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CONFIDENCE INTERVALS FOR THE RATIO OF NORMAL MEANS WITH A KNOWN COEFFICIENT OF VARIATION

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Abstract

In this paper, we propose two new confidence intervals for the ratio of normal population means with a known coefficient of variation. This situation occurs in environment and agriculture experiments where the scientist needs to know the coefficient of variation of the control group (treatment) when compared with another treatment whose a coefficient of variation is unknown. This problem is analogous to Maity and Sherman [6] who suggested the new test statistics *t*-test for the difference means with a known variance, and Niwitpong [7] who constructed the confidence interval for the difference between normal

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means with a known variance. We propose two new confidence intervals for the ratio of normal means with a known coefficient of variation. One of our confidence intervals constructed from the pivotal statistic Z, where Z follows a standard normal distribution, another confidence interval is constructed based on the generalized confidence interval (Weerahandi [8]). The performance of the proposed methods is assessed through Monte Carlo simulation studies. Coverage probabilities and expected lengths of these confidence intervals are used to assess these confidence intervals.

1. Introduction

Suppose $X_i \sim N(\mu_x, \sigma_x^2)$, $i = 1, 2, ..., n$, $Y_j \sim N(\mu_y, \sigma_y^2)$, $j = 1, 2, ..., m$ and assume that one of the coefficients of variation is known, $\tau_y = \frac{\sigma_y}{\mu_y}$, where $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2$ are, respectively, population means and population variances of X and Y and

$$\bar{X} = n^{-1}\sum_{i=1}^{n} X_i, \quad \bar{Y} = m^{-1}\sum_{i=1}^{m} Y_i, \quad S_x^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2,$$

$$S_y^2 = (m-1)^{-1}\sum_{i=1}^{m}(Y_i - \bar{Y})^2$$

are the estimators of means and variances of X and Y, respectively. We are interested in constructing the new confidence interval for a ratio of means $\theta = \frac{\mu_x}{\mu_y}$.

Recent paper of Lee and Lin [9] described the situation where the ratio of normal means arises in bioassay problem, the relative potency of a test preparation as compared with a standard is estimated by the ratio of two independent normal random variables for parallel-line assays. In biological assay problems, Fieller [2, 3] proposed Fieller's theorem to construct the confidence intervals for the ratio of means. Koschat [4] also commented that the coverage probability of confidence interval from Filler solution is exact for all the parameters when a common variance assumption is assumed. For possible unequal variances assumption and unequal sample sizes, Lee and Lin [9] showed that the generalized confidence interval (Weerahandi [8]) based on the generalized *p*-value (Tsui and Weerahandi [10]) performs well compared to the Cox's confidence interval (Cox [1]) and Filler's confidence interval. Their simulation results showed that, for unequal variances assumption, the Filler's confidence interval performs poorly in terms of its coverage probability. Also, the coverage probability of the Cox's confidence interval is not better than the generalized confidence interval. In fact, in some cases, e.g., unequal sample sizes, the generalized confidence interval performs better than the Cox's methods. We, therefore, do not consider the Filler's confidence interval and the Cox's confidence interval in our studies. Unlike Lee and Lin [9], our problem here is to construct the confidence interval for the ratio of means when we know a coefficient of variation. We proposed two new confidence intervals for the ratio of normal means with a known coefficient of variation. This kind of problem is an analogous to the works of Maity and Sherman [6] and Niwitpong [7] who investigated the t statistic test and the confidence interval for the difference between normal means with a known variance. One of our new confidence intervals constructed from the asymptotic normality of the test statistic Z, where Z follows a standard normal distribution, another confidence interval is constructed based on the generalized confidence interval (Weerahandi [8]), see Lee and Lin [9]. The performance of the proposed methods is assessed through Monte Carlo simulation studies. Coverage probabilities and expected lengths of these confidence intervals are used to assess these confidence intervals.

The paper is organized as follows: Section 2 presents two new confidence intervals for the ratio of two normal population means based on the exact method and the generalized confidence interval. Simulation design to study coverage probabilities and average length widths for each interval and their results are outlined in Section 3. Section 4 contains a discussion of the results and conclusions.

2. Confidence Interval for the Ratio of Normal Means with a Known Coefficient of Variation

2.1. Exact method

We begin this section by considering the expected lengths and variance of the estimator of θ, $\hat{\theta} = \bar{X}/\bar{Y}$.

