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FINANCIAL EXTREMES: A SHORT REVIEW

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Abstract

The paper reviews extreme value theory in finance. It discusses extreme value theory in interest rate models, stochastic volatility models and long memory models.

1. Basic Extremes

1.1. Extreme value distributions

Frechet: $\Phi_\alpha(x) = \exp(-x^{-\alpha}),\ x > 0.$

Gumbel: $\Lambda(x) = \exp(-e^{-x}),\ x \in \mathbb{R}.$

Weibull: $\Psi_\alpha(x) = \exp(-(-x)^{\alpha}),\ x \leq 0.$

1.2. Regular variation

A positive measurable function f on $(0, \infty)$ is *regularly varying* at ∞

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with index α if

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^{\alpha}, \quad x > 0.$$

We use the notation $f \in \mathcal{R}(\alpha)$ for a regularly varying function with index α.

f is said to be *slowly varying* if $\alpha = 0$, *rapidly varying* if the above limit is 0 for $x > 1$ and ∞ for $0 < x < 1$.

$F \in \mathcal{L}(\gamma)$, $\gamma \geq 0$ if for every $y \in \mathbb{R}$,

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = e^{\gamma y},$$

where $\overline{F}(u) = 1 - F(u) = P(X > u)$. See Resnick [5].

Convolution equivalent distributions

Let *X* have d.f. *F*. $F \in \mathcal{S}(\gamma)$, $\gamma \geq 0$ if $F \in \mathcal{L}(\gamma)$ and

$$\lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2\hat{f}(\gamma),$$

where $\hat{f}(\gamma) = Ee^{\gamma X}$ is the moment generating function of *X* at γ. The class $\mathcal{S} := \mathcal{S}(0)$ is the class of subexponential distributions.

Subexponential distributions are heavy tailed in the sense that no exponential moments exist. $\mathcal{S}$ contains all d.f.s *F* with regularly varying tails and is a much larger class. Distribution functions in $\mathcal{S}(\gamma)$ for some $\gamma > 0$ have exponential tails, hence are lighter tailed than subexponential distributions.

Theorem 1.1 (Fisher-Tippett Theorem)**.** *For an i.i.d. sequence, the limit distribution of the maxima is one of the three extreme value distributions*: *Frechet, Gumbel and Weibull*.

2. Diffusion Extremes

Consider the Itô stochastic differential equation model for asset price

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad t > 0, \quad X_0 = x.$$

W : standard Brownian motion,

μ : drift coefficient,

σ : diffusion coefficient or volatility,

X : diffusion process.

Running maxima:

$$M_t = \max_{0 \le s \le t} X_s, \quad t > 0.$$

Scale function:

$$s(x) = \int_z^x \exp\left(-2\int_z^y \frac{\mu(t)}{\sigma^2(t)}\,dt\right)dy, \quad x \in (l,\ r),$$

where z is any interior point in $(l,\ r)$.

Speed measure:

m has Lebesgue density

$$m'(x) = \frac{2}{\sigma^2(x)s'(x)}, \quad x \in (l,\ r)$$

and total mass $|\,m\,| = m((l,\ r))$. s' is the Lebesgue density of s. X_t is ergodic and its stationary distribution is absolutely continuous with Lebesgue density

$$h(x) = \frac{m'(x)}{|\,m\,|}, \quad x \in (l,\ r).$$

X_t satisfies the *usual conditions* which guarantees that X is ergodic with stationary density:

$$s(r) = -s(l) = \infty,$$

$$|\, m \,| < \infty.$$

Theorem 2.1 (Davis [2]). *Let* $(X_t)_{t \geq 0}$ *satisfy the usual conditions. Then for any initial value* $X_0 = y \in (l, r)$ *and any* $u_t \uparrow r$,

$$\lim_{t \to \infty} \left| P^y (M_t \leq u_t) - F^t(u_t) \right| = 0,$$

where F is a df defined for any $z \in (l, r)$ *by*

$$F(x) = \exp\left(- \frac{1}{|\, m \,| s(x)} \right), \quad x \in (z, r).$$

Outline of Proof. Diffusion can be represented as an Ornstein-Uhlenbeck process after a random time-change. Then standard theory of extremes of Gaussian processes applies

