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# ON THE OPLUS HEAT KERNEL RELATED TO THE SPECTRUM

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## Abstract

In this paper, we study the equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \oplus^k u(x, t)$$

with the initial condition $u(x, 0) = f(x)$ for $x \in \mathbb{R}^n$. The operator



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$\oplus^k$ is named the oplus operator iterated $k$ times, and is defined by

$$\oplus^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k,$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$, $u(x, t)$ is an unknown function on $\mathbb{R}^n \times (0, \infty)$, $f(x)$ is the given generalized function, $k$ is a positive integer, and $c$ is a positive constant.

We obtain the solution of such equation, which is related to the spectrum and the kernel which is so called oplus heat kernel. Moreover, such oplus heat kernel has interesting properties and also related to the kernel of an extension of the heat equation.

## 1. Introduction

The causal fundamental solution $h(x, t)$ is the particular solution of

$$\frac{\partial E}{\partial t} - a\Delta E = \delta(x)\delta(t),$$

which vanishes identically for $t < 0$. Thus $h(x, t)$ satisfies

$$\frac{\partial h}{\partial t} - a\Delta h = \delta(x)\delta(t), \quad h \equiv 0 \text{ for } t < 0.$$

The causal fundamental solution $h(x, t)$ has a direct physical interpretation; it is the temperature distribution in a medium, which is at zero temperature up to the time $t = 0$, when a concentrated source is introduced at $x = 0$, this source instantaneously releasing a unit of heat. Although $h$ is defined for all $t$ and $x$, its calculation presents a problem only for $t > 0$ $(h = 0$ for $t < 0)$. This immediately suggests a slightly different point of view; for $t > 0$ no sources are present, so that $h$ satisfies the homogeneous equation and must reduce, at $t = 0+$, to a certain initial temperature. This initial temperature is the one to which the medium has been raised just after the introduction of an instantaneous concentrated source of unit strength. We now show that this initial temperature is $\delta(x)$.

It is known that the one-dimensional diffusion equation

$$\frac{\partial u(x, t)}{\partial t} = D \frac{\partial^2 u(x, t)}{\partial x^2},$$

where $u(x, t)$ is the temperature of some object and $D$ is a constant called the "thermal diffusivity" of the material that makes up the object (we could equally well have modeled the diffusion of chemical by letting $u(x, t)$ represent the concentration of some chemical and $D$ be the constant "diffusivity" of the chemical species inside the material that makes up the object). The diffusion equation describes such physical situation as the heat conduction in a one-dimensional solid body, spread of a die in a stationary fluid, population dispersion, and other similar processes. In [2], Chou and Jiang described the diffusion onto a small surface patch on a spherical molecule with an attractive potential all around it. A similar model has been presented by Zhou, who takes into account the attractive interaction and the influence from the heterogeneous surface reactivity only in a thin spherical shell around the target molecule [23]. In this way, the interaction required to hold the reactants together long enough for them to find the reactive site can be estimated. Both of these models indicate that the short range Van der Waals' force could provide sufficient interaction to overcome the orientational constraint of the target molecule. For a recent discussion of these and some other models for heterogeneous surface reactivity see also Chou and Zhou [3]. We refer the readers to the papers [1, 4, 24, 25] for these subjects.

It is well known that for the heat equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \Delta u(x, t) \tag{1.1}$$

with the initial condition $u(x, 0) = f(x)$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

denotes the Laplace operator and $(x, t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, we have

$$u(x, t) = \frac{1}{(4c^2\pi t)^{n/2}} \int_{\mathbb{R}^n} f(x-y) e^{-|y|^2/4c^2t} dy$$

or the solution in the classical convolution form

$$u(x, t) = E(x, t) * f(x),$$

where

$$E(x, t) = \frac{1}{(4c^2\pi t)^{n/2}} e^{-|x|^2/4c^2t} \tag{1.2}$$

and the symbol * designates as the classical convolution.

