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JUSTIFICATION OF A TWO-DIMENSIONAL LINEAR SHELL MODEL OF KOITER'S TYPE

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Abstract

In this paper, motivated by Ciarlet's research work on the two-dimensional nonlinear shell model of Koiter's type, we propose a new two-dimensional linear shell model of Koiter's type, in which the flexural part is different from the classical two-dimensional linear Koiter's shell model. As the thickness of the shell goes to zero, using the corresponding Korn's inequality, we obtain a justification of the two-dimensional linear shell model of Koiter's type by the method of the asymptotic analysis.

1. Introduction

There are various methods to study the elastic shell, one of the known methods is the method of the asymptotic analysis: Under the mechanical and geometrical assumptions with the shell thickness as "small" parameter, how to

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infer a rational two-dimensional shell model from the three-dimensional shell model and to mathematically justify the obtained two-dimensional shell model in a rational way (for example, Koiter's shell model)? In this paper, we use the method of the asymptotic analysis to justify modified Koiter's shell model.

In [3], Ciarlet and Lods considered a family of linearly elastic shells with thickness 2ε, clamped along their entire lateral face, the family of linearly elastic shells have the same middle surface S which is "uniformly elliptic" in the sense that the two principal radii of curvature of S are either both positive or both negative at all points of S. Let the field $\vec{u}(\varepsilon) = (u_i(\varepsilon))$, $u_i(\varepsilon)(i = 1, 2, 3)$ denote the three covariant components of the displacement of the points of the shell given by the equations of three-dimensional elasticity. It has been shown in [3] that, if the applied body force density is $O(1)$ with respect to ε, the field $\vec{u}(\varepsilon) = (u_i(\varepsilon))$ converges to a limit $\vec{u}$ as $\varepsilon \to 0$, in which $\vec{u}$ is independent of the transverse variable and satisfies the two-dimensional equations of a "membrane shell" (for details, see [3]). In [4], Ciarlet et al. considered a family of linearly elastic shells with thickness 2ε, all having the same middle surface $S = \vec{\varphi}(\overline{\omega}) \subset R^3$, where $\omega \subset R^2$ is a bounded and connected open set with a Lipschitz-continuous boundary, and $\vec{\varphi} \in C^3(\overline{\omega}; R^3)$. The shells are clamped on a portion of their lateral face, whose middle line is $\vec{\varphi}(\gamma_0)$, where γ_0 is any portion of $\partial\omega$ with length $\gamma_0 > 0$. Ciarlet et al. made an essential geometrical assumption on the middle surface S and on the set γ_0, which states that the space of inextensional displacements

$$\begin{gathered} V_F(\omega) = \{\vec{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \gamma_{\alpha\beta}(\vec{\eta}) = 0 \text{ in } \omega\} \end{gathered}$$

contains non-zero functions, where $\gamma_{\alpha\beta}(\vec{\eta})$ are the components of the linearized change of metric tensor for the middle surface S. It showed in [4] that, if the applied body force density is $O(\varepsilon^2)$ with respect to ε, the field $\vec{u}(\varepsilon) = (u_i(\varepsilon))$ converges to a limit $\vec{u}$ as $\varepsilon \to 0$, in which $\vec{u}$ is independent of the transverse variable and satisfies the two-dimensional equations of a "flexural shell" (for details, see [4]). In [5], Ciarlet and Lods considered, as in [3] and [4], a family of linearly elastic shells with thickness 2ε, all having the same middle surface $S = \vec{\varphi}(\overline{\omega}) \subset R^3$, where $\omega \subset R^2$ is a bounded and connected open set with a Lipschitz-continuous boundary, and $\vec{\varphi} \in C^3(\overline{\omega}; R^3)$. The shells are clamped on a portion of their lateral face, whose middle line is $\vec{\varphi}(\gamma_0)$, where γ_0 is a portion of $\partial\omega$ with length $\gamma_0 > 0$. For all $\varepsilon > 0$, let u_i^ε be the covariant components of the displacement $u_i^\varepsilon \vec{g}^{i,\varepsilon}$ of the points of the shell, derived by solving the three-dimensional problem; let ζ_i^ε be the covariant components of the displacement $\zeta_i^\varepsilon \vec{a}^i$ of the points of the middle surface S, derived by solving the two-dimensional linear Koiter's shell model. Making the same assumptions as in [3], Ciarlet and Lods proved that the fields $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_i^\varepsilon \vec{g}^{i,\varepsilon} dx_3^\varepsilon$ and $\zeta_i^\varepsilon \vec{a}^i$ have the same principal part as $\varepsilon \to 0$; with the same assumptions as in [4], they also proved that the same fields again have the same principal part as $\varepsilon \to 0$, thus they verified the two-dimensional linear Koiter's shell model for "membrane" and "flexural" shells (for details, see [5]).

In this paper, motivated by Ciarlet's work on the two-dimensional nonlinear shells model of Koiter's type (see [6]), we propose a new two-dimensional linear shells model of Koiter's type in which the flexural part is different from the classical two-dimensional linear Koiter's shell model. By establishing the corresponding Korn's inequality, we prove that the new two-dimensional linear shell model of Koiter's type is asymptotically equivalent to the classical two-dimensional linear Koiter's shell model when the thickness of the shell goes to zero.

Throughout this paper, we assume that $i, j, k, \cdots$ take their values in the set $\{1, 2, 3\}$; $\alpha, \beta, \sigma, \tau, \cdots$ take their values in the set $\{1, 2\}$; the summation convention with respect to repeated indices and exponents is used. We shall denote by $\vec{a} \cdot \vec{b}$ and $\vec{a} \times \vec{b}$ the scalar and exterior products of $\vec{a}, \vec{b} \in \mathbf{R}^3$, respectively, and denote by $|\vec{a}|$ the Euclidean norm of $\vec{a} \in \mathbf{R}^3$. Let ω be a bounded open connected subset of $\mathbf{R}^2$ with a Lipschitz-continuous boundary γ, the set ω being locally on one side of γ. Let $y = (y_\alpha)$ denote a generic point in $\overline{\omega}$, the closure of ω, and $\partial_\alpha = (\partial/\partial y_\alpha)(\alpha = 1, 2), \partial_{\alpha\beta} = \partial^2/\partial y_\alpha \partial y_\beta$.

