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THE SIMILARITY AND COMMUTANT OF ANALYTIC TOEPLITZ OPERATORS ON WEIGHTED BERGMAN SPACE

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Abstract

Let D be the unit disk and $A_\alpha^2(D)$ $(\alpha > -1)$ be the weighted Bergman space. In this paper, by constructing an orthonormal basis of $A_\alpha^2(D)$, we prove that the analytic Toeplitz operator $M_{B(z)}$ is similar to $\overset{n}{\underset{1}{\oplus}} M_z$ on $A_\alpha^2(D)$ if and only if $B(z)$ has the form $B(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n$ $(0 < |a| < 1)$. Then, using the skills of matrix and operator theory techniques, we characterize the commutant of $M_{B(z)}$.

1. Introduction

Let D and $\mathbf{T}$ be the unit disk $D = \{z \in C : |z| < 1\}$ and the boundary of D, respectively. For the Hardy space $H^2(D)$, the mth power of the unilateral shift T_{z^m} is unitarily equivalent to $T_z \otimes I_m$ on $H^2(D) \otimes C^m$, since T_{z^m} is an isometry of

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multiplicity *m*. Let $A_\alpha^2(D)$ $(\alpha > -1)$ denote the weighted Bergman space of analytic functions which belong to $L_\alpha^2(D)$. It is well-known that $A_\alpha^2(D)$ is a closed subspace of $L_\alpha^2(D)$, and $A_\alpha^2(D)$ is a Hilbert space. If $f \in A_\alpha^2(D)$, then $\| f \|_{\alpha, 2}$ $= \left(\int_D | f(z) |^2 dA_\alpha(z) \right)^{\frac{1}{2}}$, where $dA_\alpha(z) = (1 + \alpha)(1 - |z|^2)^\alpha dA(z)$, and we denote by dA the ordinary Lebesgue area measure on D. When $\alpha = 0$, $A^2(D)$ is the ordinary Bergman space, and the results of the paper are certainly right on Bergman space.

Let $H^\infty(D)$ denote the algebra of bounded analytic functions on D. For $f \in H^\infty$, M_f is an analytic Toeplitz operator on Bergman space defined by $M_f g = fg$, for any $g \in A_\alpha^2(D)$. M_f is a bounded linear operator on $A_\alpha^2(D)$, with $\| M_f \| = \| f \|_\infty = \sup\{| f(z) | : z \in D\}$. Let $\mathcal{A}'(M_f)$ denote the commutant of M_f, i.e., $\mathcal{A}'(M_f) = \{T \in \mathcal{L}(A_\alpha^2(D)) : TM_f = M_f T\}$, where $\mathcal{L}(A_\alpha^2(D))$ represents the collection of all bounded linear operators on $A_\alpha^2(D)$.

In studying an operator on a Hilbert space, it is of interest to characterize the operators which commute with a given operator, for such a characterization should help in understanding the structure of the operator. The commutant of an analytic Toeplitz operator on the Hardy space and Bergman space has been studied extensively in the literature, we mention here the papers (see [1-7]), and the books (see [10, 11, 13]) including an excellent account of the knowledge of operator theory. In particular, Thomson gave an explicit description of the commutant of M_B when B is a Blaschke product with two zeros. Cowen described the commutant of M_B for a finite Blaschke product in terms of the Riemann surface generated by B. On Bergman space, to characterize the commutant of analytic Toeplitz operators is very difficult. Zhu obtained a complete description of the reducing subspaces of multiplication operators on Bergman space induced by Blaschke product with two zeros using the method of the geodesic midpoint of the two zeros of $B(z)$. In [8], Hu et al. proved that the analytic Toeplitz operator with finite Blaschke product symbol on the Bergman space has at least a reducing subspace on which the restriction of the associated Toeplitz operator is unitary equivalent to Bergman shift. In [9], Stroethoff and Zheng discussed the bounded property of product of Hankel and Toeplitz operators on the Bergman space and answered the Sarason's problem. In 2007, Jiang and Li [17] proved that each analytic Toeplitz operator $M_{B(z)}$ is similar to $n+1$ copies of the Bergman shift if and only if $B(z)$ is an $n+1$-Blaschke product. In this paper, we prove that the analytic Toeplitz operator $M_{B(z)}$ is similar to $\overset{n}{\underset{1}{\oplus}} M_z$ on $A_\alpha^2(D)$ if and only if $B(z)$ has the form $B(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n$ $(0 < |a| < 1)$. Then, using the matrix skills and operator theory techniques, we characterize the commutant of $M_{B(z)}$ on $A_\alpha^2(D)$.

The following is the Main theorem in the paper.

Main Theorem. *Let* $A_\alpha^2(D)(\alpha > -1)$ *be the weighted Bergman space. Then the analytic Toeplitz operator* $M_{B(z)}$ *is similar to* $\overset{n}{\underset{1}{\oplus}} M_z$ *on* $A_\alpha^2(D)$ *if and only if* $B(z)$ *has the form* $B(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n$ $(0 < |a| < 1)$.

2. Preparation

Lemma 1 (See [13]). *For* $-1 < \alpha < +\infty$, *let* P_α *be the orthogonal projection from* $L^2(D, dA_\alpha)$ *onto* A_α^2. *Then* $P_\alpha f(z) = \int_D \frac{f(w)dA_\alpha(w)}{(1-z\overline{w})^{2+\alpha}}$, $z \in D$. *If* $f \in A_\alpha^2(D)$, $P_\alpha f(z) = f(z)$, *then we have* $f(z) = \int_D \frac{f(w)dA_\alpha(w)}{(1-z\overline{w})^{2+\alpha}}$, $z \in D$.

Definition 2. $k_w(z) = \frac{1}{(1-\overline{w}z)^{2+\alpha}}$ $(-1 < \alpha < +\infty)$ is said to be the *reproducing kernel* of $A_\alpha^2(D)$. The kernel k_w has the property $f(w) = \langle f, k_w \rangle_\alpha$, for every $w \in D$ and $f \in A_\alpha^2(D)$. Particularly, $\| k_w(z) \|^2 = \langle k_w, k_w \rangle_\alpha = \frac{1}{(1-|w|^2)^{2+\alpha}}$, we call $\tilde{k}_w = \frac{k_w}{\| k_w \|} = \frac{(1-|w|^2)^{\frac{2+\alpha}{2}}}{(1-\overline{w}z)^{2+\alpha}}$ the *normalized reproducing kernel* of $A_\alpha^2(D)$.

Lemma 3. *Let* $B(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n$ $(0 < |a| < 1)$, $f_n = \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} B(z)\tilde{k}_a(z)$, $n = 0, 1, \ldots$. *Then* $\{f_n\}_{n=0}^{\infty}$ *form an orthonomal basis of* $A_\alpha^2(D)$.

