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MATCHING NUMBER AND EDGE COVERING NUMBER ON KRONECKER PRODUCT OF P_n

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Abstract

Let $\alpha'(G)$ and $\beta'(G)$ be the matching number and edge covering number, respectively. The Kronecker product $G_1 \otimes G_2$ of graphs of G_1 and G_2 has vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2) \mid u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$. In this paper, let G be a simple graph of order m, we prove that

$$\alpha'(P_n \otimes G) = \max\left\{n\alpha'(G),\ m\left\lfloor \frac{n}{2} \right\rfloor\right\}$$

and

$$\beta'(P_n \otimes G) = \min\left\{n\beta'(G),\ m\left\lceil \frac{n}{2} \right\rceil\right\}.$$

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1. Introduction

In this paper, graphs must be simple graphs which can be trivial graphs. Let G_1 and G_2 be graphs. Then the Kronecker product of graphs G_1 and G_2, denoted by $G_1 \otimes G_2$, is the graph that $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2) | u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$.

In [1], there are some properties about Kronecker product of graphs. We recall here.

Proposition 1.1. *Let* $H = G_1 \otimes G_2 = (V(H), E(H))$. *Then*

(i) $n(V(H)) = n(V(G_1)n(V(G_2)))$,

(ii) $n(E(H)) = 2n(E(G_1)n(E(G_2)))$,

(iii) *for every* $(u, v) \in V(H)$, $d_H((u, v)) = d_{G_1}(u)d_{G_2}(v)$.

Note that for any graph G, we have $G_1 \otimes G_2 \cong G_2 \otimes G_1$.

Theorem 1.2. *Let* G_1 *and* G_2 *be connected graphs. Then the graph* $H = G_1 \otimes G_2$ *is connected if and only if* G_1 *or* G_2 *contains an odd cycle.*

Theorem 1.3. *Let* G_1 *and* G_2 *be connected graphs with no odd cycle. Then* $G_1 \otimes G_2$ *has exactly two connected components.*

Next we get that general form of graph of Kronecker product of P_n and any simple graph.

Proposition 1.4. *Let G be a connected graph of order m. Then the graph of*

$$P_n \otimes G \text{ is } \bigcup_{i=1}^{n-1} H_i,$$

where $V(H_i) = W_i \cup W_{i+1}$ *for* $i = 1, 2, ..., n-1$; $W_i = (i, 1), (i, 2), ..., (i, m)$; $E(H_i) = \{(i, u)(i+1, v)/uv \in E(G)\}$. *Moreover, if G has no odd cycle, then for each* H_i *has exactly two connected components isomorphic to G.*

Example.

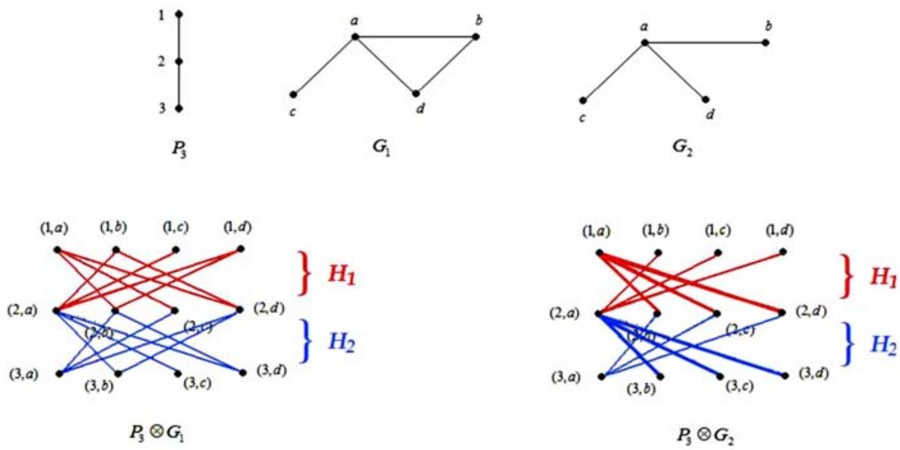

Figure 1. The graphs of $P_3 \otimes G_1$ and $P_3 \otimes G_2$.