Let $\vec{\theta} : \overline{\omega} \to \mathbf{R}^3$ be a C^3 injective mapping such that the two vectors $\vec{a}_\alpha(y) = \partial_\alpha \vec{\theta}(y)$ are linearly independent at any given point $y \in \overline{\omega}$, $\vec{a}_\alpha(y)(\alpha = 1, 2)$ constitute a covariant basis for the tangent plane to the surface $S = \vec{\theta}(\overline{\omega})$ at the point $\vec{\theta}(y)$, and the two vectors $\vec{a}^\alpha(y)(\alpha = 1, 2)$ defined by

$$\vec{a}^\alpha(y) \cdot \vec{a}_\beta(y) = \delta^\alpha_\beta$$

constitute a contravariant basis for the same tangent plane. Let

$$\vec{a}_3(y) = \vec{a}^3(y) = \frac{\vec{a}_1(y) \times \vec{a}_2(y)}{|\vec{a}_1(y) \times \vec{a}_2(y)|}.$$

For the surface S, the metric tensor in covariant or contravariant components $(a_{\alpha\beta})$ or $(a^{\alpha\beta})$, the curvature tensor in covariant or mixed components $(b_{\alpha\beta})$ or (b^β_α), and the Christoffel symbols $\Gamma^\sigma_{\alpha\beta}$, are defined, respectively, by (whenever no confusion arises, we drop the dependence on the variable $y \in \overline{\omega}$)

$$a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta, \quad a^{\alpha\beta} = \vec{a}^\alpha \cdot \vec{a}^\beta, \tag{1.1}$$

$$b_{\alpha\beta} = \vec{a}^3 \cdot \partial_\beta \vec{a}_\alpha, \quad b^\beta_\alpha = a^{\beta\sigma} \cdot b_{\sigma\alpha}, \tag{1.2}$$

and

$$\Gamma^\sigma_{\alpha\beta} = \vec{a}^\sigma \cdot \partial_\beta \vec{a}_\alpha. \tag{1.3}$$

It is easy to see that

$$a_{\alpha\beta} = a_{\beta\alpha}, \quad a^{\alpha\beta} = a^{\beta\alpha},$$

$$b_{\alpha\beta} = b_{\beta\alpha}, \quad \Gamma^\sigma_{\alpha\beta} = \Gamma^\sigma_{\beta\alpha}.$$

The area element of S is $\sqrt{a}dy$, where

$$a = det(a_{\alpha\beta}). \tag{1.4}$$

All the functions defined in (1.1)-(1.4) are at least continuous over the set $\overline{\omega}$.

Then, there exists a constant a_0 such that

$$0 < a_0 \leq a(y), \quad \forall\, y \in \overline{\omega}. \tag{1.5}$$

Let γ_0 denote a measurable subset of the boundary γ of ω with length $\gamma_0 > 0$. For more details about these geometrical preliminaries, we refer the reader to [1, Chapter 2]. Given any differentiable field $\vec{\eta} = (\eta_i) : \overline{\omega} \to \mathbf{R}^3$, define the vectors $\vec{a}_\alpha(\vec{\eta}) = \partial_\alpha(\vec{\theta} + \eta_i \vec{a}^i)$ and the functions $a_{\alpha\beta}(\vec{\eta}) = \vec{a}_\alpha(\vec{\eta}) \cdot \vec{a}_\beta(\vec{\eta})$.

If the two vectors $\vec{a}_\alpha(\vec{\eta})(y)$ are linearly independent at each point $y \in \overline{\omega}$, they thus constitute the covariant basis of the tangent plane to the deformed surface $(\vec{\theta} + \eta_i \vec{a}^i)(\overline{\omega})$ at the point $(\vec{\theta} + \eta_i \vec{a}^i)(y),\ y \in \overline{\omega}$, of this deformed surface, and the functions $a_{\alpha\beta}(\vec{\eta})$ are the covariant components of the metric tensor of the same deformed surface. If the two vectors $\vec{a}_\alpha(\vec{\eta})$ are linearly independent in $\overline{\omega}$, the vector

$$\vec{a}_3(\vec{\eta}) = \frac{\vec{a}_1(\vec{\eta}) \times \vec{a}_2(\vec{\eta})}{|\vec{a}_1(\vec{\eta}) \times \vec{a}_2(\vec{\eta})|}$$

is then normal to the deformed surface $(\vec{\theta} + \eta_i \vec{a}^i)(\overline{\omega})$ at the point $(\vec{\theta} + \eta_i \vec{a}^i)(y),\ y \in \overline{\omega}$, of this deformed surface, and the functions $b_{\alpha\beta}(\vec{\eta}) = \vec{a}_3(\vec{\eta}) \cdot \partial_\alpha \vec{a}_\beta(\vec{\eta})$ are the covariant components of the curvature tensor of the same deformed surface.

Assume that we are given a family of shells, each having the same middle surface $S = \vec{\theta}(\overline{\omega})$, whose thickness $2\varepsilon > 0$ is arbitrarily small. Then, for each $\varepsilon > 0$, the reference configuration of the shell is the set $\vec{\Theta}(\overline{\Omega}^\varepsilon)$, where $\overline{\Omega}^\varepsilon = \omega \times [-\varepsilon,\ \varepsilon]$,

$$\vec{\Theta}(y, x_3^\varepsilon) = \vec{\theta}(y) + x_3^\varepsilon \vec{a}_3(y), \tag{1.6}$$

for all $(y, x_3^\varepsilon) \in \overline{\omega} \times [-\varepsilon,\ \varepsilon]$. Let $x^\varepsilon = (x_i^\varepsilon)$ denote a generic point in the set $\overline{\Omega}^\varepsilon$ (note that $x_\alpha^\varepsilon = y_\alpha$), and let $\partial_i^\varepsilon = \partial/\partial x_i^\varepsilon$. Then for $\varepsilon > 0$ small enough, the mapping $\vec{\Theta} : \overline{\Omega}^\varepsilon \to \mathbf{R^3}$ is injective and the three vectors

$$\vec{g}_i^\varepsilon(x^\varepsilon) = \partial_i^\varepsilon \vec{\Theta}(x^\varepsilon)$$