Proof. Note that

$$B(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n = \left[\frac{(1-|a|^2)z}{1-\bar{a}z} - a\right]^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} \frac{(1-|a|^2)^k z^k}{(1-\bar{a}z)^k},$$

$$\langle f_n, f_m \rangle = \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} \sqrt{\frac{\Gamma(m+2+\alpha)}{m!\Gamma(2+\alpha)}} (1-|a|^2)^{2+\alpha}$$

$$\times \left\langle \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} \frac{(1-|a|^2)^k z^k}{(1-\bar{a}z)^{k+2+\alpha}}, \left(\frac{z-a}{1-\bar{a}z}\right)^m \frac{1}{(1-\bar{a}z)^{2+\alpha}} \right\rangle_\alpha .$$

By computing, we deduce that the function $(2+\alpha)(3+\alpha)\cdots(k+1+\alpha)$ $\frac{z^k}{(1-\bar{u}z)^{k+2+\alpha}}$ is the kernel function in $A_\alpha^2(D)$ for the functional of evaluation of the kth derivative at a, so we get

$$\overline{\left\langle \frac{z^k}{(1-\bar{a}z)^{k+2+\alpha}}, \left(\frac{z-a}{1-\bar{a}z}\right)^m \frac{1}{(1-\bar{a}z)^{2+\alpha}} \right\rangle_\alpha}$$

$$= \frac{1}{(2+\alpha)\cdots(k+1+\alpha)} \left[\frac{\partial^k}{\partial z^k}\left(\frac{(z-a)^m}{(1-\bar{a}z)^{m+2+\alpha}}\right)\right]_{z=a}$$

$$= \frac{1}{(2+\alpha)\cdots(k+1+\alpha)} \left[\sum_{j=0}^{k} \binom{k}{j} \frac{\partial^{k-j}}{\partial z^{k-j}}((z-a)^m) \frac{\partial^j}{\partial z^j}\left(\frac{1}{(1-\bar{a}z)^{m+2+\alpha}}\right)\right]_{z=a}$$

$$= \frac{1}{(2+\alpha)\cdots(k+1+\alpha)}$$

$$\times \left[\sum_{j=0}^{k} \binom{k}{j} \frac{m(m-1)\cdots(m+1-(k-j))(z-a)^{m-(k-j)}(m+2+\alpha)\cdots(m+1+\alpha+j)\bar{a}^j}{(1-\bar{a}z)^{m+2+\alpha+j}}\right]_{z=a} .$$

Therefore,

$$\langle f_n, f_m \rangle$$

$$= \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}}\sqrt{\frac{\Gamma(m+2+\alpha)}{m!\Gamma(2+\alpha)}}(1-|a|^2)^{2+\alpha}$$

$$\times \sum_{k=0}^{n}\binom{n}{k}(-a)^{n-k}(1-|a|^2)^k \frac{1}{(2+\alpha)(3+\alpha)\cdots(k+1+\alpha)}$$

$$\times \left[\sum_{j=0}^{k}\binom{k}{j}\frac{m(m-1)\cdots(m+1-(k-j))(\bar{z}-\bar{a})^{m-(k-j)}(m+2+\alpha)\cdots(m+1+\alpha+j)a^j}{(1-a\bar{z})^{m+2+\alpha+j}}\right]_{\bar{z}=\bar{a}}.$$

If $m = n$, then we obtain

$$\langle f_n, f_m \rangle$$

$$= \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}\sum_{k=0}^{n}\frac{1}{(2+\alpha)(3+\alpha)\cdots(k+1+\alpha)}\binom{n}{k}(-a)^{n-k}(1-|a|^2)^{k+2+\alpha}$$

$$\times \left\{\left[\frac{n(n-1)\cdots(n+1-k)(\bar{z}-\bar{a})^{n-k}}{(1-a\bar{z})^{n+2+\alpha}}\right]_{\bar{z}=\bar{a}}\right.$$

$$+\left[\frac{1}{(1-a\bar{z})^{n+3+\alpha}}\binom{k}{1}n(n-1)\cdots(n+2-k)(\bar{z}-\bar{a})^{n-k+1}(n+2+\alpha)a\right]_{\bar{z}=\bar{a}}$$

$$\left.+\cdots+\left[\frac{(\bar{z}-\bar{a})^n(n+2+\alpha)\cdots(n+1+k+\alpha)a^k}{(1-a\bar{z})^{n+2+k+\alpha}}\right]_{\bar{z}=\bar{a}}\right\}$$

$$= \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}\frac{(1-|a|^2)^{n+2+\alpha}}{(2+\alpha)(3+\alpha)\cdots(n+1+\alpha)}\frac{n!}{(1-|a|^2)^{n+2+\alpha}} = 1.$$

If $m > n$, then it is obvious that $\langle f_n, f_m \rangle = 0$ which implies that $\{f_n\}_{n=0}^{\infty}$ form an orthonormal basis of $A_\alpha^2(D)$. □

3. The Similarity of Analytic Toeplitz Operator

In this section, we prove that the analytic Toeplitz operator $M_{B(z)}$ is similar to $\overset{n}{\underset{1}{\oplus}} M_z$ on $A_\alpha^2(D)$.

Theorem 4. *Let* $B(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n$ $(0 < |a| < 1)$. *Then the analytic Toeplitz operator* $M_{B(z)}$ *is similar to* M_{z^n} *on* $A_\alpha^2(D)$.

Proof. Let $B(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n$ $(0 < |a| < 1)$ and $B_a = \frac{z-a}{1-\bar{a}z}$. Then define the composition operator (see [16]) $C_{B_a} : f(z) \to f(B_a)$, for any $f \in A_\alpha^2(D)$. Note that $e_n(z) = \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} z^n$ $(n = 0, 1, 2, \ldots)$ form the orthonormal basis of the weighted Bergman space $A_\alpha^2(D)$. From Lemma 3, we know that $f_n(z) = \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} B(z)\tilde{k}_a(z)$ is an orthonormal basis of $A_\alpha^2(D)$. We define an operator $X = M_{\tilde{k}_a} C_{B_a} : \{e_n(z)\}_{n=0}^\infty \to \{f_n(z)\}_{n=0}^\infty$, then X is an isometric operator. In fact, $Xz^n = M_{\tilde{k}_a} C_{B_a} z^n = \tilde{k}_a B(z)$. Since $\vee_{n=0}^\infty \{e_n(z)\} = A_\alpha^2(D)$, $\vee_{n=0}^\infty \{f_n(z)\} = A_\alpha^2(D)$, i.e., $\ker X^* = \{0\}$, so $X = M_{\tilde{k}_a} C_{B_a}$ is a unitary operator. Therefore, we have

$$M_{B(z)} X f(z) = B(z) M_{\tilde{k}_a} C_{B_a} f(z) = B(z)\tilde{k}_a f(B_a),$$

$$X M_{z^n} f(z) = M_{\tilde{k}_a} C_{B_a} M_{z^n} f(z) = \tilde{k}_a B(z) f(B_a)$$

which means that $M_{B(z)} X = X M_{z^n}$, i.e., $M_{B(z)}$ is unitarily equivalent to M_{z^n}. □

Lemma 5. *Let* $e_k(z) = \sqrt{\frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)}} z^k$ $(k = 0, 1, \ldots)$ *be the orthonormal basis of the weighted Bergman space* $A_\alpha^2(D)$ $(\alpha > -1)$. *Set* $L_j = span\{e_{nk+j}\}$

$(j = 0, 1, \ldots, n-1)$. *Then we have*

(i) $\{e_{nk+j}\}_{k=0}^{\infty}$ *form an orthonormal basis of* L_j.

(ii) $A_\alpha^2 = L_0 \oplus L_1 \oplus \cdots \oplus L_{n-1}$.

(iii) L_j *is a reducing subspace for* M_{z^n}.

Proof. (i)

$$\langle e_{nk+j}, e_{nm+j}\rangle$$

$$= \int_D \sqrt{\frac{\Gamma(nk+j+2+\alpha)}{(nk+j)!\Gamma(2+\alpha)}} z^{nk+j} \sqrt{\frac{\Gamma(nm+j+2+\alpha)}{(nm+j)!\Gamma(2+\alpha)}} \bar{z}^{nm+j}$$

$$\times (1+\alpha)(1-|z|^2)^\alpha dA(z).$$

If $k = m$, then

$$\langle e_{nk+j}, e_{nm+j}\rangle$$

$$= \frac{1}{\pi}\frac{\Gamma(nk+j+2+\alpha)}{(nk+j)!\Gamma(2+\alpha)}(1+\alpha)\int_0^1\int_0^{2\pi} r^{2(nk+j)}(1-r^2)^\alpha r dr d\theta$$

$$= \frac{\Gamma(nk+j+2+\alpha)}{(nk+j)!\Gamma(2+\alpha)}(1+\alpha)B(nk+j+1, \alpha+1) = 1,$$

where B stands for the usual Beta function.