Next, we give the definitions about some graph parameters. A subset of the edge set E of G is said to be *matching* or an *independent edge set* of G, if no two distinct edges in M have a common vertex. A matching M is *maximum matching* in G if there is no matching M' of G with $|M'| > |M|$. The cardinality of maximum matching of G is called the *matching number* of G, denoted by $\alpha'(G)$.

An edge of graph G is said *to cover the two vertices incident with it*, and an edge cover of a graph G is a set of edges covering all the vertices of G. The minimum cardinality of an edge cover of a graph G is called the *edge covering number* of G, denoted by $\beta'(G)$.

By definitions of matching number, edge covering number, clearly that $\alpha'(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$ and $\beta'(P_n) = \left\lceil \frac{n}{2} \right\rceil$.

2. Matching Number of the Graph of $P_n \otimes G$

We begin this section by giving the definition and theorem for alternating path and augmenting path, Lemma 2.2 that shows character of matching for each H_i.

Definition 2.1. Given a matching M, an *M-alternating path* is a path that alternates between edges in M and edges not in M. An M-alternating path whose endpoints are unsaturated by M is an *M-augmenting path*.

Theorem 2.1 [2]**.** *A matching M in a graph G is a maximum matching in G if and only if G has no M-augmenting path.*

Next, we give Lemma 2.2 which shows character of matching for each H_i.

Lemma 2.2. *Let* $P_n \otimes G = \bigcup_{i=1}^{n-1} H_i$. *Then for each* H_i, $\alpha'(H_i) = 2\alpha'(G)$.

Proof. Suppose G has no odd cycle, by Proposition 1.4, we get $H_i = 2G$. So $\alpha'(H_i) = 2\alpha'(G)$.

If G has odd cycle, then for each H_i, vertices $(u_i, v) \in W_i$ and $(u_{i+1}, v) \in W_{i+1}$ have $d_{H_i}((u_i, v)) = d_{H_i}((u_{i+1}, v)) = d_G(v)$. Let $\bigcup_{i=1}^{n-1} \overline{H_i} = P_n \otimes (G - \bar{e})$ when $\bar{e}$ is an edge in odd cycle and M be the maximum matching of G. We get $\overline{H_i} = 2(G - \bar{e})$, then

$$\alpha'(\overline{H_i}) = 2\alpha'(G - \bar{e}) = \begin{cases} 2[\alpha'(G) - 1], & \text{if } \bar{e} \text{ is in } M, \\ 2\alpha'(G), & \text{otherwise.} \end{cases}$$

When we add $\bar{e}$ comeback, we get $\alpha'(H_i) = \alpha'(\overline{H_i}) + 1$. Hence $\alpha'(H_i) = 2\alpha' G$. □

Next, we establish Theorem 2.3 for a matching number of $P_n \otimes G$.

Theorem 2.3. *Let G be a connected graph of order m. Then* $\alpha'(P_n \otimes G) = \max\left\{n\alpha'(G), m\left\lfloor \frac{n}{2} \right\rfloor\right\}$.

Proof. Let $V(P_n) = \{u_i, i = 1, 2, ..., n\}$, $V(G) = \{v_j, j = 1, 2, ..., m\}$, $S_i = \{(u_i, v_j) \in V(P_n \otimes G) / j = 1, 2, ..., m\}$, $i = 1, 2, ..., n$ and since $\alpha'(P_n)$

$= \left\lfloor \frac{n}{2} \right\rfloor$. Let $\alpha'(G) = k$, assume that the maximum matching of P_n, G be

$$M_1 = \left\{u_1u_2,\, u_3u_4,\, \ldots,\, u_{2\left\lfloor \frac{n}{2} \right\rfloor - 1}u_{2\left\lfloor \frac{n}{2} \right\rfloor}\right\},\quad M_2 = \{v_jv_{j+1} / j = 1, 3, \ldots, 2k-1\},$$

respectively.

