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ON THE q-EXTENSION OF SECOND KIND EULER POLYNOMIALS

C. S. Ryoo

Department of Mathematics
Hannam University
Daejeon 306-791, Korea

Abstract

In this paper, we construct the q-extension of the second kind Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$. From these numbers and polynomials, we establish some interesting identities and relations.

1. Introduction

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$.

Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number

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$q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, then we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. Throughout this paper, we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad \text{cf. [1, 2, 3, 4, 5].}$$

Hence, $\lim_{q\to 1}[x]_q = x$ for any x with $|x|_p \le 1$ in the present p-adic case.

For

$$g \in UD(\mathbb{Z}_p) = \{g \mid g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

Kim [1, 2] defined the p-adic integral on $\mathbb{Z}_p$ as follows:

$$I_1(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{0 \le x < p^N} g(x)(-1)^x. \tag{1.1}$$

From (1.1), we obtain

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} g(l), \quad \text{(see [1-3])}, \tag{1.2}$$

where $g_n(x) = g(x+n)$.

First, we introduce the second kind Euler numbers E_n and polynomials $E_n(x)$ (see [4]). The second kind Euler numbers E_n are defined by the generating function:

$$F(t) = \frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1.3}$$

We introduce the second kind Euler polynomials $E_n(x)$ as follows:

$$F(x, t) = \frac{2e^t}{e^{2t}+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.4}$$

2. q-extension of the Second Kind Euler Polynomials

In this section, we introduce the q-extension of the second kind Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ and investigate their properties. Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

For $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$, q-extension of the second kind Euler numbers $E_{n,q}$ are defined by

$$E_{n,q} = \int_{\mathbb{Z}_p} [2x+1]_q^n d\mu_{-1}(x). \tag{2.1}$$

By using p-adic integral on $\mathbb{Z}_p$, we obtain

$$\begin{aligned}\int_{\mathbb{Z}_p} [2x+1]_q^n d\mu_{-1}(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} [2x+1]_q^n (-1)^x \\ &= 2\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l \frac{1}{1+q^{2l}} \\ &= 2\sum_{m=0}^{\infty} (-1)^m [2m+1]_q^n. \end{aligned} \tag{2.2}$$

By (2.1), we have the following theorem.

Theorem 1. *For $q \in \mathbb{C}_p$ with $|q-1|_p < 1$, we have*

$$\begin{aligned} E_{n,q} &= 2\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l q^l \frac{1}{1+q^{2l}} \\ &= 2\sum_{m=0}^{\infty} (-1)^m [2m+1]_q^n. \end{aligned}$$

We set

$$F_q(t) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}.$$

By using above equation and (2.2), we have

$$F_q(t) = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}$$

$$= 2\sum_{n=0}^{\infty}\left(\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^l \frac{1}{1+q^{2l}}\right)\frac{t^n}{n!}$$

$$= 2\sum_{m=0}^{\infty}(-1)^m e^{[2m+1]_q t}. \tag{2.3}$$

Thus, q-extension of the second kind Euler numbers, $E_{n,q}$ are defined by means of the generating function

$$F_q(t) = 2\sum_{m=0}^{\infty}(-1)^m e^{[2m+1]_q t}. \tag{2.4}$$

By using (2.1), we have

$$\sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}[2x+1]_q^n d\mu_{-1}(x)\frac{t^n}{n!}$$

$$= \int_{\mathbb{Z}_p} e^{[2x+1]_q t} d\mu_{-1}(x). \tag{2.5}$$

By (2.3), (2.5), we have

$$\int_{\mathbb{Z}_p} e^{[2x+1]_q t} d\mu_{-1}(x) = 2\sum_{m=0}^{\infty}(-1)^m e^{[2m+1]_q t}.$$

Next, we introduce q-extension of the second kind Euler polynomials $E_{n,q}(x)$. The q-extension of the second kind Euler polynomials $E_{n,q}(x)$ are defined by

$$E_{n,q}(x) = \int_{\mathbb{Z}_p}[x+2y+1]_q^n d\mu_{-1}(y). \tag{2.6}$$

By using p-adic integral, we obtain

$$E_{n,q}(x) = 2\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{(x+1)l} \frac{1}{1+q^{2l}}. \tag{2.7}$$

We set

$$F_q(t, x) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{2.8}$$

By using (2.7) and (2.8), we obtain

$$F_q(t, x) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = 2\sum_{m=0}^{\infty} (-1)^m e^{[2m+1+x]_q t}. \tag{2.9}$$

Since $[x+2y+1]_q = [x]_q + q^x[2y+1]_q,$ we easily see that

$$\begin{aligned} E_{n,q}(x) &= \int_{\mathbb{Z}_p} [x+2y+1]_q^n d\mu_{-1}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{xl} E_{l,q} \\ &= ([x]_q + q^x E_q)^n \\ &= 2\sum_{m=0}^{\infty} (-1)^m [x+2m+1]_q^n, \end{aligned} \tag{2.10}$$

with the usual convention of replacing $(E_q)^n$ by $E_{n,q}$.

By (1.3), (1.4), (2.3), and (2.10), we have the following remark.

Remark 1. Note that

(1) $E_{n,q}(0) = E_{n,q}$.

(2) If $q \to 1,$ then $E_{n,q}(x) = E_n(x), \quad E_{n,q} = E_n$.

(3) If $q \to 1$, then $F_q(x, t) = F(x, t)$, $F_q(t) = F(t)$.

By (2.7), we obtain the following theorem.