Corollary 2.1.

$$\overline{F}(x) \sim \left(|\, m \,| \int_z^x s'(y) dy \right)^{-1} \sim (|\, m \,| s(x))^{-1}, \quad x \uparrow r.$$

F is in the maximum domain of attraction of $G : F \in MDA(G)$ *if*

$$\frac{M_t - b_t}{a_t} \to^{\mathcal{D}} G, \quad t \to \infty.$$

$G \in \{\Phi_\alpha, \Lambda\}, \alpha > 0.$

Φ_α *is Frechet distribution,* Λ *is Gumbel distribution.*

Theorem 2.2. *If* μ *and* σ *are differentiable in some left neighborhood of r such that*

$$\lim_{x \to r} \frac{d}{dx} \frac{\sigma^2(x)}{\mu(x)} = 0,$$

$$\lim_{x \to r} \frac{\sigma^2(x)}{\mu(x)} \exp\left(-2 \int_z^x \frac{\mu(t)}{\sigma^2(t)} dt \right) = -\infty,$$

then

$$\overline{F}(x) \sim |\mu(x)|h(x), \quad x \uparrow r.$$

3. Interest Rate Extremes

We discuss stochastic interest rate models.

Vasicek model

$$dX_t = (a - bX_t)dt + \sigma dW_t, \quad X_0 = x, \quad a \in \mathbb{R}, \quad b > 0,$$

$$X_t = \frac{a}{b} + \left(x - \frac{a}{b}\right)e^{-bt} + \sigma\int_0^t e^{-b(t-s)}dW_s,$$

$$E(X_t) = \frac{a}{b} + \left(x - \frac{a}{b}\right)e^{-bt} \to \frac{a}{b}, \quad t \to \infty,$$

$$Var(X_t) = \frac{\sigma^2}{2b}(1 - e^{-2bt}) \to \frac{\sigma^2}{2b}, \quad t \to \infty.$$

X_t has a normal stationary distribution $N\left(\frac{a}{b}, \frac{\sigma^2}{2b}\right)$.

Condition of Theorem 2.2 holds which gives

$$\overline{F}(x) \sim \frac{2b^2}{\sigma^2}\left(x - \frac{a}{b}\right)^2 \overline{H}(x), \quad x \to \infty,$$

where $\overline{H}(x)$ is the tail of a stationary normal distribution, hence F has heavier tail than H.

$F \in MDA(\Lambda)$ with norming constants

$$a_t = \frac{\sigma}{2\sqrt{b\log t}},$$

$$b_t = \frac{\sigma}{\sqrt{b}}\sqrt{\log t} + \frac{a}{b} + \frac{\sigma}{4\sqrt{b}}\frac{\log\log t + \log(\sigma^2 b/2\pi)}{\sqrt{\log t}}.$$

Cox-Ingersoll-Ross (CIR) model

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dW_t,$$

$$E(X_t) = \frac{a}{b} + \left(x - \frac{a}{b}\right)e^{-bt} \to \frac{a}{b}, \quad t \to \infty,$$

$$V(X_t) = \frac{a\sigma^2}{2b^2}\left(1 - \left(1 + \left(x - \frac{a}{b}\right)\frac{2b}{a}\right)e^{-2bt} + \left(x - \frac{a}{b}\right)\frac{2b}{a}e^{-3bt}\right) \to \frac{a\sigma^2}{2b^2},$$
$$t \to \infty.$$

X_t has a Gamma stationary distribution $\Gamma\left(\frac{2a}{\sigma^2}, \frac{2b}{\sigma^2}\right)$.