By Lemma 2.2, we have $\alpha'(H_i) = 2\alpha'(G)$. Since $P_n \otimes G$ is $\bigcup_{i=1}^{n-1} H_i$ which have matching in $H_1, H_3, \ldots, H_{2\left\lfloor \frac{n}{2} \right\rfloor - 1}$, $\alpha'(P_n \otimes G) \geq n\alpha'(G)$.

By definition of matching, we get another matching of $P_n \otimes G$ is the set of edges such that incident with vertices in S_i and S_{i+1}, $i = 1, 3, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor - 1$. So $\alpha'(P_n \otimes G) \geq m\left\lfloor \frac{n}{2} \right\rfloor$.

Hence $\alpha'(P_n \otimes G) \geq \max\left\{n\alpha'(G),\, m\left\lfloor \frac{n}{2} \right\rfloor\right\}$.

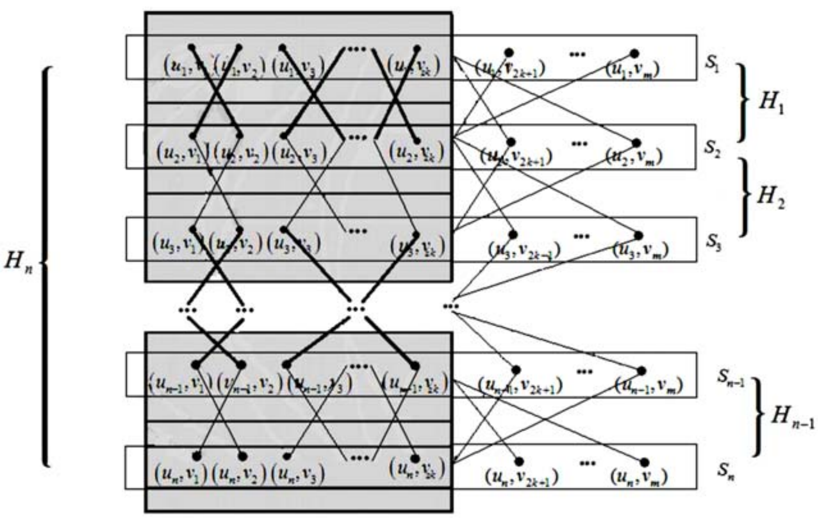

Figure 2. The matching M when $n\alpha'(G) > m\left\lfloor \frac{n}{2} \right\rfloor$ and n is odd.

If $n\alpha'(G) > m\left\lfloor \frac{n}{2} \right\rfloor$, suppose that $\alpha'(P_n \otimes G) > n\alpha'(G)$, then there exists a matching M is an augmenting path. That is not true because each vertices in $P_n \otimes G$ always incident with edges in

$$M = \left[\bigcup_{i=1,3,2\left\lfloor \frac{n}{2} \right\rfloor - 1} \{(u_i, v_j)(u_{i+1}, v_{j+1}) / j = 1, 3, \ldots, 2k-1\} \right]$$

$$\cup \left[\bigcup_{i=1,3,\ldots,2\left\lfloor \frac{n}{2} \right\rfloor - 1} \{(u_i, v_j)(u_{i+1}, v_{j-1}) / j = 2, 4, \ldots, 2k\} \right]$$

and another edges which are not in M:

$$N = \left[\bigcup_{i=2,4,2\left\lfloor \frac{n}{2} \right\rfloor} \{(u_i, v_j)(u_{i+1}, v_{j+1}) / j = 1, 3, \ldots, 2k-1\} \right]$$

$$\cup \left[\bigcup_{i=2,4,2\left\lfloor \frac{n}{2} \right\rfloor} \{(u_i, v_j)(u_{i+1}, v_{j-1}) / j = 2, 4, \ldots, 2k\} \right],$$

so the endpoints of M are saturated by M.

