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COMPLETE HOMOGENEITY AND REVERSIBILITY IN L-TOPOLOGY

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Abstract

The aim of this paper is to introduce and to study the concept of 'Complete homogeneity and reversibility' in L-topological spaces. Here we characterize all L-topological spaces which are minimum or maximum with respect to an L-topological property.

1. Introduction

In [9] we studied lattice structure of the set of all L-topologies on a given set X and proved that the lattice of L-topologies is not complemented. A related problem is to determine which subfamilies of L-topologies do possess minimum (maximum) and minimal (maximal) elements with respect to an L-topological property. In [10] Larson characterized all spaces which are minimum or maximum with respect to a topological property by introducing completely homogeneous topological spaces. In [12] Rajagopalan and Wilansky proved that a topological space is minimal or maximal for some topological property if and only if it is reversible. Here we investigate the concept 'Complete homogeneity and reversibililty' in general L-setup and L-topology.

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2. Preliminaries

Let X be a nonempty ordinary set and $L = L(\leq, \vee, \wedge, .')$ be a fuzzy lattice, i.e., a complete completely distributive lattice with smallest element 0 and largest element $1\ (0 \neq 1)$ and with an order-reversing involution $a \to a'\ (a \in L)$. L is therefore a continuous lattice [4]. We identify the constant function with value $\propto$ by $\underline{\propto}$. The fundamental definitions of L- fuzzy set theory and L-fuzzy topology are assumed to be familiar to the reader in the sense of Chang [3] and Goguen [5]. Here we call L-fuzzy subsets as L-subsets and a crisp subset $\mathcal{F}$ of L^x is called an *L-topology* if

(i) $\underline{0}, \underline{1} \in \mathcal{F}$,

(ii) $f, g \in \mathcal{F} \Rightarrow f \wedge g \in \mathcal{F}$,

(iii) $f_{\propto} \in \mathcal{F}$ for each $\propto \in A \Rightarrow V_{\propto \in A} f_{\propto} \in \mathcal{F}$.

Members of $\mathcal{F}$ are called *L-opensets*.

Definition 2.1. Let θ be a function from a set X to a set Y and f be an L-subset in Y. Then the *inverse image of f*, written as $\theta^{-1} f$, is an L-subset in X whose membership function is given by $\theta^{-1}(f)(x) = f(\theta(x))$ for all x in X. Conversely, let g be an L-subset in X. Then the *image of g*, written as $\theta(g)$, is an L-subset in Y, whose membership function is given by

$$\theta(g)(y) = \begin{cases} \sup\{g(z);\ z \in \theta^{-1}(y)\} & \text{if } \theta^{-1}(y) \neq \varnothing \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.2. A function θ from an L-topological space $(X, \mathcal{F})$ to an L-topological space $(Y, \mathcal{G})$ is *L-continuous* if and only if the inverse image of each $\mathcal{G}$ open L-subset in Y is $\mathcal{F}$ open L-subset in X. An *L-homeomorphism* is an L-continuous one to one map of an L-topological space $(X, \mathcal{F})$ onto an L-topological space $(Y, \mathcal{G})$ such that the inverse of the map is also L-continuous.

3. Reversibility and Complete Homogeneity in L-topology

In [12] Rajagopalan and Wilansky unified the notion of minimal and maximal topologies by introducing reversible topological spaces. In [10] Larson studied subfamilies of topologies possessing minimal and maximal elements and characterized all spaces which are minimum and maximum with respect to a topological property. Here we introduce in an analogous way the concept of complete homogeneity and reversibility in the general set up of L-sets and L-topology.

Definition 3.1. An L-topological space $(X, \mathcal{F})$ is called *reversible* if it has no strictly stronger (weaker) L-topology $\mathcal{F}'$ such that $(X, \mathcal{F})$ and $(X, \mathcal{F}')$ are L-homeomorphic.

Lemma 3.2. *An L-topological space $(X, \mathcal{F})$ is reversible if and only if each L-continuous one to one map of the space onto itself is an L-homeomorphism.*

Theorem 3.3. *If X is any finite set and $\mathcal{F}$ is any L-topology on X, then $(X, \mathcal{F})$ is reversible.*

Proof of Theorem 3.3 is easy.

Definition 3.4 [3]**.** An element p of L is called *prime* if $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leq p$, then $a \leq p$ or $b \leq p$. The set of all prime elements of L will be denoted by $pr(L)$.

Definition 3.5 (Warner and McLean [14])**.** The *scott topology* on L is the topology generated by the sets of the form $\{t \in L : t \nleq p\}$, where $p \in pr(L)$. Let (X, τ) be a topological space and $f : (X, \tau) \to L$ be a function, where L has its scott topology. We say that f is *scott continuous* if for every $p \in pr(L)$, $f^{-1}(\{t \in L : t \nleq p\}) \in \tau$.

The set $\omega(\tau)$ of all scott continuous functions from a topological space (X, τ) to L with its scott topology is an L-topology called the *induced L-topology* [1]. This induced L-topology is equivalent to topologically generated spaces of Lowen [11] when $L = [0, 1]$. Conversely for an L-topology $\mathcal{F}$ on X, $i(\mathcal{F})$ is the weak topology on X induced by all functions $f : X \to L$, where $f \in \mathcal{F}$ and L – with its scott topology. Then we have

Theorem 3.6. *If* (X, τ) *is a topological space, then* (X, τ) *is reversible if and only if* $(X, \omega(\tau))$ *is reversible.*

Theorem 3.7. *If* (X, τ) *is a topological space such that* (X, τ) *is not reversible, g is a scott continuous function from* $(X, \tau) \to L$ *such that g is one-one and* $\mathcal{F}$ *is the L-topology generated by* $S = \{\chi_A : A \in \tau\} \cup \{g\}$, *where* χ_A *is the characteristic function of A. Then* $\mathcal{F}$ *is reversible and* $i(\mathcal{F}) = \tau$.