If $k \neq m$, without loss of generality, we assume that $k > m$, then

$$\langle e_{nk+j}, e_{nm+j}\rangle$$

$$= \frac{1}{\pi}\sqrt{\frac{\Gamma(nk+j+2+\alpha)}{(nk+j)!\Gamma(2+\alpha)}}\sqrt{\frac{\Gamma(nm+j+2+\alpha)}{(nm+j)!\Gamma(2+\alpha)}}(1+\alpha)$$

$$\times \int_0^1\int_0^{2\pi} r^{n(k+m)+2j} e^{in(k-m)\theta}(1-r^2)^\alpha r dr d\theta = 0.$$

(ii) First, from (i), we know that $L_j \perp L_t$, $0 \leq j \neq t \leq n-1$. Next, suppose that

$$\sum_{k=0}^{\infty} a_{0k} e_{nk} + \cdots + \sum_{k=0}^{\infty} a_{n-1,\,k} e_{nk+n-1} = 0,$$

from $\left\langle \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} a_{jk} e_{nk+j},\, e_l \right\rangle = 0 \ (l = 0, 1, \ldots),$ we have $a_{jk} = 0 \ (j = 0, 1, \ldots, n-1,$ $k = 0, 1, \ldots)$ which implies that $0 = \overbrace{0 \oplus 0 \oplus \cdots \oplus 0}^{n}.$ So $A_\alpha^2 = L_0 \oplus L_1 \oplus \cdots \oplus L_{n-1}.$

(iii) It is easy to see that both L_j and $L_j^{\perp}$ are the invariant subspaces for M_{z^n}.

□

Theorem 6. *Let* $A_\alpha^2(D)\ (\alpha > -1)$ *be the weighted Bergman space. Then the analytic Toeplitz operator* M_{z^n} *is similar to* $\overset{n}{\underset{1}{\oplus}} M_z$ *on* $A_\alpha^2(D)$.

Proof. Set $T_j = M_{z^n}|_{L_j}\ (j = 0, 1, \ldots, n-1).$ Then

$$T_j e_{nk+j} = z^n \sqrt{\frac{\Gamma(nk + j + 2 + \alpha)}{(nk + j)!\,\Gamma(2 + \alpha)}} z^{nk+j}$$

$$= \sqrt{\frac{\Gamma(nk + j + 2 + \alpha)(nk + n + j)!}{(nk + j)!\,\Gamma(nk + n + j + 2 + \alpha)}} e_{nk+n+j}. \tag{3.1}$$

Define $X_j : A_\alpha^2(D) \to L_j$ such that $X_j e_k = c_k^j e_{nk+j},$ where c_k^j are given by

$$\begin{cases} c_k^j = \sqrt{\dfrac{(nk + j)!(k + 1 + \alpha)(k + \alpha)\cdots(2 + \alpha)}{(nk + j + 1 + \alpha)(nk + j + \alpha)\cdots(j + 2 + \alpha)k!\,j!}} & (k \geq 1), \\ c_0^j = 1. \end{cases} \tag{3.2}$$

In fact, we claim that $X_j M_z e_k = T_j X_j e_k.$ Indeed,

$$X_j \sqrt{\frac{k + 1}{k + 2 + \alpha}} e_{k+1} = T_j c_k^j e_{nk+j}$$

and

$$\sqrt{\frac{k+1}{k+2+\alpha}}\, c_{k+1}^{j} e_{nk+n+j} = c_k^j \sqrt{\frac{\Gamma(nk+j+2+\alpha)(nk+n+j)!}{(nk+j)!\Gamma(nk+n+j+2+\alpha)}}\, e_{nk+n+j}.$$

From

$$\frac{c_{k+1}^{j}}{c_k^j} = \sqrt{\frac{\Gamma(nk+j+2+\alpha)(nk+n+j)!(k+2+\alpha)}{\Gamma(nk+n+j+2+\alpha)(nk+j)!(k+1)}}, \tag{3.3}$$

we can get (3.2).

In the following, we will analyze the coefficient c_k^j and prove that $\lim_{k\to\infty} c_k^j \neq 0, \infty$.

Case 1. If $k \geq 1$, $\alpha \geq 0$, then we consider the equality

$$\begin{aligned}
(c_k^j)^2 &= \frac{(nk+j)!(k+1+\alpha)(k+\alpha)\cdots(2+\alpha)}{(nk+j+1+\alpha)(nk+j+\alpha)\cdots(j+2+\alpha)k!\,j!} \\
&= \frac{(nk+j)(nk+j-1)\cdots(k+1)(j+1+\alpha)\cdots(3+\alpha)(2+\alpha)}{(nk+j+1+\alpha)(nk+j+\alpha)\cdots(k+2+\alpha)\,j!} \\
&= \frac{(j+1+\alpha)\cdots(3+\alpha)(2+\alpha)(k+1)}{(nk+j+1+\alpha)\,j!}\,\frac{1}{b_k^j(\alpha)},
\end{aligned} \tag{3.4}$$

where

$$b_k^j(\alpha) = \left(1+\frac{\alpha}{nk+j}\right)\left(1+\frac{\alpha}{nk+j-1}\right)\cdots\left(1+\frac{\alpha}{k+2}\right). \tag{3.5}$$

Note that

$$\begin{aligned}
\frac{b_{k+1}^{j}(\alpha)}{b_k^j(\alpha)} &= \frac{\left(1+\frac{\alpha}{nk+n+j}\right)\left(1+\frac{\alpha}{nk+n+j-1}\right)\cdots\left(1+\frac{\alpha}{nk+j+1}\right)}{1+\frac{\alpha}{k+2}} \\
&\geq \frac{\left(1+\frac{\alpha}{nk+n+j}\right)^n}{1+\frac{\alpha}{k+2}}
\end{aligned}$$

$$\geq \frac{1 + \dfrac{n\alpha}{nk + n + j}}{1 + \dfrac{n\alpha}{nk + 2n}} \geq 1$$

which implies that $\{b_k^j(\alpha)\}$ is a monotone increasing sequence, and

$$\left(1 + \frac{\alpha}{nk + j}\right)^{(n-1)k+j-1} \leq b_k^j(\alpha) \leq \left(1 + \frac{\alpha}{k + 2}\right)^{(n-1)k+j-1}. \tag{3.6}$$

Let $f(k) = \left(1 + \dfrac{\alpha}{nk + j}\right)^{(n-1)k+j-1}$. Then

$$\ln f(k) = [(n-1)k + j - 1][\ln(nk + j + \alpha) - \ln(nk + j)],$$

$$\frac{f'(k)}{f(k)} = (n-1)\ln\left(1 + \frac{\alpha}{nk + j}\right) + [(n-1)k + j - 1]\left(\frac{n}{nk + j + \alpha} - \frac{n}{nk + j}\right)$$

$$= (n-1)\ln\left(1 + \frac{\alpha}{nk + j}\right) - \frac{[(n-1)k + j - 1]n\alpha}{(nk + j + \alpha)(nk + j)}. \tag{3.7}$$