This result shows that our estimator is an asymptotic biased estimator. Also, the variance of this estimator converges to zero for large sample size, i.e., $Var(\hat{\theta}) \to 0$, $n \to \infty$.

Theorem 1. *Suppose* $X_i \sim N(\mu_x, \sigma_x^2)$, $i = 1, 2, \ldots, n$, $Y_j \sim N(\mu_y, \sigma_y^2)$, $j = 1, 2, \ldots, m$, *where* $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2$ *are, respectively, population means and population variances of X and Y. Then the estimator of* θ *is* $\hat{\theta} = \frac{\bar{X}}{\bar{Y}}$, *the expected lengths of* $E(\hat{\theta})$ *and* $E(\hat{\theta}^2)$ *when a coefficient of variation is known,* $\tau_y = \frac{\sigma_y}{\mu_y}$, *are, respectively,*

$$\theta\left[1 + \sum_{k=1}^{\infty} \frac{(2k)!}{2^k k!}\left(\frac{\tau_y^2}{n}\right)^k\right] \quad and \quad \theta^2 \frac{\sum_{k=0}^{\infty}(2k+1)!}{2^k k!}\left(\frac{\tau_y^2}{n}\right)^k.$$

Proof of Theorem 1. Using Taylor series expansion of $\frac{1}{\bar{Y}}$ at $\bar{Y} = \mu_y$ as shown in Mahmoudvand and Hassani [5], the estimator $\hat{\theta}$ can be written as

$$\hat{\theta} = \frac{\bar{X}}{\bar{Y}} = \bar{X}\left(\frac{1}{\mu_y} + \frac{f'\left(\frac{1}{\mu_y}\right)(\bar{Y} - \mu_y)}{1!} + \frac{f''\left(\frac{1}{\mu_y}\right)(\bar{Y} - \mu_y)^2}{2!} + \ldots\right)$$

$$= \bar{X}\left(\frac{1}{\mu_y} + \left(-\frac{1}{\mu_y^2}\right)(\bar{Y} - \mu_y) + \left(\frac{2}{2\mu_y^3}\right)(\bar{Y} - \mu_y)^2$$

$$+\left(-\frac{3(2)}{3(2)\mu_y^4}\right)(\bar{Y}-\mu_y)^3+\dots\Bigg)$$

$$=\bar{X}\,\frac{\sum_{k=1}^{\infty}(-1)^{k-1}(\bar{Y}-\mu_y)^{k-1}}{\mu_y^k}. \qquad (1)$$

Hence,

$$E\left(\frac{\bar{X}}{\bar{Y}}\right)=E(\bar{X})\sum_{k=1}^{\infty}(-1)^{k-1}\frac{E(\bar{Y}-\mu_y)^{k-1}}{\mu_y^k}$$

$$=\mu_x\sum_{k=0}^{\infty}2^k\frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\mu_y^{2k+1}}\left[\frac{\sigma_y^2}{n}\right]^k$$

$$=\frac{\mu_x}{\mu_y}\left[1+\sum_{k=1}^{\infty}2^k\frac{\Gamma(k+1/2)}{\sqrt{\pi}}\left(\frac{\sigma_y^2}{\mu_y^2 n}\right)^k\right]$$

$$=\frac{\mu_x}{\mu_y}\left[1+\sum_{k=1}^{\infty}\frac{2^k\Gamma(k+1/2)}{\sqrt{\pi}}\left(\frac{\tau_y^2}{n}\right)^k\right],\ \tau_y=\frac{\sigma_y}{\mu_y}$$

$$=\frac{\mu_x}{\mu_y}\left[1+\frac{\sum_{k=1}^{\infty}2^k\Pi_{j=1}^{k}(k-j+1/2)\Gamma\left(\frac{1}{2}\right)\left(\frac{\tau_y^2}{n}\right)^k}{\sqrt{\pi}}\right]$$

$$=\frac{\mu_x}{\mu_y}\left[1+\sum_{k=1}^{\infty}2^k\Pi_{j=1}^{k}(2k-2j+1)\left(\frac{\tau_y^2}{n}\right)^k\right],\ \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

$$=\frac{\mu_x}{\mu_y}\left[1+\sum_{k=1}^{\infty}\frac{(2k)!}{2^k k!}\left(\frac{\tau_y^2}{n}\right)^k\right]. \qquad (2)$$