Proof. Suppose θ is a bijection on X such that θ is not an identity map. Then $\theta^{-1}(g) \notin \mathcal{F}$. That is θ is not L-continuous. Thus every one to one L-continuous map onto itself is an L-homeomorphism. Thus $(X, \mathcal{F})$ is reversible.

Remark 3.8. There are non reversible L-topological spaces such that its associated topology is reversible.

Definition 3.9. An L-topological space $(X, \mathcal{F})$ is called *completely homogeneous* if every one to one map of X onto itself is an L-homeomorphism.

Theorem 3.10. *If* $(X, \mathcal{F})$ *is discrete, indiscrete or generated by L-points with the same membership value, then* $(X, \mathcal{F})$ *is completely homogeneous.*

Definition 3.11. An *L-topological property* is a class of L-topological spaces which is closed under L-homeomorphism.

Theorem 3.12. *Given an L-topological space* $(X, \mathcal{F})$ *the following conditions are equivalent*:

(a) $(X, \mathcal{F})$ *is completely homogeneous.*

(b) $(X, \mathcal{F})$ *is minimum p for some L-topological property p.*

(c) $(X, \mathcal{F})$ *is maximum p for some L-topological property p.*

Proof. Suppose $(X, \mathcal{F})$ is completely homogeneous. Define p by the following: an L-topological space $(Y, \mathcal{G})$ has property p if there exists one to one, onto L-continuous mapping $\theta : (Y, \mathcal{G}) \to (X, \mathcal{F})$. Then $(X, \mathcal{F})$ has property p. Now assume $\mathcal{F}'$ is an L-topology on X which possesses property p. Then there exists a one to one, onto L-continuous mapping $\theta : (X, \mathcal{F}') \to (X, \mathcal{F})$; but then $\theta^{-1} : (X, \mathcal{F}) \to (X, \mathcal{F})$ is L-continuous since $(X, \mathcal{F})$ is completely homogeneous. Hence the identity mapping $\theta^{-1} \cdot \theta = i : (X, \mathcal{F}') \to (X, \mathcal{F})$ is L-continuous and $\mathcal{F} \subseteq \mathcal{F}'$. Thus $(X, \mathcal{F})$ is minimum for p. This proves (a) $\Rightarrow$ (b).

Now to prove (b) $\Rightarrow$ (a) assume $(X, \mathcal{F})$ is minimum for some L-topological property p. Let θ be a one to one mapping of X onto X. Define $\mathcal{F}(\theta) = \{\theta(g) : g \in \mathcal{F}\}$. Then $\mathcal{F}(\theta)$ is an L-topology on X and $\theta : (X, \mathcal{F}) \to (X, \mathcal{F}(\theta))$ is an L-homeomorphism. Then $\mathcal{F}(\theta)$ is also minimum for p; since p is an L-topological property which implies that $\mathcal{F}(\theta) \subseteq \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{F}(\theta)$. Hence $\mathcal{F}(\theta) = \mathcal{F}$ and $\theta : (X, \mathcal{F}) \to (X, \mathcal{F})$ is an L-homeomorphism. Thus $(X, \mathcal{F})$ is completely homogeneous.

In a similar way we can show that (a) and (c) are equivalent.

Theorem 3.13. *If* (X, τ) *is a topological space, then* (X, τ) *is completely homogeneous if and only if* $(X, \omega(\tau))$ *is completely homogeneous.*

Definition 3.14. An L-topology $(X, \mathcal{F})$ is *homogeneous* if for any $x, y \in X$ there exists an L-homeomorphism θ such that $\theta(x) = y$. Then we have

Theorem 3.15. *Every completely homogeneous L-topology is hereditarily homogeneous.*

Theorem 3.16. *Every completely homogeneous L-topology is hereditarily reversible.*

Remark 3.17. Complete homogeneity implies reversibility but reversibility need not imply complete homogeneity.

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References

[1] Halis Aygun, M. W. Warner and S. R. T. Kudri, Completely induced L-fuzzy topological spaces, Fuzzy Sets and Systems 103 (1999), 513-523.

[2] R. N. Bhaumik and A. Mukherjee, Completely induced fuzzy topological spaces, Fuzzy Sets and Systems 47 (1992), 387-390.

[3] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968), 191-201.

[4] G. Gierz, A Compendium of Continuous Lattices, Springer, Berlin, 1980.

[5] J. A. Goguen, L-fuzzy sets, J. Math. Anal. Appl. 18 (1967), 145-174.

[6] T. P. Johnson, On the lattice of fuzzy topologies I, Fuzzy Sets and Systems 48 (1992), 133-135.

[7] T. P. Johnson, Completely homogeneous fuzzy topological spaces, J. Fuzzy Mathematics 1(3) (1993), 495-500.

[8] T. P. Johnson, Reversible fuzzy topological spaces, J. Fuzzy Mathematics 1(3) (1993), 491-494.

[9] T. P. Johnson, On lattices of L-topologies, to appear.

[10] R. E. Larson, Minimum and maximum topological spaces, Bull. Polish. Acad. Sci. 18 (1970), 707-710.

[11] R. Lowen, Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl. 56 (1976), 621-633.

[12] M. Rajagopalan and A. Wilansky, Reversible topological spaces, J. Austral. Math. Soc. 6 (1966), 129-138.

[13] M. W. Warner, Fuzzy topology with respect to continuous lattice, Fuzzy Sets and Systems 35 (1990), 85-91.

[14] M. W. Warner and R. G. McLean, On compact Hausdorff L-fuzzy spaces, Fuzzy Sets and Systems 56 (1993), 103-110.

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