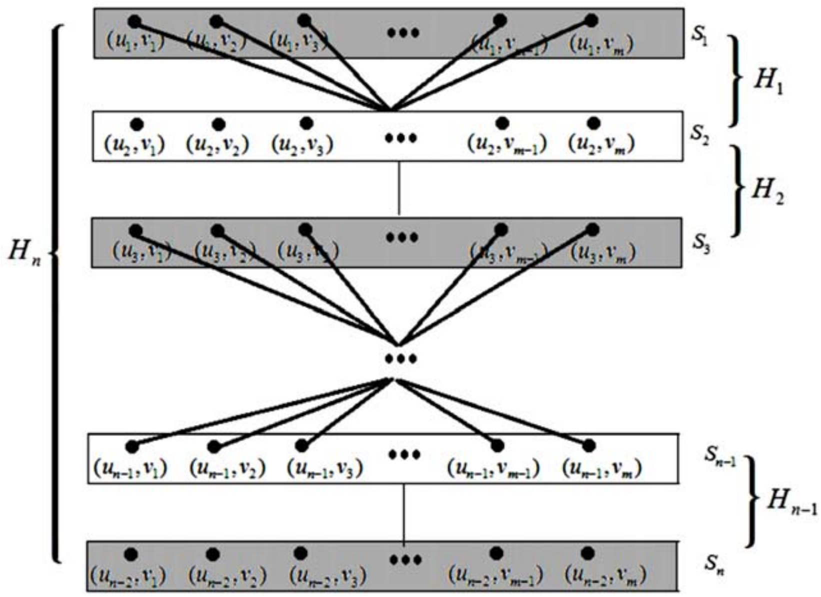

Figure 3. The matching M when $n\alpha'(G) < m\left\lfloor \frac{n}{2} \right\rfloor$ and n is odd.

If $n\alpha'(G) < m\left\lfloor \frac{n}{2} \right\rfloor$, suppose that $\alpha'(P_n \otimes G) > m\left\lfloor \frac{n}{2} \right\rfloor$, it is not true because every S_i have $|S_i| = m$. Hence $\alpha'(P_n \otimes G) = \max\left\{n\alpha'(G),\, m\left\lfloor \frac{n}{2} \right\rfloor\right\}$.

□

3. Edge Covering Number of the Graph of $P_n \otimes G$

We begin this section by giving Lemma 3.1 that shows a relation of matching number and edge covering number and Lemma 3.2 that shows character of edge cover number for each H_i.

Lemma 3.1 [2]**.** *Let G be a simple graph of order n. Then* $\alpha'(G) + \beta'(G) = n$.

Lemma 3.2. *Let* $P_n \otimes G = \bigcup_{i=1}^{n-1} H_i$. *Then for each* H_i, $\beta'(H_i) = 2\beta'(G)$.

Proof. Suppose G has no odd cycle, by Proposition 1.4, we get $H_i = 2G$. So $\beta'(H_i) = 2\beta'(G)$.

If G has odd cycle, then for each $(u_{i+1}, v) \in W_i$, $(u_{i+1}, v) \in W_{i+1}$ in $V(H_i)$ we have $d_{H_i}((u_i, v)) = d_{H_i}(u_{i+1}, v) = d_G(v)$. Let $\bigcup_{i=1}^{n-1} \overline{H_i} = Pn \otimes (G - \bar{e})$ when $\bar{e}$ is an edge in odd cycle and C be the minimum edge covering set of G. We get $\overline{H_i} = 2(G - \bar{e})$, then

$$\beta(\overline{H_i}) = 2\beta(G - \bar{e})$$

$$= \begin{cases} 2[\beta(G) + 2], & \text{if } \bar{e} = xy \in C \text{ with } d(x) > 1 \text{ and } d(y) > 1, \\ 2[\beta(G) - 1], & \text{if } \bar{e} = xy \in C \text{ with } d(x) \geq 1 \text{ or } d(y) \geq 1, \\ 2\beta(G), & \text{otherwise.} \end{cases}$$

When we add $\bar{e}$ comeback, in the case $\beta'(G - \bar{e}) = \beta'(G) - 1$, we get $\beta'(H_i) = \beta'(\overline{H_i}) + 1$. And in the case $\beta'(G - \bar{e}) = \beta'(G) + 2$, we get $\bar{e} = xy \in C$ of G replace edges ux, yv (edge cover of $G - \bar{e}$), so $\beta'(G - \bar{e}) = \beta'(G) - 2$.