In [13, 14, 15], Nonlaopon and Kananthai have studied the ultra-hyperbolic heat equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \Box^k u(x, t) \tag{1.3}$$

with the initial condition $u(x, 0) = f(x)$, where $\Box^k$ is the ultra-hyperbolic operator iterated $k$-times, and is defined by

$$\Box^k = \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k,$$

$p + q = n$ is the dimension of $\mathbb{R}^n$, $k$ is a positive integer, $u(x, t)$ is an unknown function on $\mathbb{R}^n \times (0, \infty)$, $f(x)$ is the given generalized function, and $c$ is a positive constant. The solution of (1.3) can be expressed in the form

$$u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) \exp\left( c^2 t \left( \sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{j=1}^{p} \xi_j^2 \right)^k + i(\xi, y) \right) d\xi dy \tag{1.4}$$

or the solution in the classical convolution form

$$u(x, t) = E(x, t) * f(x),$$

where

$$E(x, t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left( c^2 t \left( \sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{j=1}^{p} \xi_j^2 \right)^k + i(\xi, y) \right) d\xi, \qquad (1.5)$$

which is so called *ultra-hyperbolic heat kernel* and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x, t)$ for any fixed, $t > 0$.

In [16], Saglam et al. have studied Bessel diamond heat equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \lozenge_B^k u(x, t) \qquad (1.6)$$

with the initial condition $u(x, 0) = f(x)$, for all $x \in \mathbb{R}_n^+ = \{x : x = (x_1, ..., x_n), x_1 > 0, ..., x_n > 0\}$. The operator $\lozenge_B^k$ is first introduced by Yildirim et al. [22] and is called the *Bessel diamond operator iterated k-times*, and is defined by

$$\lozenge_B^k = [(B_{x_1} + B_{x_2} + \cdots + B_{x_p})^2 - (B_{p+1} + \cdots + B_{x_{p+q}})^2]^k,$$

where $p + q = n$, $B_{x_i} = \dfrac{\partial^2}{\partial x_i^2} + \dfrac{2v_i}{x_i} \dfrac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\dfrac{1}{2}$, $x_i > 0$, $i = 1, 2, ..., n$ and $n$ is the dimension of $\mathbb{R}_n^+$, $k$ is a positive integer, $u(x, t)$ is an unknown function on $\mathbb{R}_n^+ \times (0, \infty)$, $f(x)$ is the given generalized function and $c$ is a positive constant. The solution of (1.6) can be expressed in the form

$$u(x, t) = \int_{\mathbb{R}_n^+} \left[ C_v \int_{\mathbb{R}_n^+} e^{c^2 t V^k(z)} \prod_{i=1}^{n} J_{v_i - \frac{1}{2}}(y_i z_i) z_i^{2v_i} dz \right] T_x^y f(x) \left( \prod_{i=1}^{n} y_i^{2v_i} \right) dy, \qquad (1.7)$$

where $J_{v_i - \frac{1}{2}} x_i y_i$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator, $V(z) = (z_1^2 + z_2^2 + \cdots + z_p^2)^2 - (z_{p+1}^2 + z_{p+2}^2 + \cdots + z_{p+q}^2)^2$ and

$$C_v = \left( \prod_{i=1}^{n} 2^{v_i - \frac{1}{2}} \Gamma\left( v_i + \frac{1}{2} \right) \right)^{-1}.$$

Or the solution in the *B*-convolution form

$$u(x, t) = E(x, t) * f(x),$$

where

$$E(x, t) = C_v \int_{\Omega^+} e^{c^2 t [(y_1^2 + \cdots + y_p^2)^2 - (y_{p+1}^2 + \cdots + y_{p+q}^2)^2]^k} \prod_{i=1}^{n} J_{v_i - \frac{1}{2}}(x_i y_i) y_i^{2v_i} dy, \tag{1.8}$$

which is so called *Bessel diamond heat kernel* and $\Omega^+ \subset \mathbb{R}_n^+$ is the spectrum of $E(x, t)$ for any fixed, $t > 0$.

