JP Journal of Geometry and Topology
Volume 11, Number 3, 2011, Pages 181-191
Published Online: December 2011
Available online at http://pphmj.com/journals/jpgt.htm
Published by Pushpa Publishing House, Allahabad, INDIA

EXAMPLES OF A HASSE DIAGRAM OF FREE CIRCLE ACTIONS IN RATIONAL HOMOTOPY

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Abstract

For free S^1-actions on spaces in the rational homotopy type of a space X, a classification by a poset $\mathcal{P}_{X, S^1}$ is given. It is constructed with respect to certain subgroups of $\mathcal{E}(X_{\mathbb{Q}})$, the group of homotopy classes of homotopy self-equivalences of the rationalized space $X_{\mathbb{Q}}$, associated to S^1-equivariant structures.

1. Introduction

Puppe [8] gave a classification of S^1-actions on a space X having fixed points by Gerstenhaber's deformation [3, 4] of cohomology algebra. In this note, we give a classification of free S^1-actions on X from Klein's point of view that *geometric properties are characterized by their remaining invariant under the transformations of the principal group* [7]. Here the

2010 Mathematics Subject Classification: 55P62, 55P10.

Keywords and phrases: homotopy self-equivalences, free circle action, Sullivan minimal model, S^1-depth.

Received September 27, 2011

principal groups are given by the subgroups of $\mathcal{E}(X_{\mathbb{Q}})$ preserving S^1-actions, where $\mathcal{E}(X_{\mathbb{Q}})$ is the group [9] of homotopy classes of homotopy self-equivalences of a space $X_{\mathbb{Q}}$, the rationalization of X [6].

For a free S^1-action μ on a space Y in the rational homotopy type of a simply connected space X, we put $\mathcal{E}(p_{\mathbb{Q}})$ the set of fibrewise self-equivalences f of the rationalized Borel fibration $p_{\mathbb{Q}} : (ES^1 \times^{\mu}_{S^1} Y)_{\mathbb{Q}} \to BS^1_{\mathbb{Q}}$, which satisfies $p_{\mathbb{Q}} \circ f = p_{\mathbb{Q}}$. Put $\mathcal{E}_{\mu}(X_{\mathbb{Q}})$ the image of the natural homomorphism induced by fibre restrictions

$$F_{\mu} : \mathcal{E}(p_{\mathbb{Q}}) \to \mathcal{E}(X_{\mathbb{Q}}).$$

We are interested in the set $\mathcal{E}_{X, S^1} := \{\mathcal{E}_{\mu}(X_{\mathbb{Q}})\}_{\mu}$ of subgroups of $\mathcal{E}(X_{\mathbb{Q}})$. We define a class of free S^1-actions by $[\mu] = [\tau]$ when $\mathcal{E}_{\mu}(X_{\mathbb{Q}}) = \mathcal{E}_{\tau}(X_{\mathbb{Q}})$ in $\mathcal{E}(X_{\mathbb{Q}})$ and define $[\mu] \le [\tau]$ when there is an inclusion $i : \mathcal{E}_{\mu}(X_{\mathbb{Q}}) \to \mathcal{E}_{\tau}(X_{\mathbb{Q}})$ in $\mathcal{E}(X_{\mathbb{Q}})$. Thus we have a poset of such classes of free S^1-actions on spaces Y in the rational homotopy type of X, added with [*trivial*] for the trivial S^1-action on X,

$$\mathcal{P}_{X, S^1} := \{[\mu], \le\}.$$

Here '$[\mu] < [\tau]$' means that the action μ is 'stronger' than τ. In particular, we put $[\mu] < [\mathit{trivial}]$ for any free S^1-action μ. In this note, we consider $\mathcal{P}_{X, S^1}$ only for free S^1-actions, but it must be suitable for general S^1-actions.