are linearly independent (see Theorem 3.1-1 in [1]). They then form the covariant basis at the point $\vec{\Theta}(x^{\varepsilon}) \in \Theta(\overline{\Omega}^{\varepsilon})$, while the three vectors $\vec{g}^{i,\varepsilon}(x^{\varepsilon})$ defined by the relations

$$\vec{g}^{i,\varepsilon}(x^{\varepsilon}) \cdot \vec{g}^{\varepsilon}_j(x^{\varepsilon}) = \delta^i_j$$

form the contravariant basis at the same point $\vec{\Theta}(x^{\varepsilon})$. Let γ_0 be a subset of $\gamma = \partial\omega$ such that length $\gamma_0 > 0$. Each shell is then subjected to a boundary condition of place along the portion $\vec{\Theta}(\gamma_0 \times [-\varepsilon,\ \varepsilon])$ of its lateral face $\vec{\Theta}(\gamma \times [-\varepsilon,\ \varepsilon])$; this means that its displacement field vanishes on the set $\vec{\Theta}(\gamma_0 \times [-\varepsilon,\ \varepsilon])$. Each shell is subjected to applied body forces in its interior $\vec{\Theta}(\overline{\Omega}^{\varepsilon})$, and to applied surface forces on its "upper" and "lower" faces $\vec{\Theta}(\Gamma^{\varepsilon}_+)$ and $\vec{\Theta}(\Gamma^{\varepsilon}_-)$, where $\Gamma^{\varepsilon}_+ = \omega \times \{+\varepsilon\}$ and $\Gamma^{\varepsilon}_- = \omega \times \{-\varepsilon\}$. There forces are given by their contravariant components $f^{i,\varepsilon} \in L^2(\Omega^{\varepsilon})$ and $h^{i,\varepsilon} \in L^2(\Gamma^{\varepsilon}_+ \bigcup \Gamma^{\varepsilon}_-)$. Finally, the elastic material constituting the shell is assumed to be a St. Venant-Kirchhoff material, the same for each shell. Hence the material is characterized by its two Lame constants $\lambda > 0$ and $\mu > 0$, which are independent of ε. For each $\varepsilon > 0$, the unknown is the vector field $\vec{u}^{\varepsilon} = (u^{\varepsilon}_i) : \overline{\Omega}^{\varepsilon} \to \mathbf{R}^3$, where the functions $u^{\varepsilon}_i : \overline{\Omega}^{\varepsilon} \to \mathbf{R}$ denote the covariant components of the displacement field of the shell. This means that, for each $x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}$, the displacement vector of the point $\vec{\Theta}(x^{\varepsilon})$ is $u^{\varepsilon}_i(x^{\varepsilon})\vec{g}^{i,\varepsilon}(x^{\varepsilon})$. A two-dimensional nonlinear shell model, based on a priori assumptions of geometrical and mechanical natures, has been proposed by Koiter for modelling nonlinearly elastic shells. In the model, the unknown $\vec{\xi}^{\varepsilon}_k = (\xi^{\varepsilon}_{i,k}) : \overline{\omega} \to \mathbf{R}^3$, whose components $\xi^{\varepsilon}_{i,k} : \overline{\omega} \to \mathbf{R}^3$ are covariant components of the displacement field of the surface S, should be a stationary point of the energy j^{ε}_k defined by

$$\begin{aligned} j^{\varepsilon}_k(\vec{\eta}) = &\tfrac{\varepsilon}{2} \int_\omega a^{\alpha\beta\sigma\tau} G_{\alpha\beta}(\vec{\eta}) G_{\sigma\tau}(\vec{\eta}) \sqrt{a} dy \\ &+ \tfrac{\varepsilon^3}{6} \int_\omega a^{\alpha\beta\sigma\tau} R_{\alpha\beta}(\vec{\eta}) R_{\sigma\tau}(\vec{\eta}) \sqrt{a} dy - \int_\omega p^{i,\varepsilon}\eta_i \sqrt{a} dy, \end{aligned} \tag{1.7}$$

where

$$G_{\alpha\beta}(\vec{\eta}) = \frac{1}{2}(a_{\alpha\beta}(\vec{\eta}) - a_{\alpha\beta}),$$

$$R_{\alpha\beta}(\vec{\eta}) = b_{\alpha\beta}(\vec{\eta}) - b_{\alpha\beta},$$

respectively, denote the covariant components of the change of metric tensor and the change of curvature tensor associated with a displacement field $\eta_i\vec{a}^i$ of the surface S, and

$$p^{i,\varepsilon} = \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon} dx_3^\varepsilon + h_+^{i,\varepsilon} + h_-^{i,\varepsilon}, \tag{1.8}$$

where $h_+^{i,\varepsilon} = h^{i,\varepsilon}(\cdot, +\varepsilon), h_-^{i,\varepsilon} = h^{i,\varepsilon}(\cdot, -\varepsilon)$. But the functions $b_{\alpha\beta}(\vec{\eta})$, and thus the functions $R_{\alpha\beta}(\vec{\eta})$, are not defined at those points of $\overline{\omega}$, where the two vectors $\vec{a}_\alpha(\vec{\eta})$ are linearly dependent, for the denominator $|\vec{a}_1(\vec{\eta}) \times \vec{a}_2(\vec{\eta})| = \sqrt{a(\vec{\eta})}$, where $a(\vec{\eta}) = det(a_{\alpha\beta}(\vec{\eta}))$, vanishes at those points. In order to circumvent this difficulty, Ciarlet proposed to replace the components $R_{\alpha\beta}(\vec{\eta})$ of the "exact" change of curvature tensor by the functions

$$R^c_{\alpha\beta}(\vec{\eta}) = \frac{1}{\sqrt{a}}\partial_{\alpha\beta}(\vec{\theta} + \eta_i\vec{a}^i)[\vec{a}_1(\vec{\eta}) \times \vec{a}_2(\vec{\eta})] - b_{\alpha\beta}. \tag{1.9}$$