Hence $\beta'(H_i) = 2\beta'(G)$. □

Next, we establish Theorem 3.3 for a minimum edge covering number of $P_n \otimes G$.

Theorem 3.3. *Let G be a connected graph of order m. Then* $\beta'(P_n \otimes G) = \min\left\{n\beta'(G),\, m\left\lceil \frac{n}{2} \right\rceil\right\}$.

Proof. Let $V(P_n) = \{u_i,\, i = 1, 2, \ldots, n\}$, $V(G) = \{v_j,\, j = 1, 2, \ldots, m\}$, $S_i = \{(u_i, v_j) \in V(P_n \otimes G) / j = 1, 2, \ldots, m\}$, $i = 1, 2, \ldots, n$ and since $\beta'(P_n) = \left\lceil \frac{n}{2} \right\rceil$. Let $\beta'(G) = k$, assume that the maximum matching of G be M_2, and minimum edge covering set of P_n, G be

$$C_1 = \begin{cases} \{u_1u_2, u_3u_4, ..., u_{n-1}u_n\}, & \text{where } n \text{ is even,} \\ \{u_1u_2, u_3u_4, ..., u_{n-2}u_{n-1}, u_{n-1}u_n\}, & \text{where } n \text{ is odd,} \end{cases}$$

$$C_2 = M_2 \cup \{v_j v / j = 2k+1, 2k+2, ..., m \text{ and } v \text{ is some endvertex of matching in } M_2\}, \text{ respectively.}$$

By Lemma 3.2, we have $\beta'(H_i) = 2\beta'(G)$. Since $P_n \otimes G$ is $\left(\bigcup_{i=1}^{n-1} H_i\right)$ which have edge cover in $H_1, H_3, ..., H_{2\left\lceil \frac{n}{2} \right\rceil - 1}$, $\beta'(P_n \otimes G) \leq n\beta'(G)$.

Since definition of edge cover, we get another edge cover of $P_n \otimes G$ is set of edges, such that incident with vertices in S_i and S_{i+1}, $i = 1, 3, ..., 2\left\lceil \frac{n}{2} \right\rceil - 1$. So $\beta'(P_n \otimes G) \leq m\left\lceil \frac{n}{2} \right\rceil$.

Hence $\beta'(P_n \otimes G) \leq \min\left\{n\beta'(G), m\left\lceil \frac{n}{2} \right\rceil\right\}$.

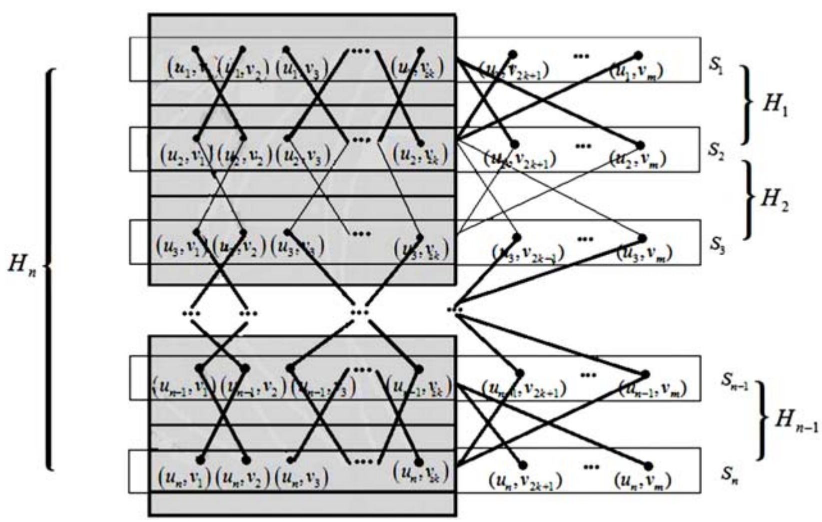

Figure 4. The edge cover when $n\beta'(G) < m\left\lceil \frac{n}{2} \right\rceil$ and n is odd.