In [17], Saglam et al. have studied Bessel ultra-hyperbolic heat equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \Box_B^k u(x, t) \tag{1.9}$$

with the initial condition $u(x, 0) = f(x)$ for $x \in \mathbb{R}_n^+$. The operator $\Box_B^k$ is called the *Bessel ultra-hyperbolic operator iterated k-times*, and is defined by

$$\Box_B^k = (B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - \cdots - B_{x_{p+q}})^k,$$

where $p + q = n$ is the dimension of the $\mathbb{R}_n^+$. The solution of (1.9) can be written in the *B*-convolution form $u(x, t) = E(x, t) * f(x)$, where

$$E(x, t) = C_v \int_{\Omega} e^{(-1)^k c^2 t V^k(y)} \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}}(x_i y_i) y_i^{2v_i} \right) dy, \tag{1.10}$$

which is so called *Bessel ultra-hyperbolic heat kernel* and $\Omega \subset \mathbb{R}_n^+$ is the spectrum of $E(x, t)$ for any fixed, $t > 0$.

In [5], Kananthai has studied diamond heat equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \lozenge u(x, t) \tag{1.11}$$

with the initial condition $u(x, 0) = f(x)$ for $x \in \mathbb{R}^n$. The operator $\lozenge$ is first introduced by Kananthai [6] and is called *diamond operator* defined by

$$\lozenge = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} \right)^2 - \left( \frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^2,$$

where $p + q = n$ is the dimension of the space $\mathbb{R}^n$. The solution of (1.11) can be expressed in the classical convolution form $u(x, t) = E(x, t) * f(x)$, where

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp\left[ c^2 t \left( \left( \sum_{i=1}^{p} \xi_i^2 \right)^2 - \left( \sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right) + i(\xi, x) \right] d\xi, \tag{1.12}$$

which is so called *diamond heat kernel* and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x, t)$ for any fixed, $t > 0$.

Furthermore, in [20], Tariboon has studied generalized diamond heat equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \lozenge^k u(x, t) \tag{1.13}$$

with the initial condition $u(x, 0) = f(x)$. The solution of (1.13) can be expressed in the classical convolution form $u(x, t) = E(x, t) * f(x)$, where

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp\left[ c^2 t \left( \left( \sum_{i=1}^{p} \xi_i^2 \right)^2 - \left( \sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right)^k + i(\xi, x) \right] d\xi, \tag{1.14}$$

and $\Omega \in \mathbb{R}^n$ is spectrum of the $E(x, t)$ for any fixed $t > 0$.

In 2000, Kananthai et al. [11] have introduced the operator $\oplus^k$, and is defined by

$$\oplus^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^4\right]^k, \tag{1.15}$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$ and $k$ is a positive integer. Next, in [10], Kananthai et al. have studied the fundamental solution of the operator $\oplus^k$ related to wave equation and Laplacian. And, in [7, 8, 9] Kananthai and Suantai have studied the convolution product, Fourier transform and inversion of the distributional kernel $K_{\alpha, \beta, \gamma, \nu}$ related to the operator $\oplus^k$. Moreover, in [21], Tariboon and Kananthai have studied the Green function of the operator $(\oplus + m^2)^k$. Recently, in [18, 19], Satsanit has studied the Green function and Fourier transform for oplus operators and he has also studied the solutions of a partial differential equation related to the oplus operator.

The purpose of this work is to study the equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \oplus^k u(x, t) \tag{1.16}$$

with the initial condition $u(x, 0) = f(x)$ for all $x \in \mathbb{R}^n$. We found that $u(x, t) = E(x, t) * f(x)$ as a solution of (1.16), where

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^4\right)^k + i(\xi, x)\right] d\xi \tag{1.17}$$

is the fundamental solution of (1.16), which is called the *oplus heat kernel*, and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x, t)$ for any fixed $t > 0$. Some properties of $E(x, t)$ will also be studied in at the end.

Before we proceed to that point, the following definitions and concepts require clarifications.