After this replacement, the unknown vector field $\vec{\xi}^\varepsilon = (\xi_i^\varepsilon) : \overline{\omega} \to \mathbf{R}^3$, where the functions $\xi_i^\varepsilon : \overline{\omega} \to \mathbf{R}$ are again the covariant components of the displacement field $\xi_i^\varepsilon\vec{a}^i$ of the surface S, should be a stationary point of the energy j^ε defined by

$$\begin{aligned} j^\varepsilon(\vec{\eta}) = {} & \tfrac{\varepsilon}{2}\int_\omega a^{\alpha\beta\sigma\tau}G_{\alpha\beta}(\vec{\eta})G_{\sigma\tau}(\vec{\eta})\sqrt{a}dy \\ & +\tfrac{\varepsilon^3}{6}\int_\omega a^{\alpha\beta\sigma\tau}R^c_{\alpha\beta}(\vec{\eta})R^c_{\sigma\tau}(\vec{\eta})\sqrt{a}dy - \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}dy. \end{aligned} \tag{1.10}$$

In other words, the unknown $\vec{\xi}^\varepsilon = (\xi_i^\varepsilon)$ satisfies the following variational problem $p^\varepsilon(\omega)$:

$$\begin{aligned} & \vec{\xi}^\varepsilon \in w(\omega) = \{\vec{\eta} = (\eta_i) \in w^{2,p}(\omega);\ \vec{\eta} = \partial_\nu\vec{\eta} = 0 \text{ on } \gamma_0\}(p > 2), \\ & \varepsilon\int_\omega a^{\alpha\beta\sigma\tau}G_{\sigma\tau}(\vec{\xi}^\varepsilon)(G'_{\alpha\beta}(\vec{\xi}^\varepsilon)\vec{\eta})\sqrt{a}dy + \tfrac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}R^c_{\sigma\tau}(\vec{\xi}^\varepsilon)((R^c_{\alpha\beta})'(\vec{\xi}^\varepsilon)\vec{\eta})\sqrt{a}dy \\ & = \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}dy \end{aligned} \tag{1.11}$$

for all $\vec{\eta} = (\eta_i) \in w(\omega)$, where

$$G_{\alpha\beta}(\vec{\eta}) = \frac{1}{2}(a_{\alpha\beta}(\vec{\eta}) - a_{\alpha\beta}) = \frac{1}{2}(\eta_{\alpha\|\beta} + \eta_{\beta\|\alpha} + a^{mn}\eta_{m\|\alpha}\eta_{n\|\beta}),$$

$$G'_{\alpha\beta}(\vec{\xi})\vec{\eta} = \frac{1}{2}(\eta_{\alpha\|\beta} + \eta_{\beta\|\alpha} + a^{mn}[\xi_{m\|\alpha}\eta_{n\|\beta} + \xi_{n\|\beta}\eta_{m\|\alpha}],$$

$$(R^c_{\alpha\beta})'(\vec{\xi})\vec{\eta} = \frac{1}{\sqrt{a}}\partial_{\alpha\beta}(\vec{\theta} + \xi_i\vec{a}^i)\cdot[\vec{a}_1(\vec{\xi})\times\partial_2(\eta_j\vec{a}^j)] \\ +\partial_1(\eta_k\vec{a}^k)\times\vec{a}_2(\vec{\xi})] + \frac{1}{\sqrt{a}}\partial_{\alpha\beta}(\eta_i\vec{a}^i)\cdot[\vec{a}_1(\vec{\xi})\times\vec{a}_2(\vec{\xi})],$$

for arbitrary fields $\vec{\xi} = (\xi_i)$ and $\vec{\eta} = (\eta_i)$ in the space $w^{2,p}(\omega)(p > 2)$. For simplicity, we consider the case when surface force vanishes, i.e.,

$$p^{i,\varepsilon} = \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon} dx_3^{\varepsilon}.$$

Remark 1.1. Compared with functions $R_{\alpha\beta}(\vec{\eta})$, by formal linearization with respect to $\vec{\eta}$ the functions $R^c_{\alpha\beta}(\vec{\eta})$ do not reduce to the covariant components $\rho_{\alpha\beta}(\vec{\eta})$ of the linearized change of curvature tensor of the middle surface for the shell. As is easily checked by direct computation they satisfy

$$\rho^{cl}_{\alpha\beta}(\vec{\eta}) = [R^c_{\alpha\beta}(\vec{\eta})]^{lin} = \rho_{\alpha\beta}(\vec{\eta}) + b_{\alpha\beta}a^{\sigma\tau}\gamma_{\sigma\tau}(\vec{\eta}),$$

where the functions $\gamma_{\sigma\tau}(\vec{\eta})$ are the covariant component of the linearized change of metric tensor, $\rho_{\sigma\tau}(\vec{\eta})$ are the covariant components of the linearized change of curvature tensor.

The linearized model equations of (1.11) are

$$\vec{\xi}^{\varepsilon} \in w(\omega) = \{\vec{\eta} = (\eta_i) \in w^{2,p}(\omega);\ \vec{\eta} = \partial_\nu\vec{\eta} = 0 \text{ on } \gamma_0\}(p > 2)$$

such that

$$\varepsilon\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{\xi}^{\varepsilon})\gamma_{\alpha\beta}(\vec{\eta})\sqrt{a}dy + \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho^{cl}_{\sigma\tau}(\vec{\xi}^{\varepsilon})\rho^{cl}_{\alpha\beta}(\vec{\eta})\sqrt{a}dy \\ = \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}dy$$

for all $\vec{\eta} = (\eta_i) \in w(\omega)$. The functions $\rho^{cl}_{\alpha\beta}(\vec{\eta})$ represent the modified linear change of curvature tensor of the middle surface for the shell. Replacing $\rho_{\alpha\beta}(\vec{\eta})$ by $\rho^{cl}_{\alpha\beta}(\vec{\eta})$ in the linear Koiter's shell model, we get a new two-dimensional linear shell model of Koiter's type, which is the linear counterpart of the famed linear Koiter's shell model (porposed by Koiter in 1970, the reader is referred to [8]),

$$\vec{\xi}^{\varepsilon} = (\xi^{\varepsilon}_i) \in V_k(\omega) = \{\vec{\eta} = (\eta_i) \in H^1(\omega)\times H^1(\omega)\times H^2(\omega);\ \eta_i = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0\}$$

such that

$$\varepsilon \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\vec{\xi}^\varepsilon)\gamma_{\alpha\beta}(\vec{\eta})\sqrt{a}dy + \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho^{cl}_{\sigma\tau}(\vec{\xi}^\varepsilon)\rho^{cl}_{\alpha\beta}(\vec{\eta})\sqrt{a}dy$$
$$= \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}dy,$$

