q}}, \tag{22}$$

where we have used the fact that $\frac{1}{2} \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \leq 1$ for $p > 1$.

By Theorem 2.1, we get

$$\int_0^{\frac{b-x}{b-a}} |\, f'(ta + (1-t)b)\,|^q\, dt \leq \left(\frac{b-x}{b-a}\right)\frac{1}{\alpha+1} M_{41}, \tag{23}$$

$$\int_{\frac{b-x}{b-a}}^1 |\, f'(ta + (1-t)b)\,|^q\, dt \leq \left(\frac{x-a}{b-a}\right)\frac{1}{\alpha+1} M_{42}, \tag{24}$$

which implies that by (22)-(24), the assertion (21) holds.

Theorem 2.6. *Let* $f : \mathbb{I} \subset [0, b^*] \to \mathbb{R}$ *be a differentiable mapping on the interior* $\mathbb{I}^0$ *of an interval* $\mathbb{I}$ *such that* $f' \in L([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$ *and* $b^* > 0$. *If* $|f'|^q$ *is* (α, m)*-convex on* $[a, b]$ *for* $(\alpha, m) \in [0, 1]^2$ *and* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then under the notations* μ_{11}, μ_{12}, ν_{11} *and* ν_{12} *in Theorem* 2.4 *the following inequality holds*:

$$\frac{1}{b-a}\left| f(x) + \frac{1}{b-a}\int_a^b f(u)du - \{f(a) + f(b)\}\right|$$

$$\leq \left\{\left(\frac{b-x}{b-a}\right)\left(1 - \frac{b-x}{2(b-a)}\right)\right\}^{\frac{1}{p}} M_{51}^{\frac{1}{q}} + \left\{\frac{1}{2}\left(1 - \left(\frac{x-a}{b-a}\right)^2\right)\right\}^{\frac{1}{p}} M_{52}^{\frac{1}{q}}, \qquad (25)$$

where

$$M_{51} = \min\left\{\begin{matrix} \mu_{11}|f'(a)|^q + \nu_{11}m\left|f'\left(\frac{b}{m}\right)\right|^q, \\ \mu_{11}m\left|f'\left(\frac{a}{m}\right)\right|^q + \nu_{11}|f'(b)|^q \end{matrix}\right\}$$

$$M_{52} = \min\left\{\begin{matrix} \mu_{12}|f'(a)|^q + \nu_{12}m\left|f'\left(\frac{b}{m}\right)\right|^q, \\ \mu_{12}m\left|f'\left(\frac{a}{m}\right)\right|^q + \nu_{12}|f'(b)|^q \end{matrix}\right\}.$$

Proof. From Lemma 2 and using the well-known power mean inequality, we get

$$\frac{1}{b-a}\left| f(x) + \frac{1}{b-a}\int_a^b f(u)du - \{f(a) + f(b)\}\right|$$

$$\leq \left\{\int_0^{\frac{b-x}{b-a}} (1-t)dt\right\}^{\frac{1}{p}} \left\{\int_0^{\frac{b-x}{b-a}} (1-t)|f'(ta + (1-t)b)|^q dt\right\}^{\frac{1}{q}}$$

$$+\left\{\int_{\frac{b-x}{b-a}}^{1} tdt\right\}^{\frac{1}{p}}\left\{\int_{\frac{b-x}{b-a}}^{1} t|\,f'(ta+(1-t)b)\,|^{q}\,dt\right\}^{\frac{1}{q}}$$

$$=\left\{\left(\frac{b-x}{b-a}\right)\left(1-\frac{b-x}{2(b-a)}\right)\right\}^{\frac{1}{p}}\left\{\int_{0}^{\frac{b-x}{b-a}}(1-t)|\,f'(ta+(1-t)b)\,|^{q}\,dt\right\}^{\frac{1}{q}}$$

$$+\left\{\frac{1}{2}\left(1-\left(\frac{x-a}{b-a}\right)^{2}\right)\right\}^{\frac{1}{p}}\left\{\int_{\frac{b-x}{b-a}}^{1} t|\,f'(ta+(1-t)b)\,|^{q}\,dt\right\}^{\frac{1}{q}}. \tag{26}$$