A, *B* and *C* are positive constants.

Condition of Theorem 2.2 holds which gives

$$\overline{F}(x) \sim \frac{2ab}{\sigma^2}\overline{G}(x) \sim Ax\overline{H}(x), \quad x \to \infty,$$

where $\overline{G}(x)$ is the tail of a stationary gamma distribution $\Gamma\left(\frac{2a}{\sigma^2} + 1, \frac{2b}{\sigma^2}\right)$ hence *F* has heavier tail than *H*.

$F \in MDA(\Lambda)$ with norming constants

$$a_t = \frac{\sigma^2}{2b},$$

$$b_t = \frac{\sigma^2}{2b}\left(\log t + \frac{2a}{\sigma^2}\log\log t + \log\left(\frac{b}{\Gamma(2a/\sigma^2)}\right)\right).$$

Chan-Karloyi-Longstaff-Sanders (CKLS) model

$$dX_t = (a - bX_t)dt + \sigma X_t^\gamma dW_t, \quad \gamma \in [1/2, \infty).$$

Case I. $\frac{1}{2} < \gamma < 1$:

$$E(x_t) = \frac{a}{b} + \left(x - \frac{a}{b}\right)e^{-bt} \to \frac{a}{b}, \quad t \to \infty, \ b > 0,$$

$$E(x_t) = \frac{a}{b} + \left(x - \frac{a}{b}\right)e^{-bt} \to \infty, \quad t \to \infty, \ b < 0,$$

$$E(x_t) = x + at \to \infty, \quad t \to \infty, \ b = 0.$$

The lack of first moment indicates that for certain parameter values the model can capture very large fluctuations in the data, which will reflect also in the maxima.

Stationary density is

$$h(x) = \frac{2}{A\sigma^2} x^{-2\gamma} \exp\left(- \frac{2}{\sigma^2} \left(\frac{a}{2\gamma - 1} x^{-(2\gamma-1)} + \frac{b}{2 - 2\gamma} x^{2-2\gamma} \right) \right)$$

for some constant $A > 0$.

Condition of Theorem 2.2 holds.

$$\overline{F}(x) \sim bxh(x) \sim Bx^{2(1-\gamma)} \overline{H}(x), \quad x \to \infty.$$

$F \in MDA(\Lambda)$ with norming constants

$$a_t = \frac{\sigma^2}{2b} \left(\frac{\sigma^2(1-\gamma)}{b} \log t \right)^{\frac{2\gamma-1}{2-2\gamma}},$$

$$b_t = \left(\frac{\sigma^2(1-\gamma)}{b} \log t \right)^{\frac{1}{2-2\gamma}} \left(1 - \frac{2\gamma - 1}{(2-2\gamma)^2} \frac{\log\left(\frac{\sigma^2(1-\gamma)}{b} \log t \right)}{\log t} \right)$$

$$+ a_t \log\left(\frac{2b}{A\sigma^2} \right).$$

Case II. $\gamma = 1$:

The model has an explicit solution

$$X_t = e^{-\left(b + \frac{\sigma^2}{2}\right)t + \sigma W_t} \left(x + a \int_0^t e^{\left(b + \frac{\sigma^2}{2}\right)s - \sigma W_s} ds \right).$$

The stationary density is inverse gamma:

$$h(x) = \left(\frac{\sigma^2}{2a} \right)^{-\frac{2b}{\sigma^2} - 1} \left(\Gamma\left(\frac{2b}{\sigma^2} + 1 \right) \right)^{-1} x^{-2b/\sigma^2 - 2} \exp\left(- \frac{2a}{\sigma^2} x^{-1} \right),$$

$x > 0$, $h \in \mathcal{R}(-2b/\sigma^2 - 2)$ and the tail $\overline{H}$ of the stationary distribution is regularly varying.

$$\overline{F}(x) \sim Bx^{-2b/\sigma^2 - 1}, \quad x \to \infty.$$

$F \in MDA(\Phi_{1+2b/\sigma^2})$ with norming constants

$$a_t \sim Ct^{1/(1+2b/\sigma^2)},$$

$$b_t = 0.$$

Case III. $\gamma > 1$:

h has the same form as in the case $\frac{1}{2} < \gamma < 1$ and $\overline{H} \in \mathcal{R}(-2\gamma + 1)$,

$$\overline{F}(x) \sim (Ax)^{-1}, \quad x \to \infty.$$

CKLS pointed out, most plausible value of $\gamma = 1.5$.