## 2. Preliminaries

**Definition 2.1.** Let $f(x) \in L_1(\mathbb{R}^n)$ be the space of integrable function in $\mathbb{R}^n$. Then the Fourier transform of $f(x)$ is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi, x)} f(x) dx, \tag{2.1}$$

where $\xi = (\xi_1, \xi_2, ..., \xi_n)$ and $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ and $dx = dx_1 dx_2 \cdots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} \hat{f}(\xi) d\xi. \tag{2.2}$$

**Definition 2.2.** The spectrum of the kernel $E(x, t)$, which is defined by (1.17), is the bounded support of the Fourier transform $\hat{E}(\xi, t)$ for any fixed $t > 0$.

**Definition 2.3.** Let $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$ and denote by

$$\Gamma_+ = \{\xi \in \mathbb{R}^n : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 > 0 \text{ and } \xi_1 > 0\}$$

the set of an interior of the forward cone, and $\overline{\Gamma}_+$ denote the closure of $\Gamma_+$.

Let $\Omega$ be spectrum of $E(x, t)$, defined by Definition 2.2 for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $\hat{E}(\xi, t)$ be the Fourier transform of $E(x, t)$ and we define

$$\hat{E}(\xi, t) = \begin{cases} \dfrac{1}{(2\pi)^{n/2}} \exp\left[ c^2 t \left( \left( \sum_{i=1}^{p} \xi_i^2 \right)^4 - \left( \sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k \right], & \text{for } \xi \in \Omega, \\ 0, & \text{for } \xi \notin \Omega. \end{cases} \tag{2.3}$$

**Lemma 2.1.** *Let L be the operator defined by*

$$L = \frac{\partial}{\partial t} - c^2 \oplus^k, \tag{2.4}$$

*where* $\oplus^k$ *is oplus operator iterated k-times, and is defined by*

$$\oplus^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^4\right]^k,$$

$p + q = n$ *is the dimension of* $\mathbb{R}^n$, *k is a positive integer,* $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $t \in (0, \infty)$ *and c is a positive constant. Then we obtain*

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^4\right)^k + i(\xi, x)\right] d\xi \tag{2.5}$$

*as a fundamental solution of* (2.4) *in the spectrum* $\Omega \subset \mathbb{R}^n$ *for* $t > 0$.

**Proof.** Let $E(x, t)$, where is the kernel or the fundamental solution of operator $L$ and $\delta$ is the Dirac-delta distribution. Thus, we have

$$\frac{\partial}{\partial t} E(x, t) - c^2 \oplus^k E(x, t) = \delta(x)\delta(t).$$

Applying the Fourier transform, which is defined by (2.1), to the both sides of the above equation, considering $\hat{\delta}(x) = 1/(2\pi)^{n/2}$, we obtain

$$\frac{\partial}{\partial t} \hat{E}(\xi, t) - c^2 \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^4\right)^k \hat{E}(\xi, t) = \frac{1}{(2\pi)^{n/2}} \delta(t).$$

Thus, we get

$$\hat{E}(\xi, t) = \frac{H(t)}{(2\pi)^{n/2}} \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^4\right)^k\right],$$

where $H(t)$ is the Heaviside function, because $H(t) = 1$ holds for $t > 0$. It follows that

$$\hat{E}(\xi, t) = \frac{1}{(2\pi)^{n/2}} \exp\left[c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^4\right)^k\right]$$

which has been already by (2.3). Thus from (2.2), we have

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp\left[c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^4\right)^k + i(\xi, x)\right] d\xi.$$

Thus, we have

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp\left[c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^4\right)^k + i(\xi, x)\right] d\xi,$$

where $\Omega$ is the spectrum of $E(x, t)$ and $t > 0$. □

## 3. Main Results

**Theorem 3.1.** *Let us consider the equation*

$$\frac{\partial}{\partial t} u(x, t) = c^2 \oplus^k u(x, t) \tag{3.1}$$

*with the initial condition*

$$u(x, 0) = f(x) \tag{3.2}$$

*where* $\oplus^k$ *is oplus operator iterated k-times, and is defined by* (1.15), *k is a positive integer,* $u(x, t)$ *is an unknown function on* $\mathbb{R}^n \times (0, \infty)$, $f(x)$ *is the given generalized function, and c is a positive constant. Then, we obtain*

$$u(x, t) = E(x, t) * f(x)$$

*as the solution of* (3.1), *which satisfies* (3.2), *where* $E(x, t)$ *is given by* (2.5).