Note that

(i) $$\int_{0}^{\frac{b-x}{b-a}}(1-t)|\,f'(ta+(1-t)b)\,|^{q}\,dt$$

$$\leq \mu_{11}|\,f'(a)\,|^{q}+m\nu_{11}\left|\,f'\left(\frac{b}{m}\right)\right|^{q}, \tag{27}$$

(ii) $$\int_{\frac{b-x}{b-a}}^{1} t|\,f'(ta+(1-t)b)\,|^{q}\,dt$$

$$\leq \mu_{12}|\,f'(a)\,|^{q}+\nu_{12}m\left|\,f'\left(\frac{b}{m}\right)\right|^{q}. \tag{28}$$

Analogously, also we have

(i)′ $$\int_{0}^{\frac{b-x}{b-a}}(1-t)|\,f'(ta+(1-t)b)\,|^{q}\,dt$$

$$\leq \mu_{11}m\left|\,f'\left(\frac{a}{m}\right)\right|^{q}+m\nu_{11}|\,f'(b)\,|^{q}, \tag{29}$$

(ii)′ $$\int_{\frac{b-x}{b-a}}^{1} t|\, f'(ta + (1-t)b)\,|^{q}\, dt$$

$$\leq \mu_{12} m \left| f'\left(\frac{a}{m}\right)\right|^{q} + \nu_{12} m |\, f'(b)\,|^{q}. \tag{30}$$

By the inequalities (27)-(30), we have

$$\int_{0}^{\frac{b-x}{b-a}} (1-t)|\, f'(ta + (1-t)b)\,|^{q}\, dt \leq M_{51}, \tag{31}$$

$$\int_{\frac{b-x}{b-a}}^{1} t|\, f'(ta + (1-t)b)\,|^{q}\, dt \leq M_{52}. \tag{32}$$

By the inequalities (26) and (31)-(32), the assertion (25) holds.

References

[1] M. Avci, Havva Kavurmaci and M. E. Özdemir, New inequalities of Hermite-Hadamard type via *s*-convex functions in the second sense with applications, Appl. Math. Comput. 217 (2011), 5171-5176.

[2] S. S. Dragomir, On some new inequalities of Hermite-Hadamard type for *m*-convex functions, Tamkang J. Math. 33(1) (2002), 45-55.

[3] M. E. Özdemir, Ahmet Ocak Akdemir and Erhan Set, On (*h*-*m*)-convexity and Hadamard-type inequalities, Classical Analysis and ODEs [math.CA], ArXiv:1103.6163v1, 31 Mar 2011.

[4] J. Park, Refinements of Hermite-Hadamard-type inequalities for α-star *s*-convex functions, Far East J. Math. Sci. (FJMS) 41(1) (2010), 97-113.

[5] J. Park, Hermite-Hadamard type inequalities for differentiable (s, m)-convex mappings in the second sense, Far East J. Math. Sci. (FJMS) 52(1) (2011), 57-73.

[6] J. Park, Generalization of Simpson-like type inequalities via differentiable real *s*-convex mappings in the second sense, Inter. J. Math. Math. Sci. (2011), Article in press, ID 493531 doi:10.1155/2011/493531.

[7] J. Park, Some inequalities of Hermite-Hadamard type via differentiable (s, m)-convex mappings, Far East J. Math. Sci. (FJMS) 52(2) (2011), 209-221.

[8] J. Park, Inequalities for some differentiable convex mappings, Far East J. Math. Sci. (FJMS) 44(2) (2010), 251-259.

[9] Erhan Set, M. E. Özdemir and Mehmet Zeki Sarikaya, Inequalities of Hermite-Hadamard's type for functions whose derivatives absolute values are *m*-convex, AIP Conf. Proc. 1309(1) (2010), 861-873 doi:10-1063/1.3525219.

[10] Mehmet Zeki Sarikaya, Erhan Set and M. E. Özdemir, On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex, RGMIA Research Report Coll. 13(1) (2010), Art. No. 1.

[11] Gh. Toader, Some generalizations of the convexity, Proc. Colloq. Approx. Optim. Cluj-Napoca (Romania), 1984, pp. 329-338.

[12] Gh. Toader, The hierarchy of convexity and some classic inequalities, J. Math. Inequal. 3(3) (2009), 305-313.