Condition of Theorem 2.2 holds.

$F \in MDA(\Phi_1)$ with norming constants

$$a_t \sim t/A,$$

$$b_t = 0.$$

4. Stochastic Volatility Extremes

Levy-Ornstein-Uhlenbeck volatility

Empirical volatility changes in time and exhibits tails which are heavier than normal. Empirical volatility has upward jumps and clusters on high levels. Levy driven Ornstein-Uhlenbeck models can capture heavy tails and volatility jumps and have volatility clusters if the driving Levy process has regularly varying tails.

Black-Scholes model

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Heston model

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t,$$

$$dV_t = \lambda(a - V_t) dt + \sigma\sqrt{V_t} dB_t,$$

$\lambda, a, \sigma > 0,\ \ \lambda a \geq \frac{\sigma^2}{2}$, a is the long run mean, λ is the rate of mean reversion and $r > 0$ is the risk-free interest rate under the risk-neutral world. Volatility is a CIR Process, W and B are two independent Brownian motions for simplicity, they could be correlated to include *leverage*.

GARCH model

$$dS_t = \sqrt{V_t} dW_t,$$

$$dV_t = \lambda(a - V_t) dt + \sigma V_t dB_t.$$

Volatility is a CKLS model with elasticity $\gamma = 1$.

Our focus is on Levy driven stock price and volatility model.

Barndorff-Neilsen-Shephard model

$$dS_t = (\mu + rS_t) dt + \sqrt{V_t} dW_t + \rho dL_{\lambda t},$$

$$dV_t = -\lambda V_t dt + \sigma dL_{\lambda t},$$

$$V_t = e^{-\lambda t} V_0 + \int_0^t e^{-\lambda(t-s)} dL_{\lambda s}$$

is a cadlag process.

If V_0 is independent of the driving Levy process L and $V_0 =^d \int_0^\infty e^{-s} dL_s$, then the process is stationary. The stationary solution is

$$V_t = e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} dL_{\lambda s}.$$

We are concerned with processes L which are heavy or semi-heavy tailed, i.e., whose tails decrease no faster than exponentially.

Define

$$M_h := \sup_{0 \le t \le h} V_t.$$

Theorem 4.1. (a) *If* $L_1 \in \mathcal{S} \cap MDA(\Phi_\alpha)$, *then*

$$P(M_h > x) \sim (\lambda h + \alpha^{-1}) P(L_1 > x), \quad x \to \infty.$$

(b) *If* $L_1 \in \mathcal{S} \cap MDA(\Lambda)$, *then*

$$P(M_h > x) \sim \lambda h P(L_1 > x), \quad x \to \infty.$$

Running maxima theorem

(a) If $L_1 \in \mathcal{S} \cap MDA(\Phi_\alpha)$, then

$$\lim_{T\to\infty} P(a_{\lambda T}^{-1} M_T \le x) = e^{-x^{-\alpha}}, \quad x > 0,$$

where a_T is such that

$$\lim_{T\to\infty} TP(L_1 > a_T x) = x^{-\alpha}, \quad x > 0.$$

(b) If $L_1 \in \mathcal{S} \cap MDA(\Lambda)$, then

$$\lim_{T\to\infty} P(a_{\lambda T}^{-1}(M_T - b_{\lambda T}) \le x) = e^{-e^{-x}}, \quad x \in \mathbb{R},$$