for all $\vec{\eta} = (\eta_i) \in V_k(\omega)$, where $a^{\alpha\beta\sigma\tau}$ are the contravariant components of the two-dimensional elasticity tensor of the middle surface,

$$p^{i,\varepsilon} = \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}dx_3^\varepsilon, \tag{1.12}$$

$$\gamma_{\alpha\beta}(\vec{\eta}) = \frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3, \tag{1.13}$$

$$\begin{aligned}\rho_{\alpha\beta}(\vec{\eta}) = \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 - b^\sigma_\alpha b_{\sigma\beta}\eta_3 + b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau)\\ +b^\tau_\beta(\partial_\alpha\eta_\tau - \Gamma^\sigma_{\alpha\tau}\eta_\sigma) + (\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma)\eta_\tau,\end{aligned} \tag{1.14}$$

$$\rho^{cl}_{\alpha\beta}(\vec{\eta}) = \rho_{\alpha\beta}(\vec{\eta}) + b_{\alpha\beta}a^{\sigma\tau}\gamma_{\sigma\tau}(\vec{\eta}). \tag{1.15}$$

2. Korn's Inequality

Similar to Theorem 2.1 in [2] (also see [1]), we have the following Theorem 2.1 (its proof is just a modification of Theorem 2.1 in [2]).

Theorem 2.1. (Inequality of Korn's type without Boundary conditions): *Let ω be a domain in $\mathbf{R}^2$ and $\vec{\theta} \in C^3(\overline{\omega}; \mathbf{R}^3)$ be an injective mapping such that the two vectors $\vec{a}_\alpha = \partial_\alpha\vec{\theta}$ are linearly independent at all points of $\overline{\omega}$.*

Given $\vec{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$, let $\gamma_{\alpha\beta}(\vec{\eta}) \in L^2(\omega), \rho_{\alpha\beta}(\vec{\eta}) \in L^2(\omega)$ denote the covariant components of the linearized change of metric tensors and linearized change of curvature tensors associated with the displacement fields $\eta_i\vec{a}^i$ (from $\gamma_{\alpha\beta}(\vec{\eta}), \rho_{\alpha\beta}(\vec{\eta}) \in L^2(\omega)$ it concludes that $\rho^{cl}_{\alpha\beta}(\vec{\eta}) \in L^2(\omega)$).

Then, there exists a positive constant $c_0 = c_0(\omega, \vec{\theta})$ *such that*

$$\left\{ \sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega} \right\}^{1/2}$$
$$\leq c_0 \left\{ \sum_\alpha \|\eta_\alpha\|^2_{0,\omega} + \|\eta_3\|^2_{1,\omega} + \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\vec{\eta})|^2_{0,\omega} + \sum_{\alpha,\beta} |\rho^{cl}_{\alpha,\beta}(\vec{\eta})|^2_{0,\omega} \right\}^{1/2}$$

for all $\vec{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$.

Since $\rho^{cl}_{\alpha\beta}(\vec{\eta}) = \rho_{\alpha\beta}(\vec{\eta}) + b_{\alpha\beta} a^{\sigma\tau} \gamma_{\sigma\tau}(\vec{\eta})$, it is easy to see that

$$\gamma_{\alpha\beta}(\vec{\eta}) = \rho^{cl}_{\alpha\beta}(\vec{\eta}) = 0 \iff \gamma_{\alpha\beta}(\vec{\eta}) = \rho_{\alpha\beta}(\vec{\eta}) = 0,$$

for all $\vec{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$.

Similar to the proof of Theorem 2.3 in [2], we also get the following theorem.

Theorem 2.2. (Infinitesimal Rigid Displacement Lemma): *Let the mapping* $\vec{\theta} : \overline{\omega} \to \mathbf{R}^3$ *be assumed as in Theorem 2.1. Then, we have the following:*

(a) *Let* $\vec{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ *be such that*

$$\gamma_{\alpha\beta}(\vec{\eta}) = \rho^{cl}_{\alpha\beta}(\vec{\eta}) = 0 \text{ in } \omega.$$

Then, the vector field $\eta_i \vec{a}^i : \overline{\omega} \to \mathbf{R}^3$ *is an infinitesimal rigid displacement, in the sense that there exist two vectors* $\vec{c}, \vec{d} \in \mathbf{R}^3$ *such that*

$$\eta_i(y)\vec{a}^i(y) = \vec{c} + \vec{d} \times \vec{\theta}(y) \text{ for all } y \in \overline{\omega}.$$

(b) *Let* γ_0 *be a* $d\gamma$*-measurable subset of* $\gamma = \partial\omega$ *that satisfies length* $\gamma_0 > 0$, *and let* ∂_ν *be the outer normal derivative operator along* γ.

Then,

$$\left.\begin{array}{l} \vec{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega), \\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \\ \gamma_{\alpha\beta}(\vec{\eta}) = \rho^{cl}_{\alpha\beta}(\vec{\eta}) = 0 \text{ in } \omega. \end{array}\right\} \Longrightarrow \vec{\eta} = \vec{O} \text{ in } \omega.$$

Based on Theorem 2.1 and Theorem 2.2, we can conclude the following Theorem 2.3 (its proof is similar to Theorem 2.4 in [2]).

Theorem 2.3. (Inequality of Korn's type with Boundary conditions)*: Let the mapping* $\vec{\theta}: \overline{\omega} \to \mathbf{R}^3$ *be assumed as in Theorem 2.1, and let* γ_0 *be a* $d\gamma$*-measurable subset of* $\gamma = \partial\omega$ *that satisfies length* $\gamma_0 > 0$*, the space* $V_k(\omega)$ *be defined by*

$$V_k(\omega) = \{\vec{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\}.$$

Then, there exists a positive constant $c_1 = c_1(\omega, \gamma_0, \vec{\theta})$ *such that*

$$\begin{aligned}&\{\sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{2,\omega}^2\}^{1/2}\\ &\leq c_1\{\sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\vec{\eta})|_{0,\omega}^2 + \sum_{\alpha,\beta} |\rho_{\alpha\beta}^{cl}(\vec{\eta})|_{0,\omega}^2\}^{1/2}\end{aligned}$$

for all $\vec{\eta} \in V_k(\omega)$*.*

3. Asymptotic Analysis

Based on Theorem 2.3, we conclude the following existence and uniqueness theorem from the Lax-Milgram theorem.