Applying the Lagrange mean value theorem, we have

$$\frac{f'(k)}{f(k)} \geq (n-1)\frac{\dfrac{\alpha}{nk + j}}{1 + \dfrac{\alpha}{nk + j}} - [(n-1)k + j - 1]\frac{n\alpha}{(nk + j + \alpha)(nk + j)}$$

$$= \frac{(n-1)\alpha}{nk + j + \alpha} - \frac{n[(n-1)k + j - 1]\alpha}{(nk + j + \alpha)(nk + j)}$$

$$= \frac{\alpha}{nk + j + \alpha}\frac{n - j}{nk + j} \geq 0. \tag{3.8}$$

So $f'(k) \geq 0$, and $f(k)$ is a monotone increasing sequence,

$$f(k) \geq f(1) = \left(1 + \frac{\alpha}{n + j}\right)^{n+j-2}. \tag{3.9}$$

Let $g(k) = \left(1 + \dfrac{\alpha}{k + 2}\right)^{(n-1)k+j-1}$. Then

$$\ln g(k) = [(n-1)k + j - 1][\ln(k + 2 + \alpha) - \ln(k + 2)],$$

$$\frac{g'(k)}{g(k)} = (n-1)\ln\left(1+\frac{\alpha}{k+2}\right) + [(n-1)k + j - 1]\left(\frac{1}{k+2+\alpha} - \frac{1}{k+2}\right)$$

$$= (n-1)\ln\left(1+\frac{\alpha}{k+2}\right) - \frac{[(n-1)k + j - 1]\alpha}{(k+2+\alpha)(k+2)}. \tag{3.10}$$

Applying the Lagrange mean value theorem again, we have

$$\frac{g'(k)}{g(k)} \geq (n-1)\frac{\frac{\alpha}{k+2}}{1+\frac{\alpha}{k+2}} - [(n-1)k + j - 1]\frac{\alpha}{(k+2+\alpha)(k+2)}$$

$$= \frac{(n-1)\alpha}{k+2+\alpha} - \frac{[(n-1)k + j - 1]\alpha}{(k+2+\alpha)(k+2)}$$

$$= \frac{\alpha}{k+2+\alpha}\frac{2n - j - 1}{k+2} \geq 0. \tag{3.11}$$

Thus $g'(k) \geq 0,$ and we get

$$g(k) \leq \lim_{k\to\infty}\left(1+\frac{\alpha}{k+2}\right)^{(n-1)k+j-1} = e^{(n-1)\alpha}. \tag{3.12}$$

And we obtain

$$\left(1+\frac{\alpha}{n+j}\right)^{n+j-2} \leq \lim_{k\to\infty} b_k^j(\alpha) \leq e^{(n-1)\alpha}. \tag{3.13}$$

Combining (3.4) with (3.13), we arrive at

$$\frac{(j+1+\alpha)\cdots(2+\alpha)}{nj!\,e^{(n-1)\alpha}} \leq \lim_{k\to\infty}(c_k^j)^2$$

$$\leq \frac{(j+1+\alpha)\cdots(2+\alpha)}{nj!}\left(\frac{n+j}{n+j+\alpha}\right)^{n+j-2} \quad (j \geq 1). \tag{3.14}$$

If $j = 0,$ then we get

$$\frac{1}{ne^{(n-1)\alpha}} \leq \lim_{k\to\infty}(c_k^0)^2 \leq \frac{1}{n}\left(\frac{n}{n+\alpha}\right)^{n-2}. \tag{3.15}$$

Case 2. If $k \geq 1,\ -1 < \alpha < 0,$ then

$$\frac{b_{k+1}^{j}(\alpha)}{b_{k}^{j}(\alpha)} = \frac{\left(1+\frac{\alpha}{nk+n+j}\right)\left(1+\frac{\alpha}{nk+n+j-1}\right)\cdots\left(1+\frac{\alpha}{nk+j+1}\right)}{1+\frac{\alpha}{k+2}}$$

$$< \frac{\left(1+\frac{\alpha}{nk+n+j}\right)^{n}}{1+\frac{\alpha}{k+2}}$$

$$< \frac{\left(1+\frac{\alpha}{nk+2n+j}\right)^{n}}{1+\frac{\alpha}{k+2}}. \tag{3.16}$$

We consider the following inequality:

$$\frac{\ln\left(1+\frac{\alpha}{nk+2n+j}\right)^{n}}{\ln\left(1+\frac{\alpha}{k+2}\right)} = \frac{n\ln\left(1+\frac{\alpha}{nk+2n+j}\right)}{\ln\left(1+\frac{\alpha}{k+2}\right)}$$

$$< \frac{\frac{n\alpha}{nk+2n+j}}{\frac{\frac{\alpha}{k+2}}{1+\frac{\alpha}{k+2}}}$$

$$= \frac{nk+2n+n\alpha}{nk+2n+j} < 1 \tag{3.17}$$

which implies that

$$\left(1+\frac{\alpha}{nk+2n+j}\right)^{n} < 1+\frac{\alpha}{k+2}.$$

So $b_{k+1}^{j}(\alpha) < b_{k}^{j}(\alpha),$ and we have

$$\left(1+\frac{\alpha}{k+2}\right)^{(n-1)k+j-1} < b_{k}^{j}(\alpha) < \left(1+\frac{\alpha}{nk+j}\right)^{(n-1)k+j-1}. \tag{3.18}$$

Let $h(k) = \left(1 + \frac{\alpha}{k+2}\right)^{(n-1)k+j-1}$. Then

$$\ln h(k) = [(n-1)k + j - 1][\ln(k + 2 + \alpha) - \ln(k+2)],$$

and

$$\frac{h'(k)}{h(k)} = (n-1)\ln\left(1 + \frac{\alpha}{k+2}\right) + [(n-1)k + j - 1]\left(\frac{1}{k+2+\alpha} - \frac{1}{k+2}\right)$$

$$< (n-1)\frac{\alpha}{k+2} - \frac{\alpha[(n-1)k + j - 1]}{(k+2+\alpha)(k+2)}$$

$$= \frac{\alpha}{k+2}\frac{n - j + (n-1)(1+\alpha)}{k+2+\alpha} < 0. \tag{3.19}$$

Thus $h'(k) < 0$, and $\{h(k)\}_{k=1}^{\infty}$ is a monotone decreasing sequence. We have

$$h(k) > \lim_{k\to\infty}\left(1 + \frac{\alpha}{k+2}\right)^{(n-1)k+j-1} = e^{(n-1)\alpha}. \tag{3.20}$$

Note that

$$\lim_{k\to\infty}\left(1 + \frac{\alpha}{nk+j}\right)^{(n-1)k+j-1} = e^{\left(1-\frac{1}{n}\right)\alpha}. \tag{3.21}$$

We obtain

$$e^{(n-1)\alpha} \le \lim_{k\to\infty} b_k^j(\alpha) \le e^{\left(1-\frac{1}{n}\right)\alpha}. \tag{3.22}$$

Therefore,

$$\frac{(j+1+\alpha)\cdots(2+\alpha)}{nj!\,e^{\left(1-\frac{1}{n}\right)\alpha}} \le \lim_{k\to\infty}(c_k^j)^2 \le \frac{(j+1+\alpha)\cdots(2+\alpha)}{nj!\,e^{(n-1)\alpha}}\ (j \ge 1). \tag{3.23}$$

If $j = 0$, then we have

$$\frac{1}{ne^{\left(1-\frac{1}{n}\right)\alpha}} \le \lim_{k\to\infty}(c_k^0)^2 \le \frac{1}{ne^{(n-1)\alpha}}. \tag{3.24}$$

Finally, according to the above analysis, we deduce that the operator X_j $(j = 0, 1, ..., n-1)$ is bounded and invertible. We obtain

$$T_j \sim M_z \ (j = 0, 1, ..., n-1). \tag{3.25}$$

Furthermore, we have

$$M_{z^n}|_{A^2_\alpha(D)} = T_0 \oplus T_1 \oplus \cdots \oplus T_{n-1} \sim \bigoplus_1^n M_z. \tag{3.26}$$