where a_T and b_T are such that

$$\lim_{T\to\infty} TP(L_1 > a_T x + b_T) = e^{-x}, \quad x \in \mathbb{R}.$$

Example: Positive shot noise process

Let L be a positive compound Poisson process

$$L_t = \sum_{j=1}^{N_t} \xi_j,$$

where $(N_t)_{t\ge 0}$ is a Poisson process on $\mathbb{R}_+$ with intensity $\mu > 0$ and jump times $(\Gamma_k)_{k\in\mathbb{N}}$. The process N is independent of the i.i.d. sequence of positive r.v.s $(\xi_k)_{k\in\mathbb{N}}$ with d.f. F. The resulting volatility process is then the positive shot noise process

$$V_t = e^{-\lambda t}V_0 + \int_0^t e^{-\lambda(t-s)}dL_{\lambda s}$$
$$= e^{-\lambda t}V_0 + \sum_{i=1}^{N_{\lambda t}} e^{-\lambda t+\Gamma_j}\xi_j.$$

Running maxima theorem

(a) Let V be a stationary version of the OU process, where L is a positive, compound Poisson process. Assume $L_1 \in \mathcal{S}(\gamma)$, $\gamma > 0$. Then

$$\lim_{T\to\infty} P(a_{\lambda T}^{-1}(M_T - b_{\lambda T}) \le x) = e^{-e^{-x}}, \quad x \in \mathbb{R},$$

where $a_T > 0$ and $b_T \in \mathbb{R}$ are such that

$$\lim_{T\to\infty} TP(L_1 > a_T x + b_T) = \frac{Ee^{\gamma L_1}}{Ee^{\gamma V_0}} e^{-x}, \quad x \in \mathbb{R}.$$

(b) Assume that V is a $\Gamma(\mu, \gamma)$-OU process. Then

$$\lim_{T\to\infty} P(a_{\lambda T}^{-1}(M_T - b_{\lambda T}) \le x) = e^{-e^{-x}}, \quad x \in \mathbb{R},$$

where $a_T > 0$ and $b_T \in \mathbb{R}$ are the norming constants of a $\Gamma(\mu + 1, \gamma)$ distributed r.v. Y such that

$$\lim_{T\to\infty} TP(Y > a_T x + b_T) = \mu^{-1}e^{-x}, \quad x \in \mathbb{R}.$$

5. Long Memory Extremes

A stationary process with correlation function ρ exhibits long range dependence, if there exists a $H \in (0, 1/2)$ and l is a slowly varying function such that

$$\rho(h) \sim l(h)h^{-2H}, \quad h \to \infty.$$

Long range dependence implies that

$$\int_0^\infty \rho(h)dh = \infty.$$

5.1. Superposition of Ornstein-Uhlenbeck (supOU) processes

Barndorff-Neilsen and Shephard proposed supOU processes as volatility models. Empirical volatility has long memory in the sense that the empirical autocorrelation function decreases slower than exponential. The class of supOU processes can capture extremal clusters and long range dependence.

$$V_t = \int_{\mathbb{R}_+ \times \mathbb{R}} e^{-r(t-s)} I_{[0,\infty)}(t-s)\, d\Lambda(r, \lambda s), \quad t \geq 0,$$

where $\lambda > 0$ and Λ is an *infinitely divisible independently scattered random measure* (i.d.i.s.r.m.) which are extensions of OU type processes of the form

$$V_t = \int_{-\infty}^{t} e^{-\lambda(t-s)} dL_{\lambda s},$$

where $\lambda > 0$ and L is a Levy process. The time change by λ yields marginal distributions independent of λ. To guarantee that the volatility process V is positive, the Levy process L is chosen as a subordinator. The resulting price process has martingale term $dS_t = \sqrt{V_t}\, dB_t$, where B is a Brownian motion independent of L.