Theorem 3.1. *Suppose that the mapping* $\vec{\theta}: \overline{\omega} \to \mathbf{R}^3$ *is given as in Theorem 2.1, let* γ_0 *be a* $d\gamma$*-measurable subset of* $\gamma = \partial\omega$ *that satisfies length* $\gamma_0 > 0$*, and let the space* $V_k(\omega)$ *be defined by*

$$V_k(\omega) = \{\vec{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\}.$$

Then, there exists a unique solution to the problem

$$\vec{\xi}^\varepsilon = (\xi_i^\varepsilon) \in V_k(\omega) = \{\vec{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\}$$

such that

$$\begin{aligned}&\varepsilon \int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\vec{\xi}^\varepsilon) \gamma_{\alpha\beta}(\vec{\eta}) \sqrt{a} dy + \frac{\varepsilon^3}{3} \int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}^{cl}(\vec{\xi}^\varepsilon) \rho_{\alpha\beta}^{cl}(\vec{\eta}) \sqrt{a} dy\\ &= \int_\omega p^{i,\varepsilon} \eta_i \sqrt{a} dy \ \text{ for all } \vec{\eta} = (\eta_i) \in V_k(\omega).\end{aligned} \tag{3.1}$$

In what follows we will discuss the asymptotic analysis of problem (3.1) as $\varepsilon \to 0$. First, we discuss the relationship between the linear membrane shell model and the linear shell model (3.1) of Koiter's type.

Theorem 3.2. *Assume that $\gamma = \gamma_0$ and that there exists a positive constant c such that*

$$\left\{ \sum_{\alpha} \|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{0,\omega}^2 \right\}^{1/2} \leq c \left\{ \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\vec{\eta})|_{0,\omega}^2 \right\}^{1/2} \tag{3.2}$$

for all $\vec{\eta} = (\eta_i) \in V_M(\omega)$, where the space $V_M(\omega)$ is defined by

$$V_M(\omega) = \{\vec{\eta} = (\eta_i);\ \eta_\alpha \in H_0^1(\omega),\ \eta_3 \in L^2(\omega)\} = H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega), \tag{3.3}$$

and assume that there exist functions $f^i \in L^2(\Omega)$ independent of ε such that

$$f^{i,\varepsilon}(x^\varepsilon) = f^i(x)\ for\ all\ x \in \Omega = \omega \times (-1, 1). \tag{3.4}$$

Let $\vec{\xi} \in V_M(\omega)$ denote the unique solution of the two-dimensional membrane shell equation:

$$\int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\vec{\xi}) \gamma_{\alpha\beta}(\vec{\eta}) \sqrt{a} dy = \int_\omega \{ \int_{-1}^1 f^i(x) dx_3 \} \eta_i \sqrt{a} dy \tag{3.5}$$

for all $\vec{\eta} = (\eta_i) \in V_M(\omega)$.

Finally, let $\vec{\xi}^\varepsilon = (\xi_i^\varepsilon) \in V_K(\omega)$ denote the solution of the two-dimensional shell equations (3.1) of Koiter's type. Then,

$$\xi_\alpha^\varepsilon \to \xi_\alpha\ in\ H^1(\omega) (\alpha = 1, 2),\ \xi_3^\varepsilon \to \xi_3\ in\ L^2(\omega) \tag{3.6}$$

as $\varepsilon \to 0$.

Remark 3.1. In the case when the middle surface of the elastic shell is elliptic, if the boundary γ of ω is of class C^4 and $\vec{\theta} \in C^5(\overline{\omega}; \mathbf{R}^3)$, then inequality (3.2) is satisfied.

Proof. There exists a positive constant c_0 such that (the reader is referred to Lemma 2.1 of Bernadou et al. in [7])

$$\sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \le c_0 a^{\alpha\beta\sigma\tau}(y) t_{\sigma\tau} t_{\alpha\beta} \tag{3.7}$$

for all $y \in \overline{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$. That equation (3.5) has a unique solution $\vec{\xi} \in V_M(\omega)$ is then a consequence of (3.7) combined with assumption (3.2).

For brevity, denote

$$B_M(\vec{\xi}, \vec{\eta}) = \int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\vec{\xi}) \gamma_{\alpha\beta}(\vec{\eta}) \sqrt{a} dy, \tag{3.8}$$

$$B_F^{cl}(\vec{\xi}, \vec{\eta}) = \frac{1}{3} \int_\omega a^{\alpha\beta\sigma\tau} \rho^{cl}_{\sigma\tau}(\vec{\xi}) \rho^{cl}_{\alpha\beta}(\vec{\eta}) \sqrt{a} dy, \tag{3.9}$$

$$L(\vec{\eta}) = \int_\omega \{ \int_{-1}^{1} f^i(x) dx_3 \} \eta_i \sqrt{a} dy, \tag{3.10}$$

$$\|L\| = \{ \sum_i \| \int_{-1}^{1} f^i dx_3 \|^2_{0,\omega} \}^{1/2}, \tag{3.11}$$

$$\|\vec{\eta}\|_{V_M(\omega)} = \left\{ \sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{0,\omega} \right\}^{1/2}. \tag{3.12}$$

Note that the space $V_k(\omega)$ is contained in the space $V_M(\omega)$. By assumption (3.4) (note that $x_1^\varepsilon = x_1,\ x_2^\varepsilon = x_2,\ x_3^\varepsilon = \varepsilon x_3,\ y = (x_1, x_2)$), the solution $\vec{\xi}^\varepsilon$ of (3.1) also satisfies

$$B_M(\vec{\xi}^\varepsilon, \vec{\eta}) + \varepsilon^2 B_F^{cl}(\vec{\xi}^\varepsilon, \vec{\eta}) = L(\vec{\eta}) \text{ for all } \vec{\eta} \in V_k(\omega). \tag{3.13}$$