□

In order to prove the Main theorem, we introduce the following concepts, an operator $T \in \mathcal{L}(\mathcal{H})$ is compact if the image of the unit ball in complex separable Hilbert space $\mathcal{H}$ under T has compact closure. Let $\mathcal{K}(\mathcal{H})$ be the ideal of all compact operators acting on $\mathcal{H}$ and $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{A}(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the canonical projection of $\mathcal{L}(\mathcal{H})$ to Calkin algebra, $\sigma_e(T)$ is called the *essential spectrum* of T, which is the spectrum of $\pi(T)$ in $\mathcal{A}(\mathcal{H})$ and $C \backslash \sigma_e(T)$ is called the *Fredholm domain* of T and is denoted by $\rho_F(T)$, and $\sigma_e(T) = \sigma_{le}(T) \cup \sigma_{re}(T)$ (left and right essential spectrum of T). T is a *semi-Fredholm operator* if ran T is closed and either $\operatorname{nul} T = \dim\ker T$ or $\operatorname{nul} T^* = \dim\ker T^*$ is finite. In this case, the index of T is defined by $\operatorname{ind} T = \operatorname{nul} T - \operatorname{nul} T^*$ and $\sigma_{lre}(T) = \sigma_{le}(T) \cap \sigma_{re}(T)$ is called *Wolf spectrum* of T.

Now we prove the Main theorem.

Theorem 7 (Main Theorem)**.** *Let* $A^2_\alpha(D)$ $(\alpha > -1)$ *be the weighted Bergman space*. *Then the analytic Toeplitz operator* $M_{B(z)}$ *is similar to* $\bigoplus_1^n M_z$ *on* $A^2_\alpha(D)$ *if and only if* $B(z)$ *has the form* $B(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n$ $(0 < |a| < 1)$.

Proof. Suppose that $M_B^* \sim M_{z^n}^*$, then M_B^* is a Cowen-Douglas operator (see [7]) with index n, and $\sigma_{lre}(M_B^*) = B(\mathbf{T}) = \mathbf{T}$. Therefore, $B(z)$ is an inner function, but the only inner function in disk algebra is the finite Blaschke product (see [15]).

From $\operatorname{ind}(\lambda - M_B)^* = n\ (\lambda \in D)$, $B(z)$ is an n-Blaschke product. Moreover, $B(z)$ has the form $B(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n\ (0 < |a| < 1)$.

Conversely, let $B(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n\ (0 < |a| < 1)$. By Theorems 4 and 6, $M_{B(z)} \sim M_{z^n}$ and $M_{z^n} \sim \overset{n}{\underset{1}{\oplus}} M_z$. Therefore, we have $M_{B(z)} \sim \overset{n}{\underset{1}{\oplus}} M_z$. □

4. The Commutant of Some Analytic Toeplitz Operators

In this section, we will characterize the commutant of some analytic Toeplitz operators using the Main theorem.

Theorem 8. *Let $F \in H^\infty(D)$ and $F = \varphi B(z)$ be its outer-inner factorization, where $B(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n\ (0 < |a| < 1)$. If for some $0 \neq \lambda \in C$, $\varphi - \lambda$ is divisible by $B(z)$, then $\mathcal{A}'(M_F) = \mathcal{A}'(M_{B(z)}) \cap \mathcal{A}'(M_\varphi)$.*

Proof. We consider $M_F^* = M_{B(z)}^* M_\varphi^*$, by Main theorem, we can find an invertible operator Y such that $YM_{B(z)}^*Y^{-1} = \overset{n}{\underset{1}{\oplus}} M_z^*$. So $A = YM_F^*Y^{-1} = YM_{B(z)}^*Y^{-1}YM_\varphi^*Y^{-1} = \overset{n}{\underset{1}{\oplus}} M_z^*T$, where $T = YM_\varphi^*Y^{-1}$. For two operators T_1 and T_2, if $T_1 \sim T_2$, then $\mathcal{A}'(T_1) \cong \mathcal{A}'(T_2)$. We only need to prove that $\mathcal{A}'(A) = \mathcal{A}'\left(\overset{n}{\underset{1}{\oplus}} M_z^*\right) \cap \mathcal{A}'(T)$. From the condition of theorem, there exists a $\lambda \neq 0$ such that $\varphi - \lambda = B(z) f$, then $T - \tilde{\lambda} I = \overset{n}{\underset{1}{\oplus}} M_z^* S$, and $S \in \mathcal{A}'\left(\overset{n}{\underset{1}{\oplus}} M_z^*\right)$. So

$$A = \overset{n}{\underset{1}{\oplus}} M_z^* \left(\overset{n}{\underset{1}{\oplus}} M_z^* S + \bar{\lambda} I\right) = \left(\overset{n}{\underset{1}{\oplus}} M_z^*\right)^2 S + \bar{\lambda} \overset{n}{\underset{1}{\oplus}} M_z^*.$$

Denote the orthonormal basis of $A_\alpha^2(D)$ by $\left\{1, \sqrt{2+\alpha}\, z, \sqrt{\frac{(3+\alpha)(2+\alpha)}{2!}} z^2, \ldots\right\}$.

Suppose A is defined on $\overset{n}{\underset{1}{\oplus}} A_\alpha^2(D)$, then $\overset{n}{\underset{1}{\oplus}} A_\alpha^2(D)$ has an orthonormal basis

$$\left\{\left.\begin{pmatrix} \sqrt{\dfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} z^2 \oplus 0 \oplus 0 \oplus \cdots \oplus 0 \\ 0 \oplus \sqrt{\dfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} z^2 \oplus 0 \oplus \cdots \oplus 0 \\ \cdots \\ 0 \oplus 0 \oplus 0 \oplus \cdots \oplus \sqrt{\dfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} z^n \end{pmatrix} \right| n = 0, 1, 2, \ldots \right\}.$$

The operator $\overset{n}{\underset{1}{\oplus}} M_z^*$ admits the following matrix representation with respect to the above basis:

$$\overset{n}{\underset{1}{\oplus}} M_z^* = \begin{bmatrix} 0 & \sqrt{\dfrac{1}{2+\alpha}} I_n & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \sqrt{\dfrac{2}{3+\alpha}} I_n & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{\dfrac{3}{4+\alpha}} I_n & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{\dfrac{n}{n+1+\alpha}} I_n & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}. \tag{4.1}$$

Note that $G \in \mathcal{A}'\left(\overset{n}{\underset{1}{\oplus}} M_z^*\right)$ if and only if

$$G = \begin{bmatrix} G_1 & \sqrt{\frac{1}{2+\alpha}} G_{12} & \sqrt{\frac{2\cdot 1}{(3+\alpha)(2+\alpha)}} G_{13} & \sqrt{\frac{3\cdot 2\cdot 1}{(4+a)(3+\alpha)(2+\alpha)}} G_{14} & \cdots & \sqrt{\frac{(n-1)!\Gamma(2+\alpha)}{\Gamma(n+1+\alpha)0!}} G_{1n} & \cdots \\ 0 & G_1 & \sqrt{\frac{2}{3+\alpha}} G_{12} & \sqrt{\frac{3\cdot 2}{(4+\alpha)(3+\alpha)}} G_{13} & \cdots & \sqrt{\frac{(n-1)!\Gamma(3+\alpha)}{\Gamma(n+1+\alpha)1!}} G_{1,n-1} & \cdots \\ 0 & 0 & G_1 & \sqrt{\frac{3}{4+\alpha}} G_{12} & \cdots & \sqrt{\frac{(n-1)!\Gamma(4+\alpha)}{\Gamma(n+1+\alpha)2!}} G_{1,n-2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{\frac{(n-1)!\Gamma(n+\alpha)}{\Gamma(n+1+\alpha)(n-2)!}} G_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}. \tag{4.2}$$