In Section 3, we give the examples in cases

$$(1)\ S^2 \times S^3, \quad (2)\ S^3 \times S^5 \times S^9, \quad (3)\ S^4 \times S^6 \times S^9$$

for X. In Section 4, we define the S^1-depth of a space as a numerical invariant in rational homotopy.

2. Sullivan Model

Let X be a simply connected finite CW complex with the Sullivan minimal model $M(X) = (\Lambda V, d)$ [11]. It is a free $\mathbb{Q}$-commutative differential graded algebra (DGA) generated by the $\mathbb{Q}$-graded vector space $V = \oplus_{i \geq 2} V^i$ of finite type, in which the differential d is decomposable; i.e., $d(V) \subset \Lambda^{\geq 2} V$ and $d \circ d = 0$. Denote the degree of a homogeneous element x of a graded algebra as $|x|$. Note that $M(X)$ determines the rational homotopy type of X. In particular, $H^*(M(X)) \cong H^*(X; \mathbb{Q})$ and $V^n \cong Hom(\pi_n(X), \mathbb{Q})$ for any n. Refer to [1] for details.

When the circle S^1 acts on X by $\mu : S^1 \times X \to X$, the model of the Borel fibration $X \to ES^1 \times^{\mu}_{S^1} X \to BS^1$ is given by a relative Sullivan algebra

$$(\mathbb{Q}[t], 0) \to \left(\mathbb{Q}[t] \otimes \wedge V, D\right) \xrightarrow{p_t} (\wedge V, d) \qquad (*)$$

with $|t| = 2$, $Dt = 0$ and $Dv \equiv dv$ modulo the ideal (t) for $v \in V$.

Proposition 2.1 [5, Proposition 4.2]**.** *For a finite simply connected complex X, there is a free S^1-action on a finite simply connected complex Y with $Y_{\mathbb{Q}} \simeq X_{\mathbb{Q}}$ if and only if there is a relative Sullivan algebra $(*)$ satisfying* $\dim H^*(\mathbb{Q}[t] \otimes \wedge V, D) < \infty$.

For the group of DGA-homotopy classes of DGA-automorphisms $Aut(\Lambda V, d)$ of $(\Lambda V, d)$, it folds that

$$\mathcal{E}(X_{\mathbb{Q}}) = Aut(\Lambda V, d)$$

[11]. Denote by $Aut_t(\mathbb{Q}[t] \otimes \Lambda V, D)$ the group of DGA-homotopy classes of DGA-automorphisms f of $(\mathbb{Q}[t] \otimes \Lambda V, D)$ with $f(t) = t$. Then

$$\mathcal{E}(p_{\mathbb{Q}}) = Aut_t(\mathbb{Q}[t] \otimes \Lambda V, D)$$

and $F_\mu : \mathcal{E}(p_{\mathbb{Q}}) \to \mathcal{E}(X_{\mathbb{Q}})$ is equivalent to

$$F'_\mu : Aut_t(\mathbb{Q}[t] \otimes \Lambda V,\, D) \to Aut(\Lambda V,\, d)$$

with $F'_\mu(f)(v) = p_t(f(v))$ for $v \in V$.

3. The Examples

(1) When $X = S^2 \times S^3$, $M(X) = (\Lambda(v, u_1, u_2), d)$ with $|v| = 2$, $|u_1| = |u_2| = 3$, $dv = du_2 = 0$ and $du_1 = v^2$. Put a relative Sullivan algebra $(*)$ by

$$Du_1 = v^2 + \alpha t^2, \quad Du_2 = vt, \quad \alpha \in \mathbb{Q}^* = \mathbb{Q} - \{0\}$$

and suppose that certain S^1-actions μ_1 and μ_2 make $\alpha \in (\mathbb{Q}^*)^2$ and $\alpha \notin (\mathbb{Q}^*)^2$, respectively. Then the Hasse diagram

$$\begin{tikzcd} & {[trivial]} & \\ {[\mu_1]} \arrow[ur] & & {[\mu_2]} \arrow[ul] \end{tikzcd}$$

is induced by the inclusions

$$\begin{tikzcd} & \mathcal{E}(X_{\mathbb{Q}}) & \\ \mathcal{E}_{\mu_1}(X_{\mathbb{Q}}) \arrow[ur] & & \mathcal{E}_{\mu_2}(X_{\mathbb{Q}}). \arrow[ul] \end{tikzcd}$$