**Proof.** Taking the Fourier transform, which is defined by (2.1), to both sides of (3.1), we obtain

$$\frac{\partial}{\partial t}\hat{u}(\xi, t) = c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^4\right]^k \hat{u}(\xi, t).$$

Thus, we get

$$\hat{u}(\xi, t) = K(\xi)\exp\left[c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^4\right)\right]^k, \tag{3.3}$$

where $K(\xi)$ is constant and $\hat{u}(\xi, 0) = K(\xi)$.

Now, by (3.2) we have

$$K(\xi) = \hat{u}(\xi, 0) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i(\xi, x)} f(x)dx \tag{3.4}$$

and by the inversion in (2.2), (3.3) and (3.4), we obtain

$$\begin{aligned} u(x, t) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi, x)}\hat{u}(\xi, t)d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left[c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^4\right.\right. \\ &\qquad \left.\left. - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^4\right)^k + i(\xi, x-y)\right] f(y)dyd\xi. \end{aligned} \tag{3.5}$$

Set

$$E(x, t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp\left[c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^4\right)^k + i(\xi, x)\right]d\xi. \tag{3.6}$$

Since the integral of (3.6) is divergent, therefore we choose $\Omega \subset \mathbb{R}^n$ be the spectrum of $E(x, t)$ and by (2.5), we have

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp\left[ c^2 t \left( \left( \sum_{i=1}^{p} \xi_i^2 \right)^4 - \left( \sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k + i(\xi, x) \right] d\xi. \quad (3.7)$$

So, (3.6) can be written in the convolution form

$$u(x, t) = E(x, t) * f(x).$$

Since $E(x, t)$ exists,

$$\begin{aligned} \lim_{t \to 0} E(x, t) &= \frac{1}{(2\pi)^n} \int_{\Omega} e^{i(\xi, x)} d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\xi, x)} d\xi \\ &= \delta(x), \text{ for } x \in \mathbb{R}^n, \end{aligned} \quad (3.8)$$

see [12, p. 64, Equation (4)].

Thus for the solution $u(x, t) = E(x, t) * f(x)$ of (3.1), then

$$\lim_{t \to 0} u(x, t) = u(x, 0) = \delta(x) * f(x) = f(x)$$

which satisfies (3.2). This completes the proof. □

**Theorem 3.2.** *The kernel* $E(x, t)$ *defined by* (3.7) *has the following properties*:

(1) $E(x, t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0, \infty))$ *is the space of continuous function with infinitely differentiable*.

(2) $\left( \frac{\partial}{\partial t} - c^2 \oplus^k \right) E(x, t) = 0$ *for* $t > 0$.

(3) $\lim_{t \to 0} E(x, t) = \delta(x)$.

**Proof.** (1) From (3.7), and

$$\frac{\partial^n}{\partial x^n} E(x, t) = \frac{1}{(2\pi)^n} \int_\Omega \frac{\partial^n}{\partial x^n} \exp\left[ c^2 t \left( \left( \sum_{i=1}^{p} \xi_i^2 \right)^4 - \left( \sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k + i(\xi, x) \right] d\xi,$$

we have $E(x, t) \in \mathcal{C}^\infty$ for $x \in \mathbb{R}^n$, $t > 0$.

(2) From $u(x, t) = E(x, t) * f(x)$, we have following equality for $f(x) = \delta(x)$ by Fourier transformation

$$u(x, t) = E(x, t).$$

Then by direct computation, we obtain

$$\left( \frac{\partial}{\partial t} - c^2 \oplus^k \right) E(x, t) = 0.$$

(3) This case is obvious by (3.8). □

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