Since $T,\ S \in \mathcal{A}'\left(\bigoplus_1^n M_z^*\right)$, we have

$$T=\begin{bmatrix} T_1 & \sqrt{\frac{1}{2+\alpha}}T_{12} & \sqrt{\frac{2\cdot 1}{(3+\alpha)(2+\alpha)}}T_{13} & \sqrt{\frac{3\cdot 2\cdot 1}{(4+a)(3+\alpha)(2+\alpha)}}T_{14} & \cdots & \sqrt{\frac{(n-1)!\Gamma(2+\alpha)}{\Gamma(n+1+\alpha)0!}}T_{1n} & \cdots \\ 0 & T_1 & \sqrt{\frac{2}{3+\alpha}}T_{12} & \sqrt{\frac{3\cdot 2}{(4+\alpha)(3+\alpha)}}T_{13} & \cdots & \sqrt{\frac{(n-1)!\Gamma(3+\alpha)}{\Gamma(n+1+\alpha)1!}}T_{1,n-1} & \cdots \\ 0 & 0 & T_1 & \sqrt{\frac{3}{4+\alpha}}T_{12} & \cdots & \sqrt{\frac{(n-1)!\Gamma(4+\alpha)}{\Gamma(n+1+\alpha)2!}}T_{1,n-2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{\frac{(n-1)!\Gamma(n+\alpha)}{\Gamma(n+1+\alpha)(n-2)!}}T_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}, \tag{4.3}$$

$$S=\begin{bmatrix} S_1 & \sqrt{\frac{1}{2+\alpha}}S_{12} & \sqrt{\frac{2\cdot 1}{(3+\alpha)(2+\alpha)}}S_{13} & \sqrt{\frac{3\cdot 2\cdot 1}{(4+a)(3+\alpha)(2+\alpha)}}S_{14} & \cdots & \sqrt{\frac{(n-1)!\Gamma(2+\alpha)}{\Gamma(n+1+\alpha)0!}}S_{1n} & \cdots \\ 0 & S_1 & \sqrt{\frac{2}{3+\alpha}}S_{12} & \sqrt{\frac{3\cdot 2}{(4+\alpha)(3+\alpha)}}S_{13} & \cdots & \sqrt{\frac{(n-1)!\Gamma(3+\alpha)}{\Gamma(n+1+\alpha)1!}}S_{1,n-1} & \cdots \\ 0 & 0 & S_1 & \sqrt{\frac{3}{4+\alpha}}S_{12} & \cdots & \sqrt{\frac{(n-1)!\Gamma(4+\alpha)}{\Gamma(n+1+\alpha)2!}}S_{1,n-2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{\frac{(n-1)!\Gamma(n+\alpha)}{\Gamma(n+1+\alpha)(n-2)!}}S_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}, \tag{4.4}$$

$$A=\bigoplus_1^n M_z^* T$$

$$=\begin{bmatrix} 0 & \sqrt{\frac{1}{2+\alpha}}T_1 & \sqrt{\frac{2\cdot 1}{(3+\alpha)(2+\alpha)}}T_{12} & \sqrt{\frac{3\cdot 2\cdot 1}{(4+a)(3+\alpha)(2+\alpha)}}T_{13} & \cdots & \sqrt{\frac{(n-1)!\Gamma(2+\alpha)}{\Gamma(n+1+\alpha)0!}}T_{1,n-1} & \cdots \\ 0 & 0 & \sqrt{\frac{2}{3+\alpha}}T_1 & \sqrt{\frac{3\cdot 2}{(4+\alpha)(3+\alpha)}}T_{12} & \cdots & \sqrt{\frac{(n-1)!\Gamma(3+\alpha)}{\Gamma(n+1+\alpha)1!}}T_{1,n-2} & \cdots \\ 0 & 0 & 0 & \sqrt{\frac{3}{4+\alpha}}T_1 & \cdots & \sqrt{\frac{(n-1)!\Gamma(4+\alpha)}{\Gamma(n+1+\alpha)2!}}T_{1,n-3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{\frac{(n-1)!\Gamma(n+\alpha)}{\Gamma(n+1+\alpha)(n-2)!}}T_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}, \tag{4.5}$$

$$\left(\bigoplus_1^n M_z^*\right)^2 S =$$

$$\begin{bmatrix} 0 & 0 & \sqrt{\frac{2\cdot 1}{(3+\alpha)(2+\alpha)}}S_1 & \sqrt{\frac{3\cdot 2\cdot 1}{(4+\alpha)(3+\alpha)(2+\alpha)}}S_{12} & \cdots & \sqrt{\frac{(n-1)!\Gamma(2+\alpha)}{\Gamma(n+1+\alpha)0!}}S_{1,n-2} & \cdots \\ 0 & 0 & 0 & \sqrt{\frac{3\cdot 2}{(4+\alpha)(3+\alpha)}}S_1 & \cdots & \sqrt{\frac{(n-1)!\Gamma(3+\alpha)}{\Gamma(n+1+\alpha)1!}}S_{1,n-3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{\frac{(n-1)!\Gamma(n-1+\alpha)}{\Gamma(n+1+\alpha)(n-3)!}}S_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}, \tag{4.6}$$

$$A = \left(\bigoplus_1^n M_z^*\right)^2 S + \overline{\lambda}\bigoplus_1^n M_z^*$$

$$= \begin{bmatrix} 0 & \overline{\lambda}\sqrt{\frac{1}{2+\alpha}}I_n & * & * & * & \cdots \\ 0 & 0 & \overline{\lambda}\sqrt{\frac{2}{3+\alpha}}I_n & * & * & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & \cdots & \overline{\lambda}\sqrt{\frac{n}{n+1+\alpha}}I_n & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}. \tag{4.7}$$

Comparing (4.5) with (4.7), we have $T_1 = \overline{\lambda} I_n$. Thus

$$A = \begin{bmatrix} 0 & \overline{\lambda}\sqrt{\frac{1}{2+\alpha}}I_n & \sqrt{\frac{2\cdot 1}{(3+\alpha)(2+\alpha)}}T_{12} & \sqrt{\frac{3\cdot 2\cdot 1}{(4+\alpha)(3+\alpha)(2+\alpha)}}T_{13} & \cdots & \sqrt{\frac{(n-1)!\Gamma(2+\alpha)}{\Gamma(n+1+\alpha)0!}}T_{1,n-1} & \cdots \\ 0 & 0 & \overline{\lambda}\sqrt{\frac{2}{3+\alpha}}I_n & \sqrt{\frac{3\cdot 2}{(4+\alpha)(3+\alpha)}}T_{12} & \cdots & \sqrt{\frac{(n-1)!\Gamma(3+\alpha)}{\Gamma(n+1+\alpha)1!}}T_{1,n-2} & \cdots \\ 0 & 0 & 0 & \overline{\lambda}\sqrt{\frac{3}{4+\alpha}}I_n & \cdots & \sqrt{\frac{(n-1)!\Gamma(4+\alpha)}{\Gamma(n+1+\alpha)2!}}T_{1,n-3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \overline{\lambda}\sqrt{\frac{(n-1)!\Gamma(n+\alpha)}{\Gamma(n+1+\alpha)(n-2)!}}I_n & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}. \tag{4.8}$$

On one hand, if $Q \in \mathcal{A}'\left(\overset{n}{\underset{1}{\oplus}} M_z^*\right) \cap \mathcal{A}'(T)$, then we have

$$Q \overset{n}{\underset{1}{\bigoplus}} M_z^* = \overset{n}{\underset{1}{\bigoplus}} M_z^* Q \Rightarrow Q \overset{n}{\underset{1}{\bigoplus}} M_z^* T = \overset{n}{\underset{1}{\bigoplus}} M_z^* QT$$

$$\Rightarrow Q \overset{n}{\underset{1}{\bigoplus}} M_z^* T = \overset{n}{\underset{1}{\bigoplus}} M_z^* TQ \Rightarrow QA = AQ \Rightarrow Q \in \mathcal{A}'(A)$$

which means that $\mathcal{A}'\left(\overset{n}{\underset{1}{\oplus}} M_z^*\right) \cap \mathcal{A}'(T) \subseteq \mathcal{A}'(A)$.