Hence, taking $\vec{\eta} = \vec{\xi}^\varepsilon$ in (3.13) concludes

$$\frac{1}{c^2} \|\vec{\xi}^\varepsilon\|^2_{V_M(\omega)} + \frac{1}{3} \sum_{\alpha,\beta} \|\varepsilon \rho^{cl}_{\alpha\beta}(\vec{\xi}^\varepsilon)\|^2_{0,\omega} \le c_0 \|L\| \|\vec{\xi}^\varepsilon\|_{V_M(\omega)}.$$

From the above inequality, we can conclude that the family $(\vec{\xi}^\varepsilon)_{\varepsilon>0}$ is bounded in $V_M(\omega)$ and the families $(\varepsilon\rho^{cl}_{\alpha\beta}(\vec{\xi}^\varepsilon))_{\varepsilon>0}$ are bounded in $L^2(\omega)$; there exists a subsequence (again by $(\vec{\xi}^\varepsilon)_{\varepsilon>0}$ for convenience) and there exist functions $\widetilde{\vec{\xi}} \in V_M(\omega)$ and $\rho^{-1}_{\alpha\beta} \in L^2(\omega)$ such that

$$\vec{\xi}^\varepsilon \rightharpoonup \widetilde{\vec{\xi}} \text{ in } V_M(\omega), \tag{3.14}$$

$$\varepsilon\rho^{cl}_{\alpha\beta}(\vec{\xi}^\varepsilon) \rightharpoonup \rho^{-1}_{\alpha\beta} \text{ in } L^2(\omega). \tag{3.15}$$

($\rightarrow$ and $\rightharpoonup$ denote strong and weak convergences, respectively.)

Fix $\vec{\eta} \in V_K(\omega)$ in (3.13) and let $\varepsilon \to 0$; from the weak convergence (3.14) and (3.15), we get $B_M(\widetilde{\vec{\xi}}, \vec{\eta}) = L(\vec{\eta})$.

Because the space $V_K(\omega)$ is dense in $V_M(\omega)$, we conclude that $B_M(\widetilde{\vec{\xi}}, \vec{\eta}) = L(\vec{\eta})$ for all $\vec{\eta} \in V_M(\omega)$. Therefore,

$$\widetilde{\vec{\xi}} = \vec{\xi},$$

where $\vec{\xi} \in V_M(\omega)$ is the unique solution to the membrane shell equation (3.5).

Because the solution to the membrane shell equation (3.5) is unique, the weak convergence $\vec{\xi}^\varepsilon \rightharpoonup \vec{\xi}$ in $V_M(\omega)$ holds for the whole family $(\vec{\xi}^\varepsilon)_{\varepsilon>0}$.

By inequalities (3.2) and (3.7), the following strong convergence

$$\vec{\xi}^\varepsilon \to \vec{\xi} \quad \text{in} \quad V_M(\omega),$$

is equivalent to the convergence

$$B_M(\vec{\xi}^\varepsilon - \vec{\xi}, \vec{\xi}^\varepsilon - \vec{\xi}) \to 0,$$

which can be established from the following relations,

$$0 \le B_M(\vec{\xi}^\varepsilon - \vec{\xi}, \vec{\xi}^\varepsilon - \vec{\xi}) = B_M(\vec{\xi}^\varepsilon, \vec{\xi}^\varepsilon) - 2B_M(\vec{\xi}^\varepsilon, \vec{\xi}) + B_M(\vec{\xi}, \vec{\xi}),$$

$$B_M(\vec{\xi}^\varepsilon, \vec{\xi}^\varepsilon) \le L(\vec{\xi}^\varepsilon),$$

$$B_M(\vec{\xi}^\varepsilon, \vec{\xi}) \to B_M(\vec{\xi}, \vec{\xi}),\ L(\vec{\xi}^\varepsilon) \to L(\vec{\xi}),$$

$$B_M(\vec{\xi}, \vec{\xi}) = L(\vec{\xi}).$$

The proof is completed.

Theorem 3.3. *Assume that length* $\gamma_0 > 0$, *and that the space of inextensional displacements*

$$V_F(\omega) = \{\vec{\eta} \in V_K(\omega);\ \gamma_{\alpha\beta}(\vec{\eta}) = 0 \ \ in \ \ \omega\}$$

is not equal to $\{\vec{o}\}$, *assume that there exist functions* $f^i(x) \in L^2(\Omega)$ *such that*

$$f^{i,\varepsilon}(x^\varepsilon) = \varepsilon^2 f^i(x) \tag{3.16}$$

for all $x = (x_1, x_2, x_3) \in \Omega = \omega \times (-1, 1)$, *where* $(x_1^\varepsilon, x_2^\varepsilon) = (x_1, x_2) = y, x_3^\varepsilon = \varepsilon x_3$. *We denote by* $\vec{\xi} \in V_F(\omega)$ *the unique solution to the two-dimensional flexural shell equations*

$$\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{\xi})\rho_{\alpha\beta}(\vec{\eta})\sqrt{a}dy = \int_\omega \{\int_{-1}^{1} f^i(x)dx_3\}\eta_i\sqrt{a}dy \tag{3.17}$$

for all $\vec{\eta} = (\eta_i) \in V_F(\omega)$.

Denote by $\vec{\xi}^\varepsilon \in V_K(\omega)$ *the solution to the two-dimensional shell equations* (3.1) *of Koiter's type.*
Then,

$$\xi_\alpha^\varepsilon \to \xi_\alpha \quad in \quad H^1(\omega)(\alpha = 1, 2),$$

$$\xi_3^\varepsilon \to \xi_3 \quad in \quad H^2(\omega),$$

as $\varepsilon \to 0$.

Proof. From the inequality (3.7), Theorem 2.3, and the definition of the space $V_F(\omega)$, we can conclude that there exists a unique solution to the two-dimensional flexural shell equations (3.17).

Based on the notation (3.8)-(3.12), we denote

$$\|\vec{\eta}\|_{V_K(\omega)} = \{\sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega}\}^{1/2},$$

$$B_F(\vec{\xi},\vec{\eta}) = \frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\vec{\xi})\rho_{\alpha\beta}(\vec{\eta})\sqrt{a}dy.$$

From Theorem 2.3, there exists a positive constant c_1 such that

$$\|\vec{\eta}\|_{V_K(\omega)} \leq c_1\{\sum_{\alpha,\beta}\|\gamma_{\alpha\beta}(\vec{\eta})\|^2_{0,\omega} + \sum_{\alpha,\beta}\|\rho^{cl}_{\alpha\beta}(\vec{\eta})\|^2_{0,\omega}\}^{1/2} \tag{3.18}$$

for all $\vec{\eta} \in V_K(\omega)$.