From (1),

$$E\left(\frac{\bar{X}}{\bar{Y}}\right) \to \theta, \quad n \to \infty \tag{3}$$

and $E\left(\frac{1}{w}\hat{\theta}\right) = \theta$, where $w = \left[1 + \sum_{k=1}^{\infty} \frac{(2k)!}{2^k k!}\left(\frac{\tau_y^2}{n}\right)^k\right]$. Therefore, the estimation $\left(\frac{\bar{X}}{w\bar{Y}}\right)$ is an unbiased estimator of $\theta = \frac{\mu_x}{\mu_y}$.

Define the variance of $\hat{\theta}$ by $Var(\hat{\theta})$, where

$$Var(\hat{\theta}) = E(\hat{\theta}^2) - (E(\hat{\theta})). \tag{4}$$

From (2) and (3), we have $Var(\hat{\theta}) \approx E(\hat{\theta}^2)$, $n \to \infty$ and

$$E(\hat{\theta}^2) = E\left(\left(\frac{\bar{X}}{\bar{Y}}\right)^2\right) = E(\bar{X}^2)\frac{\sum_{k=1}^{\infty}(-1)^{k-1}kE(\bar{Y} - \mu_y)^{k-1}}{\mu_y^{k+1}}$$

$$= \mu_x^2 \frac{\sum_{k=0}^{\infty}(2k+1)2^k\Gamma(k+1/2)\left(\frac{\sigma^2}{n}\right)^k}{\sqrt{\pi}\mu_y^{2k+2}}$$

$$= \frac{\mu_x^2}{\mu_y^2}\frac{\sum_{k=0}^{\infty}(2k+1)!}{2^k k!}\left(\frac{\tau_y^2}{n}\right)^k. \tag{5}$$

From (2)-(5), we have

$$Var(\hat{\theta}) \to 0, \quad n \to \infty. \qquad \square$$

Corollary 1. *From Theorem* 1, $Var(\hat{\theta}) \approx \frac{\sigma_x^2}{n\mu_y^2} + \left(\frac{\theta}{m} + \frac{\sigma_x^2}{nm\mu_y^2}\right)\tau_y^2.$

Proof of Corollary 1. Now consider only the first two terms of the right hand side of (1),

$$\hat{\theta} = \frac{\overline{X}}{\overline{Y}} \approx \frac{\overline{X}}{\mu_y} - \frac{\overline{X}(\overline{Y} - \mu_y)}{\mu_y^2},$$

$$Var(\hat{\theta}) = \frac{Var(\overline{X})}{\mu_y^2} + \frac{Var(\overline{X}(\overline{Y} - \mu_y))}{\mu_y^4} - \frac{2Cov(\overline{X},\ \overline{X}(\overline{Y} - \mu_y))}{\mu_y^3}$$

$$= \frac{\sigma_x^2}{n\mu_y^2} + \left(\frac{(E(\overline{Y} - \mu_y))^2 \dfrac{\sigma_x^2}{n} + (E(\overline{X}))^2 Var(\overline{Y} - \mu_y) + \dfrac{\sigma_x^2}{n}\dfrac{\sigma_y^2}{m}}{\mu_y^4} \right)$$

$$= \frac{\sigma_x^2}{n\mu_y^2} + \left(\frac{0 + \mu_x^2\left(\dfrac{\sigma_y^2}{m}\right) + \dfrac{\sigma_x^2}{n}\dfrac{\sigma_y^2}{m}}{\mu_y^4} \right)$$

$$= \frac{\sigma_x^2}{n\mu_y^2} + \frac{\left(\dfrac{\mu_x^2\sigma_y^2}{m} + \dfrac{\sigma_x^2}{n}\dfrac{\sigma_y^2}{m}\right)}{\mu_y^4}$$

$$= \frac{\sigma_x^2}{n\mu_y^2} + \left(\mu_x^2 + \frac{\sigma_x^2}{n}\right)\frac{\sigma_y^2}{n\mu_y^4}$$

$$= \frac{\sigma_x^2}{n\mu_y^2} + \left(\frac{\theta}{m} + \frac{\sigma_x^2}{nm\mu_y^2}\right)\tau_y^2. \tag{6}$$