If $n\beta'(G) < m\left\lceil \frac{n}{2} \right\rceil$, suppose that $\beta'(P_n \otimes G) < n\beta'(G)$, then there exist edges xy in edge covering of each $H_1, H_3, \ldots, H_{2\left\lceil \frac{n}{2} \right\rceil - 1}$, which is endvertex x and y incident with another edges in edge covering of each $H_1, H_3, \ldots, H_{2\left\lceil \frac{n}{2} \right\rceil - 1}$, it not impossible.

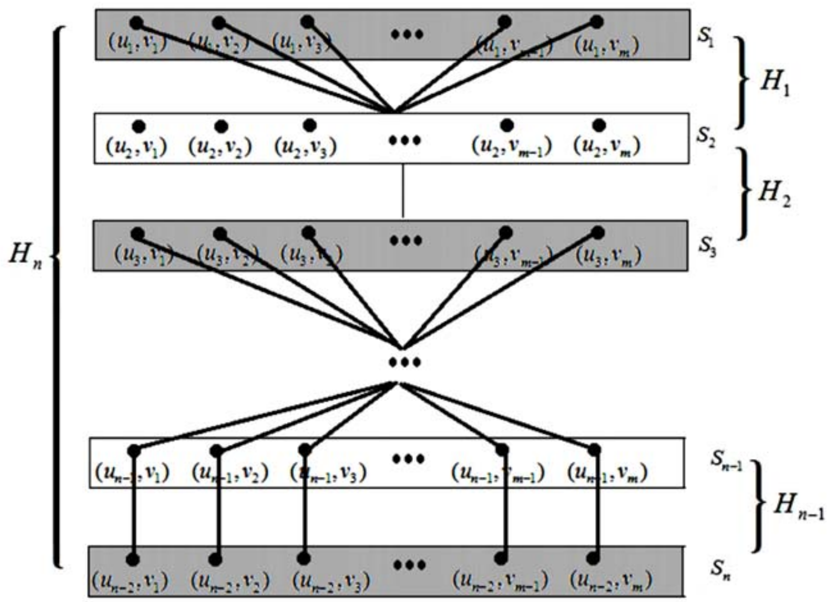

Figure 5. The edge cover when $n\beta'(G) > m\left\lceil \frac{n}{2} \right\rceil$ and n is odd.

If $n\beta'(G) > m\left\lceil \frac{n}{2} \right\rceil$, suppose that $\beta'(P_n \otimes G) > m\left\lceil \frac{n}{2} \right\rceil$, that is not true because every S_i have $|S_i| = m$.

Hence $\beta'(P_n \otimes G) = \min\left\{ n\beta'(G), m\left\lceil \frac{n}{2} \right\rceil \right\}$. □

By Theorem 2.3 and Lemma 3.1, we can also show that

$$\alpha'(P_n \otimes G) + \beta'(P_n \otimes G) = mn$$

$$\max\left\{n\alpha'(G),\ m\left\lfloor\frac{n}{2}\right\rfloor\right\} + \beta'(P_n \otimes G) = mn$$

$$\begin{aligned}\beta'(P_n \otimes G) &= mn - \max\left\{n\alpha'(G),\ m\left\lfloor\frac{n}{2}\right\rfloor\right\}\\ &= mn + \min\left\{-n\alpha'(G),\ -m\left\lfloor\frac{n}{2}\right\rfloor\right\}\\ &= \min\left\{n(m - \alpha'(G)),\ m\left(n - \left\lfloor\frac{n}{2}\right\rfloor\right)\right\}\\ &= \min\left\{n\beta'(G),\ m\left\lceil\frac{n}{2}\right\rceil\right\}.\end{aligned}$$

□

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