Indeed, for the basis v, u_1, u_2, we can represent as

$$\mathcal{E}(X_{\mathbb{Q}}) = \left\{ \begin{pmatrix} a & & \\ & a^2 & c \\ & & b \end{pmatrix}; a, b \in \mathbb{Q}^*, c \in \mathbb{Q} \right\} \cong \mathbb{Q}^* \times \mathbb{Q}^* \ltimes \mathbb{Q},$$

$$\mathcal{E}_{\mu_1}(X_{\mathbb{Q}}) = \left\{ \begin{pmatrix} a & & \\ & a^2 & \\ & & 1 \end{pmatrix}; a \in \mathbb{Q}^* \right\} \cong \mathbb{Q}^* \text{ when } \alpha \in (\mathbb{Q}^*)^2,$$

$$\mathcal{E}_{\mu_2}(X_{\mathbb{Q}}) = \begin{pmatrix} \pm 1 & & \\ & 1 & \\ & & \pm 1 \end{pmatrix} \cong \mathbb{Z}_2 (:= \{\pm 1\}) \text{ when } \alpha \notin (\mathbb{Q}^*)^2$$

since $\alpha \in (\mathbb{Q}^*)^2$ if and only if $(\mathbb{Q}[t] \otimes \wedge V, D) \cong (\mathbb{Q}[t] \otimes \wedge V, D')$ with $D'u_1 = v^2$ and $D'u_2 = t^2$. (Mimura-Shiga, Bull. Belg. M.S.S.).

(2) When $X = S^3 \times S^5 \times S^9$, $M(X) = (\Lambda(v_1, v_2, v_3), 0)$ with $|v_1| = 3$, $|v_2| = 5$, $|v_3| = 9$. Then the group of DGA-automorphisms $Aut(\Lambda V, 0)$ $= \mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}^* = \{(a_1, a_2, a_3) | a_i \in \mathbb{Q}^*\}$ with $f(v_i) = a_i v_i$ for $i = 1, 2, 3$. Then the Hasse diagram of inclusions of $\mathcal{E}_{X, S^1}$ is

$$\begin{tikzcd}
& \mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}^* & \\
\mathbb{Q}^* \times \mathbb{Q}^* \arrow[ur, "i_1"] & \mathbb{Q}^* \times \mathbb{Q}^* \arrow[u, "i_2"] & \mathbb{Q}^* \times \mathbb{Q}^* \arrow[ul, "i_3"'] \\
& & \mathbb{Q}^* \arrow[u, "i_4"]
\end{tikzcd}$$

where $i_1(a, b) = (1, a, b)$, $i_2(a, b) = (a, 1, b)$, $i_3(a, b) = (a, b, 1)$ and $i_4(a) =$ (a, a^{-1}) for $a, b \in \mathbb{Q}^*$. The actions $\mu_i : S^1 \times X \to X$ with $\mathcal{E}_{\mu_i}(X_{\mathbb{Q}})$ $= \mathbb{Q}^* \times \mathbb{Q}^*$ are given by $Dv_i = t^{n_i}$ and $Dv_j = 0$ for $j \neq i$ $(i = 1, 2, 3)$ with $n_1 = 2$, $n_2 = 3$, $n_3 = 5$, respectively. Also, from Proposition 2.1, a free action μ_4 is guaranteed by the model of $Dv_3 = v_1 v_2 t + t^5$, where $\mathcal{E}_{\mu_4}(X_{\mathbb{Q}}) = \mathbb{Q}^*$. The elements of $\mathcal{E}_{\mu_4}(X_{\mathbb{Q}})$ are represented by $a \in \mathbb{Q}^*$ such that $f(v_1) = av_1$, $f(v_2) = a^{-1}v_2$ and $f(v_3) = v_3$. (It is given by a perturbation of the differential $Dv_3 = t^5$ of the model of the action μ_3.)