This ends the proof. □

Now we will use the fact that, from Central Limit Theorem,

$$Z = \frac{\hat{\theta} - \theta}{\sqrt{Var(\hat{\theta})}} \sim N(0,1). \tag{7}$$

Plugging (6) to (7), we have

$$Z = \frac{\dfrac{\hat{\theta}}{w} - \theta}{\sqrt{\dfrac{\sigma_x^2}{n\mu_y^2} + \left(\dfrac{\theta^2}{m} + \dfrac{\sigma_x^2}{nm\mu_y^2}\right)\tau_y^2}} \sim N(0, 1).$$

It is therefore easily seen that the $100(1-\alpha)\%$ confidence interval for θ is

$$CI_{exact} = \left[\frac{\hat{\theta}}{w} \pm Z_{1-\alpha/2}\sqrt{\frac{S_x^2}{n\bar{X}^2} + \left(\frac{\hat{\theta}^2}{m} + \frac{S_x^2}{nm\bar{Y}^2}\right)\tau_y^2}\right], \tag{8}$$

where $Z_{1-\alpha/2}$ is an upper $1-\alpha/2$ quantile of the standard normal distribution.

2.2. Generalized confidence interval (GCI)

We now give a brief introduction to the GCI idea, based on Weerahandi [8] and the generalized *p*-value, Tsui and Weerahandi [10]. These two papers mentioned that the method of GCI can be used whenever 'standard' pivotal quantity is difficult to obtain. They introduced the concept of GCI as follows:

G1. Let *X* be a random variable with probability distribution $f(X, \theta, \delta)$, where θ is the parameter of interest and δ is a nuisance parameter.

G2. Let *x* denote the observed value of *X*. In order to obtain a generalized confidence interval for θ, we start with a generalized pivotal quantity $R(X, x, \theta, \delta)$, which is a function of the random variable *X*, its observed value *x*, and the parameters θ and δ.

G3. Also, $R(X, x, \theta, \delta)$ is required to satisfy the following conditions:

C1. For a fixed *x*, the probability distribution of $R(X, x, \theta, \delta)$ is free of unknown parameters.

C2. The observed value of $R(X, x, \theta, \delta)$, namely, $R(x, x, \theta, \delta)$ is simply θ.

C3. For fixed x and δ, $P(R(X, x, \theta, \delta) > t \mid \theta)$ is a non-decreasing in δ.

A $100(1-\alpha/2)\%$ generalized lower confidence limit for θ is then given by $R_{1-\alpha}$, the $100(1-\alpha)$th percentiles of $R(X, x, \theta, \delta)$.

Further, given the observed value x, let t_1 and $t_2\theta$ be such values that $P(t_1 < R(X, x, \theta, \delta) < t_2 \mid \theta) = 1-\alpha$ for chosen significant level $\alpha \in (0, 1)$. Then the confidence interval for parameter θ defined by $\{\theta : t_1 < R(X, x, \theta, \delta) < t_2\}$ is a $100(1-\alpha)\%$ generalized confidence interval for θ.

We now begin to construct the confidence interval for $\theta = \dfrac{\mu_x}{\mu_y}$ with a known coefficient variation $\tau_y = \dfrac{\sigma_y}{\mu_y}$.

Consider

$$\mu_x = \bar{x} - \frac{(\bar{X} - \mu_x)}{\frac{\sigma_x}{\sqrt{n}}}\frac{\sigma_x}{\sqrt{n}}$$

$$= \bar{x} - Z\sqrt{\frac{(n-1)s_x^2}{U}}\frac{1}{\sqrt{n}}, \quad Z \sim N(0, 1), \quad U \sim \chi_{n-1}^2, \tag{9}$$

$$\mu_y = \bar{y} - \frac{(\bar{Y} - \mu_y)}{\left(\frac{\tau_y\mu_y}{\sqrt{m}}\right)}\left(\frac{\tau_y\mu_y}{\sqrt{m}}\right), \quad \tau_y = \frac{\sigma_y}{\mu_y}, \quad \sigma_y = \tau_y\mu_y,$$