Theorem 2 (Property of complement)**.**

$$E_{n, q^{-1}}(-x) = (-1)^n q^n E_{n, q}(x).$$

By (2.7), we have the following distribution relation:

Theorem 3. *For any positive integer* $m(= odd)$, *we have*

$$E_{n, q}(x) = [m]_q^n \sum_{a=0}^{m-1} (-1)^a E_{n, q^m}\left(\frac{2a + x + 1 - m}{m}\right), \; n \in \mathbb{Z}_+.$$

By (1.2), (2.1), and (2.6), we easily see that

$$E_{m, q}(2n) + (-1)^{n-1} E_{m, q} = 2\sum_{l=0}^{n-1} (-1)^{n-1-l} [2l + 1]_q^m.$$

Hence, we obtain the following theorem.

Theorem 4. *Let* $m \in \mathbb{Z}_+$. *If* $n \equiv 0 \,(\mathrm{mod}\, 2)$, *then*

$$E_{m, q}(2n) - E_{m, q} = 2\sum_{l=0}^{n-1} (-1)^{l+1} [2l + 1]_q^m.$$

If $n \equiv 1 (\mathrm{mod}\, 2)$, *then*

$$E_{m, q}(2n) + E_{m, q} = 2\sum_{l=0}^{n-1} (-1)^{l} [2l + 1]_q^m.$$

From (1.2), we note that

$$\begin{aligned} 2e^t &= \int_{\mathbb{Z}_p} e^{[2x+3]_q t} \, d\mu_{-1}(x) + \int_{\mathbb{Z}_p} e^{[2x+1]_q t} \, d\mu_{-1}(x) \\ &= \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} [2x + 3]_q^n d\mu_{-1}(x) + \int_{\mathbb{Z}_p} [2x + 1]_q^n d\mu_{-1}(x) \right) \frac{t^n}{n!} \end{aligned}$$

$$= \sum_{n=0}^{\infty} (E_{n,q}(2) + E_{n,q}) \frac{t^n}{n!}.$$

Therefore, we obtain the following theorem.

Theorem 5. *For* $n \in \mathbb{Z}_+$, *we have*

$$E_{n,q}(2) + E_{n,q} = 2.$$

By Theorem 5 and (2.10), we have the following corollary.

Corollary 6. *For* $n \in \mathbb{Z}_+$, *we have*

$$(q^2 E_q + [2]_q)^n + E_{n,q} = 2,$$

with the usual convention of replacing $(E_q)^n$ *by* $E_{n,q}$.

3. The Analogue of the Euler Zeta Function

By using q-extension of second kind Euler numbers and polynomials, q-Euler zeta function and Hurwitz q-Euler zeta functions are defined. These functions interpolate the q-extension of second kind Euler numbers $E_{n,q}$, and polynomials $E_{n,q}(x)$, respectively. Let q be a complex number with $|q| < 1$ and $h \in \mathbb{Z}$. From (2.4), we note that

$$\left.\frac{d^k}{dt^k} F_q(t)\right|_{t=0} = 2\sum_{m=0}^{\infty} (-1)^n [2m+1]_q^k$$

$$= E_{k,q}, \ (k \in \mathbb{N}).$$

By using the above equation, we are now ready to define q-Euler zeta functions.

Definition 7. Let $s \in \mathbb{C}$. Then

$$\zeta_q(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{[2n+1]_q^s}. \tag{3.1}$$

Note that $\zeta_q(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $q \to 1$, then $\zeta_q(s) = \zeta(s)$ which is the Euler zeta function (see [5]). Relation between $\zeta_q(s)$ and $E_{k,q}$ is given by the following theorem.

Theorem 8. *For* $k \in \mathbb{N}$, *we have*

$$\zeta_q(-k) = E_{k,q}.$$

Observe that $\zeta_q(s)$ function interpolates $E_{k,q}$ numbers at non-negative integers. By using (2.9), we note that

$$\frac{d^k}{dt^k} F_q(t, x)\bigg|_{t=0} = 2\sum_{m=0}^{\infty}(-1)^m [2x+1+m]_q^k \tag{3.2}$$

and

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}\right)\Bigg|_{t=0} = E_{k,q}(x), \text{ for } k \in \mathbb{N}. \tag{3.3}$$

By (3.2) and (3.3), we are now ready to define the Hurwitz q-Euler zeta functions.

Definition 9. Let $s \in \mathbb{C}$. Then

$$\zeta_q(s, x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{[n+2x+1]_q^s}. \tag{3.4}$$

Note that $\zeta_q(s, x)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $q \to 1$, then $\zeta_q(s, x) = \zeta(s, x)$ which is the Hurwitz Euler zeta function (see [5]). Relation between $\zeta_q(s, x)$ and $E_{k,q}(x)$ is given by the following theorem.

Theorem 10. *For* $k \in \mathbb{N}$, *we have*

$$\zeta_q(-k, x) = E_{k,q}(x).$$

Observe that $\zeta_q(-k, x)$ function interpolates $E_{k,q}(x)$ numbers at non-negative integers.

References

[1] T. Kim, q-Euler numbers and polynomials associated with p-adic q-integrals, J. Nonlinear Math. Phys. 14 (2007), 15-27.

[2] T. Kim, q-Volkenborn integration, Russ. J. Math. Phys. 9 (2002), 288-299.

[3] T. Kim, L. C. Jang and H. K. Pak, A note on q-Euler and Genocchi numbers, Proc. Japan Acad. 77A (2001), 139-141.

[4] C. S. Ryoo, Calculating zeros of the second kind Euler polynomials, J. Comput. Anal. Appl. 12 (2010), 828-833.

[5] C. S. Ryoo, A note on the q-Hurwitz Euler zeta functions, J. Comput. Anal. Appl. 13 (2011), 1012-1018.

[6] C. S. Ryoo, Calculating zeros of the q-Euler polynomials, Proc. Jangjeon Math. Soc. 12 (2009), 253-259.