On the other hand, if $Q \in \mathcal{A}'(A)$, then in order to prove $\mathcal{A}'(A) \subseteq \mathcal{A}'\left(\overset{n}{\underset{1}{\oplus}} M_z^*\right)$ $\cap\, \mathcal{A}'(T)$, we only need to prove $Q \in \mathcal{A}'\left(\overset{n}{\underset{1}{\oplus}} M_z^*\right)$, and we complete the proof of Theorem 8. In fact,

$$QA = AQ \Rightarrow Q \overset{n}{\underset{1}{\bigoplus}} M_z^* T = \overset{n}{\underset{1}{\bigoplus}} M_z^* TQ$$

$$\Rightarrow QT \overset{n}{\underset{1}{\bigoplus}} M_z^* = T \overset{n}{\underset{1}{\bigoplus}} M_z^* Q$$

$$\Rightarrow QT \overset{n}{\underset{1}{\bigoplus}} M_z^* = TQ \overset{n}{\underset{1}{\bigoplus}} M_z^*$$

$$\Rightarrow (QT - TQ) \overset{n}{\underset{1}{\bigoplus}} M_z^* = 0$$

$$\Rightarrow \overset{n}{\underset{1}{\bigoplus}} M_z (T^* Q^* - Q^* T^*) = 0.$$

Noting that $\overset{n}{\underset{1}{\oplus}} M_z$ is an injective operator, we have $QT = TQ$. Suppose that $Q \in \mathcal{A}'(A)$, then Q admits the following matrix representation:

$$Q = \begin{bmatrix} Q_1 & Q_{12} & Q_{13} & Q_{14} & \cdots \\ 0 & Q_2 & Q_{23} & Q_{24} & \cdots \\ 0 & 0 & Q_3 & Q_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{4.9}$$

From $QA = AQ$, we can get

$$\begin{bmatrix} 0 & \bar{\lambda}\sqrt{\frac{1}{2+\alpha}}Q_1 & \sqrt{\frac{2\cdot 1}{(3+\alpha)(2+\alpha)}}Q_1T_{12} + \bar{\lambda}\sqrt{\frac{2}{3+\alpha}}Q_{12} & QA(1,\ 4) & \cdots \\ 0 & 0 & \bar{\lambda}\sqrt{\frac{2}{3+\alpha}}Q_2 & QA(2,\ 4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \bar{\lambda}\sqrt{\frac{1}{2+\alpha}}Q_2 & \bar{\lambda}\sqrt{\frac{2}{3+\alpha}}Q_{23} + \sqrt{\frac{2\cdot 1}{(3+\alpha)(2+\alpha)}}T_{12}Q_3 & QA(1,\ 4) & \cdots \\ 0 & 0 & \bar{\lambda}\sqrt{\frac{2}{3+\alpha}}Q_3 & QA(2,\ 4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

where

$$QA(1,\ 4) = \sqrt{\frac{3\cdot 2\cdot 1}{(4+\alpha)(3+\alpha)(2+\alpha)}}Q_1T_{13} + \sqrt{\frac{3\cdot 2}{(4+\alpha)(3+\alpha)}}Q_{12}T_{12} + \bar{\lambda}\sqrt{\frac{3}{4+\alpha}}Q_{13},$$

$$QA(2,\ 4) = \sqrt{\frac{3\cdot 2}{(4+\alpha)(3+\alpha)}}Q_2T_{12} + \bar{\lambda}\sqrt{\frac{3}{4+\alpha}}Q_{23},$$

$$AQ(1,\ 4) = \bar{\lambda}\sqrt{\frac{1}{2+\alpha}}Q_{24} + \sqrt{\frac{2\cdot 1}{(3+\alpha)(2+\alpha)}}T_{12}Q_{34} + \sqrt{\frac{3\cdot 2\cdot 1}{(4+\alpha)(3+\alpha)(2+\alpha)}}T_{13}Q_4,$$

$$AQ(2,\ 4) = \bar{\lambda}\sqrt{\frac{2}{3+\alpha}}Q_{34} + \sqrt{\frac{3\cdot 2}{(4+\alpha)(3+\alpha)}}T_{12}Q_4.$$

Comparing $(n,\ n+1)$ $(n \geq 1)$ entries of QA with that of AQ, we obtain $Q_1 = Q_2 = \cdots = Q_n$. Comparing $(n,\ n+2)$ $(n \geq 1)$ entries of QA with that of AQ, we have

$$\overline{\lambda}\left(\sqrt{\frac{(n-1)!\Gamma(n+\alpha)}{\Gamma(n+1+\alpha)(n-2)!}}Q_{n,n+1}-\sqrt{\frac{n!\Gamma(n+1+\alpha)}{\Gamma(n+2+\alpha)(n-1)!}}Q_{n-1,n}\right)$$

$$=\sqrt{\frac{n!\Gamma(n+\alpha)}{\Gamma(n+2+\alpha)(n-2)!}}(Q_1T_{12}-T_{12}Q_1). \tag{4.10}$$

Therefore,

$$\overline{\lambda}\left(\sqrt{\frac{n-1}{n+\alpha}}Q_{n,n+1}-\sqrt{\frac{n}{n+1+\alpha}}Q_{n-1,n}\right)$$

$$=\sqrt{\frac{n(n-1)}{(n+1+\alpha)(n+\alpha)}}(Q_1T_{12}-T_{12}Q_1)\quad(n\geq 2). \tag{4.11}$$

Dividing the two sides of the equality by $\sqrt{\frac{n(n-1)}{(n+1+\alpha)(n+\alpha)}}$, we get

$$\overline{\lambda}\left(\sqrt{\frac{n+1+\alpha}{n}}Q_{n,n+1}-\sqrt{\frac{n+\alpha}{n-1}}Q_{n-1,n}\right)=Q_1T_{12}-T_{12}Q_1\quad(n\geq 2). \tag{4.12}$$

So

$$\overline{\lambda}\left(\sqrt{\frac{n+1+\alpha}{n}}Q_{n,n+1}-\sqrt{\frac{2+\alpha}{1}}Q_{12}\right)=(n-1)(Q_1T_{12}-T_{12}Q_1)\quad(n\geq 2). \tag{4.13}$$