$$\mu_y = \bar{y} - \frac{Z\tau_y\mu_y}{\sqrt{m}},$$

$$\mu_y\left(1 - \frac{Z\tau_y}{\sqrt{m}}\right) = \bar{y},$$

$$\mu_y = \frac{\bar{y}}{\left(1 - \frac{Z\tau_y}{\sqrt{m}}\right)}. \tag{10}$$

From (9) and (10),

$$\therefore \quad \frac{\mu_x}{\mu_y} = \frac{\bar{x} - Z\sqrt{\frac{(n-1)s_x^2}{U}} \cdot \frac{1}{\sqrt{n}}}{\frac{\bar{y}}{\left(1 - Z\frac{\tau_y}{\sqrt{m}}\right)}} = \frac{\frac{\left(\sqrt{n}\,\bar{x} - Zs_x\sqrt{\frac{(n-1)}{U}}\right)}{\sqrt{n}}}{\frac{\bar{y}}{\frac{(\sqrt{m} - Z\tau_y)}{\sqrt{m}}}},$$

$$\frac{\mu_x}{\mu_y} = \frac{\sqrt{n}\,\bar{x} - Zs_x\sqrt{\frac{(n-1)}{U}}}{\sqrt{n}} \times \frac{(\sqrt{m} - Z\tau_y)}{\sqrt{m}\bar{y}}, \tag{11}$$

$$R(X,\ x,\ Y,\ y,\ \theta,\ \sigma_x^2,\ \sigma_y^2) = \frac{\sqrt{n}\bar{x} - Z_{1-\alpha/2}S_x\sqrt{\frac{(n-1)}{U}}}{\sqrt{n}} \times \frac{(\sqrt{m} - Z_{1-\alpha/2}\tau_y)}{\sqrt{m}\bar{y}}. \tag{12}$$

It is easily seen that $R(X,\ x,\ Y,\ y,\ \theta,\ \sigma_x^2,\ \sigma_y^2)$ satisfies Conditions C1-C3. Hence, a $100(1-\alpha)\%$ generalized confidence interval for θ is

$$CI_{gci} = [R_{\alpha/2},\ R_{1-\alpha/2}], \tag{13}$$

where $R_{1-\alpha/2}$ is a $(1-\alpha/2)100\%$ upper quantile of $R(X,\ x,\ Y,\ y,\ \theta,\ \sigma_x^2,\ \sigma_y^2)$.

3. Simulation Studies

In this section, we use Monte Carlo simulation to assess two confidence intervals notified in the previous section: CI_{exact} and CI_{gci} based on their coverage probabilities and average length widths. We design a simulation, without losing generality, by setting $\mu_1 = \mu_2 = 1$, the ratio of variances $\sigma_1^2/\sigma_2^2 = 0.25, 0.5, 0.8, 1, 2, 3, 4, 5, 10$ and the samples sizes ($n = m = 10$),

(n = 10, m = 20), (n = 20, m = 10), ($n = m$ = 20), (n = 20, m = 40), (n = 40, m = 20), ($n = m$ = 40). We wrote the function in R program to generate the data which is normally distributed with means and variances which are mentioned previously to construct confidence intervals CI_{exact} and CI_{gci} and then compute coverage probability and average length width of each confidence interval. All results are illustrated in Table I with a number of simulation runs, $M = 10{,}000$ and the nominal level $(1 - \alpha = 0.95)$. From Table I, we found that the coverage probability of the CI_{exact} confidence interval performs as well as the confidence interval CI_{gci} for the ratio of variances, $\sigma_1^2/\sigma_2^2 =$ 0.25, 0.5, 0.8, 1, 2, 3, 4, 5, 10. Further, the coverage probability of these two intervals, for small ratio of variances, is below 0.95 and increasing far beyond 0.95 otherwise. These results suggest us that the more ratio of variances, the more coverage probability of these two confidence intervals are. In other words, these two confidence intervals are wider than usual when the ratio of variances is large. Table I also shows that the ratio of the expected lengths of the two intervals, $E(CI_{gci}/CI_{exact})$ is slightly greater than 1 for moderate and large ratio of variances. This means the length of confidence interval CI_{exact} is shorter than the length of the confidence interval CI_{gci} for almost cases except two cases for small values of sample sizes, e.g., $(n = m = 10)$ and $(n = 10,\ m = 20)$.