Let π be a probability measure on $\mathbb{R}_+$ with $\lambda^{-1} := \int_{\mathbb{R}_+} r^{-1} \pi(dr) < \infty$.

The *generating quadruple* (m, σ^2, ν, π) determines completely the distribution of Λ. The underlying driving Levy process

$$L_t = \Lambda(\mathbb{R}_+ \times [0, t])$$

has *generating triplet* (m, σ^2, ν).

Define the probability measure $\overline{\pi}(dr) := \lambda / r \pi(dr)$ and the idisrm $\overline{\Lambda}$ with generating quadruple $(m/\lambda, \sigma^2/\lambda, \nu\lambda, \overline{\pi})$.

Thus $\overline{\pi}$ is a probability measure on $\mathbb{R}_+$ with $\lambda := \int_{\mathbb{R}_+} r\overline{\pi}(dr)$. The distribution π governs the long range dependence of the model. Essentially the measure π needs sufficient mass near 0. We write $\pi(r) := \pi((0, r])$.

Then

$$X_t = \int_{-\infty}^{\infty} e^{-rt} \int_{-\infty}^{rt} e^s d\overline{\Lambda}(r, s).$$

Then $X = V$ a.s.

$$dX_t = \int_{\mathbb{R}_+} \{-rX(t, dr)dt + d\overline{\Lambda}(t, r)\},$$

where

$$X(t, B) = \int_B e^{-rt} \int_{-\infty}^{rt} e^s d\overline{\Lambda}(r, \lambda s).$$

Example 5.1. Let π be gamma distribution with density

$$\pi(dr) = \Gamma(2H + 1)^{-1} r^{2H} e^{-r} dr$$

for $r > 0$ and $H > 0$. Then $\lambda = 2H$ and

$$\rho(h) = \Gamma(2H)^{-1} \int_0^{\infty} r^{2H-1} e^{-r(h+1)} dr = (h+1)^{-2H}, \quad h \geq 0.$$

The following theorem shows how long range dependence can be introduced in the supOU models.

Theorem 5.1. *Suppose l is slowly varying*. *Then*

$$\overline{\pi}(r) \sim (2H)^{-1} l(r^{-1}) r^{2H}, \quad r \to 0$$

if and only if

$$\rho(h) \sim \Gamma(2H) l(h) h^{-2H}, \quad h \to \infty.$$

Theorem 5.2. *Define* $M_T := \sup_{0 \leq t \leq T} V_t$.

(a) *Let* $L_1 \in \mathcal{R}_{-\alpha}$ *with norming constants* $a_T > 0$ *such that*

$$\lim_{T \to \infty} TP(L_1 > a_T x) = x^{-\alpha}, \quad x > 0.$$

Then

$$\lim_{T \to \infty} P(a_{\lambda T}^{-1} M_T \leq x) = e^{-x^{-\alpha}}, \quad x > 0.$$

(b) *Let* $L_1 \in \mathcal{S}(\gamma) \cap MDA(\Lambda)$ *with norming constants* $a_T > 0$ *and* $b_T \in \mathbb{R}$ *such that*

$$\lim_{T \to \infty} TP(L_1 > a_T x + b_T) = e^{-x}, \quad x \in \mathbb{R}.$$

Then

$$\lim_{T \to \infty} P(a_{\lambda T}^{-1}(M_T - b_{\lambda T}) \le x) = e^{-[Ee^{\gamma L_1}]^{-1} Ee^{\gamma V_0} e^{-x}}, \quad x \in \mathbb{R}.$$

Typical examples of d.f.s in $\mathcal{S}(\gamma) \cap MDA(\Lambda)$ are GIG, NIG, GH, CGMY. All these distributions are self-decomposable, which means that they are possible stationary distributions of OU-type processes and hence also supOU processes. For proofs of theorems see [1-4].

Remark. More general stock price models should have both stochastic interest rate and stochastic volatility with jumps and long-memory driving terms. However, analyses of three factor models are difficult.

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