By assumptions (3.16), the solution $\vec{\xi}^\varepsilon$ to the two-dimensional shell equations (3.1) also satisfies

$$\frac{1}{\varepsilon^2}B_M(\vec{\xi}^\varepsilon,\vec{\eta}) + B^{cl}_F(\vec{\xi}^\varepsilon,\vec{\eta}) = L(\vec{\eta}) \tag{3.19}$$

for all $\vec{\eta} \in V_k(\omega)$.

Taking $\vec{\eta} = \vec{\xi}^\varepsilon$ in (3.19) and combining inequalities (3.18) and (3.7), we obtain (without loss of generality, we may suppose $\varepsilon \leq 1$):

$$c_2\|\vec{\xi}^\varepsilon\|^2_{V_K(\omega)} \leq \sum_{\alpha,\beta}\|\frac{1}{\varepsilon}\gamma_{\alpha\beta}(\vec{\xi}^\varepsilon)\|^2_{0,\omega} + \frac{1}{3}\sum_{\alpha,\beta}\|\rho^{cl}_{\alpha\beta}(\vec{\xi}^\varepsilon)\|^2_{0,\omega} \leq c_0\|L\|\|\vec{\xi}^\varepsilon\|_{V_K(\omega)},$$

where c_0, c_2 are positive constants. Hence, we can select a subsequence, still by $(\vec{\xi}^\varepsilon)_{\varepsilon>0}$ for convenience, and there exists a function $\widetilde{\vec{\xi}} \in V_K(\omega)$ such that

$$\vec{\xi}^\varepsilon \rightharpoonup \widetilde{\vec{\xi}} \text{ in } V_K(\omega), \tag{3.20}$$

$$\gamma_{\alpha\beta}(\vec{\xi}^\varepsilon) \to 0 \text{ in } L^2(\omega). \tag{3.21}$$

From the weak convergence (3.20), we can conclude that $\gamma_{\alpha\beta}(\vec{\xi}^\varepsilon) \rightharpoonup \gamma_{\alpha\beta}(\widetilde{\vec{\xi}})$ in $L^2(\omega)$. Hence, $\gamma_{\alpha\beta}(\widetilde{\vec{\xi}}) = 0$ by (3.21), and thus $\widetilde{\vec{\xi}} \in V_F(\omega)$. Fix $\vec{\eta} \in V_F(\omega)$ in

(3.19) and let $\varepsilon \to 0$; then the weak convergence (3.20) yields $B_F^{cl}(\widetilde{\vec{\xi}}, \vec{\eta}) = L(\vec{\eta})$.

Since $\widetilde{\vec{\xi}}, \vec{\eta} \in V_F(\omega)$ and

$$\rho_{\alpha\beta}^{cl}(\vec{\eta}) = \rho_{\alpha\beta}(\vec{\eta}) + b_{\alpha\beta}a^{\sigma\tau}\gamma_{\sigma\tau}(\vec{\eta}),$$

$$\begin{aligned} B_F^{cl}(\widetilde{\vec{\xi}}, \vec{\eta}) &= \tfrac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}^{cl}(\widetilde{\vec{\xi}})\rho_{\alpha\beta}^{cl}(\vec{\eta})\sqrt{a}dy \\ &= \tfrac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\widetilde{\vec{\xi}})\rho_{\alpha\beta}(\vec{\eta})\sqrt{a}dy = B_F(\widetilde{\vec{\xi}}, \vec{\eta}). \end{aligned}$$

We have $B_F(\widetilde{\vec{\xi}}, \vec{\eta}) = L(\vec{\eta})$, thus $\widetilde{\vec{\xi}} = \vec{\xi}$, where $\vec{\xi} \in V_F(\omega)$ is the unique solution to the two-dimensional flexural shell equations (3.17).

The weak convergence (3.20) holds for the whole family $(\vec{\xi}^\varepsilon)_{\varepsilon>0}$.

By inequality (3.18) combined with the strong convergence (3.21) and the relations $\gamma_{\alpha\beta}(\vec{\xi}) = 0$, the following strong convergence

$$\vec{\xi}^\varepsilon \to \vec{\xi} \ \text{ in } \ V_K(\omega),$$

is equivalent to the convergence

$$B_F^{cl}(\vec{\xi}^\varepsilon - \vec{\xi}, \vec{\xi}^\varepsilon - \vec{\xi}) \to 0,$$

which can be verified from the relations

$$0 \le B_F^{cl}(\vec{\xi}^\varepsilon - \vec{\xi}, \vec{\xi}^\varepsilon - \vec{\xi}) = B_F^{cl}(\vec{\xi}^\varepsilon, \vec{\xi}^\varepsilon) - 2B_F^{cl}(\vec{\xi}^\varepsilon, \vec{\xi}) + B_F^{cl}(\vec{\xi}, \vec{\xi}),$$

$$B_F^{cl}(\vec{\xi}^\varepsilon, \vec{\xi}^\varepsilon) \le L(\vec{\xi}^\varepsilon),$$

$$B_F^{cl}(\vec{\xi}^\varepsilon, \vec{\xi}) \to B_F^{cl}(\vec{\xi}, \vec{\xi}), \ L(\vec{\xi}^\varepsilon) \to L(\vec{\xi}),$$

$$B_F^{cl}(\vec{\xi}, \vec{\xi}) = L(\vec{\xi}).$$

The proof is completed.

The method used in this paper is similar to that used in [5]. Since the two-dimensional linear shell model considered in this paper is different from the classical two-dimensional linear Koiter's shells model considered in [5] (the difference is the flexural part), it is necessary to establish the new Korn's inequality on the new two-dimensional linear shell model of Koiter's type before the asymptotic analysis.

In fact, from Theorem 3.2 and Theorem 3.3 in this paper and the conclusions of Theorem 2.1 and Theorem 2.2 in [5], we have proved that the new two-dimensional linear shell model of Koiter's type is asymptotically equivalent to the classical two-dimensional linear Koiter's shells model considered in [5], this is a justification of the new two-dimensional linear shell model (3.1) of Koiter's type.

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