4. Discussion and Conclusion

We propose two new confidence intervals for the ratio of normal means with a known coefficient of variation. One of these intervals, CI_{exact} is constructed using the result from the pivotal statistic Z, a standard normal distribution. Another confidence interval based on the generalized confidence interval is derived for the first time for this problem. It is shown, by means of simulation, that coverage probabilities of both intervals are not significantly different. The coverage probabilities of both intervals are slightly increased to one when a ratio of variances is large. This shows both confidence intervals are wider when a ratio of variances is large. We may argue that a good confidence interval should have a large coverage probability and a short expected length. In this case, it is therefore to assess these two intervals using a ratio of expected lengths as shown in the last column of Table I. In sum, from the simulation studies, the confidence interval based on exact method, CI_{exact} is shorter than the confidence interval based on the generalized confidence interval. The confidence interval CI_{exact} is also easy to use more than the confidence interval CI_{gci} which is based on a computational approach. We, therefore, recommend the confidence interval CI_{exact} when a coefficient of variation is known for practitioner.

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Table I. Coverage probability and a ratio of expected length of the intervals CI_{exact} and CI_{gci}

n	m	τ_y	Coverage probability		Ratio of lengths
10	10		CI_{gci}	CI_{exact}	$E(CI_{gci}/CI_{exact})$
10	10	0.25	0.9286	0.8986	0.9713
		0.5	0.9270	0.9208	0.9766
		0.8	0.9424	0.9566	1.0175
		1	0.9552	0.9742	1.0419
		2	0.9850	0.9990	1.0467
		3	0.9890	0.9999	1.0722
		4	0.9952	0.9999	1.0638
		5	0.9960	0.9999	1.0599
		10	0.9998	0.9999	1.1152
10	20	0.25	0.9414	0.8986	0.9442
		0.5	0.9460	0.9200	0.9557
		0.8	0.9508	0.9500	0.9688
		1	0.9594	0.9662	0.9048
		2	0.9768	0.9962	1.0037
		3	0.9872	0.9996	1.0163
		4	0.9926	0.9999	1.1189
		5	0.9954	0.9999	1.0787
		10	0.9992	0.9999	1.0769
20	10	0.25	0.9104	0.9074	1.0202
		0.5	0.9132	0.9278	1.0279
		0.8	0.9356	0.9608	1.0576
		1	0.9536	0.9756	1.0211

		2	0.9828	0.9970	1.0252
		3	0.9926	0.9992	1.0569
		4	0.9966	0.9999	1.0335
		5	0.9986	0.9999	1.0869
		10	0.9999	0.9999	1.0477
20	20	0.25	0.9262	0.9132	1.0171
		0.5	0.9302	0.9268	1.0175
		0.8	0.9416	0.9552	1.0147
		1	0.9514	0.9698	1.0102
		2	0.9806	0.9966	1.0132
		3	0.9934	0.9992	1.0194
		4	0.9974	0.9998	1.0325
		5	0.9992	0.9999	1.0335
		10	0.9994	0.9999	1.0266
20	40	0.25	0.9330	0.9146	1.0152
		0.5	0.9424	0.9350	1.0178
		0.8	0.9448	0.9518	1.0112
		1	0.9520	0.9626	1.0125
		2	0.9838	0.9966	1.0008
		3	0.9904	0.9994	1.0049
		4	0.9968	0.9998	1.0404
		5	0.9976	0.9999	1.0260
		10	0.9996	0.9999	1.0397
40	20	0.25	0.9134	0.9158	1.0199
		0.5	0.9074	0.9264	1.0069
		0.8	0.9316	0.9498	1.0069

		1	0.9546	0.9706	1.0097
		2	0.9884	0.9964	1.0294
		3	0.9954	0.9998	1.0213
		4	0.99861	0.9999	1.0563
		5	0.9998	0.9999	1.0301
		10	0.9998	0.9999	1.0718
40	40	0.25	0.9248	0.9214	1.0154
		0.5	0.9234	0.9340	1.0093
		0.8	0.9338	0.9498	1.0061
		1	0.9482	0.9648	1.0048
		2	0.9868	0.9954	1.0045
		3	0.9950	0.9982	1.0105
		4	0.9972	0.9999	1.0173
		5	0.9988	0.9994	1.0170
		10	0.9999	0.9999	1.0143