If $Q_1T_{12}-T_{12}Q_1\neq 0$, then $\|(n-1)(Q_1T_{12}-T_{12}Q_1)\|\to+\infty$. But the left side of the equality is bounded. It is a contradiction. Therefore, $Q_1T_{12}=T_{12}Q_1$. Furthermore, $\sqrt{\frac{n+1+\alpha}{n+\alpha}}Q_{n,n+1}=\sqrt{\frac{n}{n-1}}Q_{n-1,n}$ $(n\geq 2)$. So $Q_{12}=\sqrt{\frac{1}{2+\alpha}}Q'_{12}$, $Q_{23}=\sqrt{\frac{2}{3+\alpha}}Q'_{12}$, ..., $Q_{n,n+1}=\sqrt{\frac{n}{n+1+\alpha}}Q'_{12}$. Now we suppose that $(n, n+k)$ entry of Q is $Q_{n,n+k}=\sqrt{\frac{(n+k-1)!\Gamma(n+1+\alpha)}{\Gamma(n+k+1+\alpha)(n-1)!}}Q'_{1,k+1}$. Comparing $(n, n+k+1)$ entry of QA with that of AQ, we have

$$\overline{\lambda}\left(\sqrt{\frac{\Gamma(n+k+2+\alpha)(n-1)!}{(n+k)!\Gamma(n+1+\alpha)}}Q_{n,n+k+1}-\sqrt{\frac{\Gamma(n+k+1+\alpha)(n-2)!}{(n+k-1)!\Gamma(n+\alpha)}}Q_{n-1,n+k}\right)$$

$$=(Q_1T_{1,k+1}-T_{1,k+1}Q_1)+(Q'_{12}T_{1k}-T_{1k}Q'_{12})+\cdots+(Q'_{1k}T_{12}-T_{12}Q'_{1k}). \tag{4.14}$$

Similarly, we can prove that

$$\sqrt{\frac{\Gamma(n+k+2+\alpha)(n-1)!}{(n+k)!\Gamma(n+1+\alpha)}}Q_{n,\,n+k+1} = \sqrt{\frac{\Gamma(n+k+1+\alpha)(n-2)!}{(n+k-1)!\Gamma(n+\alpha)}}Q_{n-1,\,n+k}. \quad (4.15)$$

So

$$\sqrt{\frac{n+k+1+\alpha}{n+\alpha}}Q_{n,\,n+k+1} = \sqrt{\frac{n+k}{n-1}}Q_{n-1,\,n+k} \quad (n \geq 2). \quad (4.16)$$

Therefore,

$$Q_{n,\,n+k+1} = \sqrt{\frac{(n+k)!\Gamma(n+1+\alpha)}{\Gamma(n+k+2+\alpha)(n-1)!}}Q'_{1,\,k+2}. \quad (4.17)$$

Now we complete the proof of Theorem 8.

Lemma 9 (See [3]). *Let* $N \in \mathcal{L}(\mathcal{H})$ *be a nilpotent operator,* $X_0 = \lambda I + N$, $0 \neq \lambda \in C$, *if* $B, A_0, A_1, A_2, \cdots \in \mathcal{L}(\mathcal{H})$ *satisfy*

(a) $\| A_k \| \leq M$, $k = 0, 1, 2, \ldots$ *and*

(b) $A_k X_0 = X_0 A_{k-1} + B$, $k = 1, 2, 3, \ldots$.

Then $A_0 = A_1 = A_2 = \cdots$.

Theorem 10. *Let* $F \in H^\infty(D)$ *and* $F = \varphi B(z)$ *be its outer-inner factorization, where* $B(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n$ $(0 < |a| < 1)$. *If for some invertible operator Y,* $YM_\varphi^* Y^{-1}$ *admits the matrix representation* $T = \{T : T_{jj} = \lambda I_n + N,\ 0 \neq \lambda \in C,\ N$ *is a nilpotent matrix,* $j = 1, 2, \ldots\}$ *with respect to the basis of* $\bigoplus_1^n A_\alpha^2(D)$, *then* $\mathcal{A}'(M_F) = \mathcal{A}'(M_{B(z)}) \cap \mathcal{A}'(M_\varphi)$.

Proof. By the Main Theorem, there exists an invertible operator Y such that $YM_{B(z)}^* Y^{-1} = \bigoplus_1^n M_z^*$. So $A = YM_F^* Y^{-1} = YM_{B(z)}^* Y^{-1} YM_\varphi^* Y^{-1} = \bigoplus_1^n M_z^* T$, where

$T = YM_{\varphi}^{*}Y^{-1}$. Note that $T \in \mathcal{A}'\left(\overset{n}{\underset{1}{\oplus}} M_z^*\right)$. It is easy to see that $\mathcal{A}'(A) \supseteq \mathcal{A}'\left(\overset{n}{\underset{1}{\oplus}} M_z^*\right) \cap \mathcal{A}'(T)$. Next we prove that $\mathcal{A}'(A) \subseteq \mathcal{A}'\left(\overset{n}{\underset{1}{\oplus}} M_z^*\right) \cap \mathcal{A}'(T)$. According to the method from Theorem 8, we have

$$A = \overset{n}{\underset{1}{\oplus}} M_z^* T$$

$$= \begin{bmatrix} 0 & \sqrt{\frac{1}{2+\alpha}}T_1 & \sqrt{\frac{2\cdot 1}{(3+\alpha)(2+\alpha)}}T_{12} & \sqrt{\frac{3\cdot 2\cdot 1}{(4+\alpha)(3+\alpha)(2+\alpha)}}T_{13} & \cdots & \sqrt{\frac{(n-1)!\Gamma(2+\alpha)}{\Gamma(n+1+\alpha)0!}}T_{1,n-1} & \cdots \\ 0 & 0 & \sqrt{\frac{2}{3+\alpha}}T_1 & \sqrt{\frac{3\cdot 2}{(4+\alpha)(3+\alpha)}}T_{12} & \cdots & \sqrt{\frac{(n-1)!\Gamma(3+\alpha)}{\Gamma(n+1+\alpha)1!}}T_{1,n-2} & \cdots \\ 0 & 0 & 0 & \sqrt{\frac{3}{4+\alpha}}T_1 & \cdots & \sqrt{\frac{(n-1)!\Gamma(4+\alpha)}{\Gamma(n+1+\alpha)2!}}T_{1,n-3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{\frac{(n-1)!\Gamma(n+\alpha)}{\Gamma(n+1+\alpha)(n-2)!}}T_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}, \tag{4.18}$$

where $T_1 = \lambda I_n + N$, N is a nilpotent matrix. Suppose that $Q \in \mathcal{A}'(A)$, and Q has the following form:

$$Q = \begin{bmatrix} Q_1 & Q_{12} & Q_{13} & Q_{14} & \cdots \\ 0 & Q_2 & Q_{23} & Q_{24} & \cdots \\ 0 & 0 & Q_3 & Q_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{4.19}$$

From $QA = AQ$, comparing $(n, n+1)(n \geq 1)$ entries of QA with that of AQ, we have $\sqrt{\frac{n}{n+1+\alpha}}Q_nT_1 = \sqrt{\frac{n}{n+1+\alpha}}T_1Q_{n+1}$ $(n \geq 1)$. Since $T_1 = \lambda I_n + N$, applying Lemma 9, we know that $Q_n = Q_{n+1}$ $(n \geq 1)$. The following process is the same as Theorem 8, we omit it. Therefore, we complete the proof of Theorem 10. □

Corollary 11. *Let* $F \in H^{\infty}(D)$ *and* $F = \varphi B(z)$ *be its outer-inner factorization, where* $B(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n$ $(0 < |a| < 1)$. *If for some invertible* *operator* Y, $YM_{\varphi}^{*}Y^{-1}$ *admits the matrix representation* $T = \{T : T_{jj}$ *is a Jordan block*, $j = 1, 2, \ldots\}$ *with respect to the basis of* $\bigoplus_{1}^{n} A_{\alpha}^{2}(D)$, *then* $\mathcal{A}'(M_F) = \mathcal{A}'(M_{B(z)}) \cap \mathcal{A}'(M_{\varphi})$.

Proof. Since $T_{jj} = T_1$ is a Jordan block, there exists some $0 \neq \lambda \in C$ such that $T_1 = \lambda I_n + N$, where N is a nilpotent matrix, using Theorem 10, we can get the result of Corollary 11. □

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