Thus there is given the Hasse diagram of the poset $\mathcal{P}_{X, S^1} =$ $\{[trivial], [\mu_1], [\mu_2], [\mu_3], [\mu_4]\}$ as

$$\begin{tikzcd}
 & {[trivial]} & \\
{[\mu_1]} \arrow[ur] & {[\mu_2]} \arrow[u] & {[\mu_3]} \arrow[ul] \\
 & & {[\mu_4].} \arrow[u]
\end{tikzcd}$$

(3) When $X = S^4 \times S^6 \times S^9$,

$$M(X) = (\Lambda V, d) = (\Lambda(v_1, v_2, u_1, u_2, u_3), d),$$

where $|v_1| = 4$, $|v_2| = 6$, $|u_1| = 7$, $|u_2| = 11$, $|u_3| = 9$, $dv_i = du_3 = 0$, $du_1 = v_1^2$ and $du_2 = v_2^2$. The model of the Borel space $ES^1 \times_{S^1}^{\mu} Y$ is given by $Dv_1 = Dv_2 = 0$ and

$$Du_1 = v_1^2 + a_{1,1}v_2t + a_{1,2}v_1t^2 + a_{1,3}t^4,$$

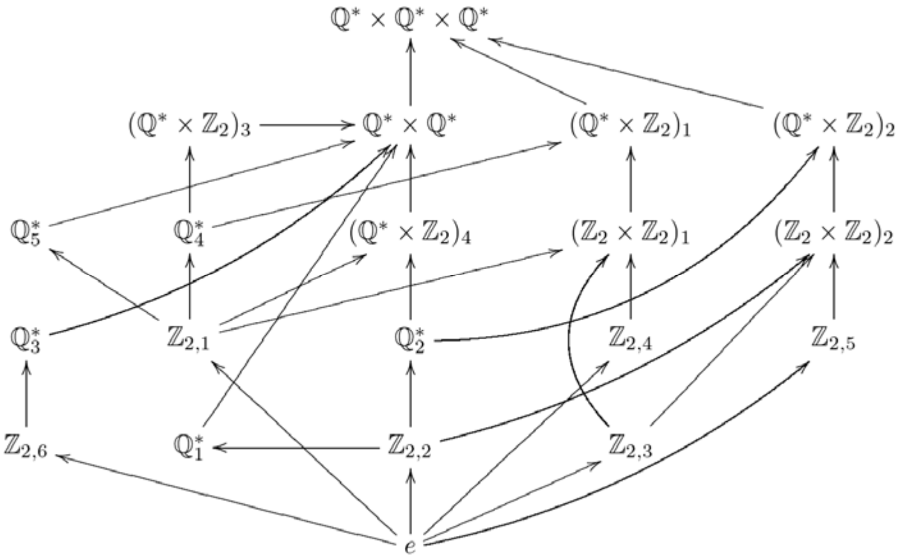
$$Du_2 = v_2^2 + a_{2,1}v_2t^3 + a_{2,1}v_1t^4 + a_{2,3}t^6,$$

$$Du_3 = a_{3,1}v_2t^2 + a_{3,2}v_1t^3 + a_{3,3}t^5,$$

where Du_1, Du_2, Du_3 is a regular sequence in $\mathbb{Q}[v_1, v_2, t]$ and $a_{ij} \in \mathbb{Q}$. That is, the necessary and sufficient conditions for $\dim H^*(\mathbb{Q}[t] \otimes \wedge V, D) < \infty$. Then the algebraic set $V(Du_1, Du_2, Du_3)$ in $\mathbb{C}^3$ is $(0, 0, 0)$ [10, Lemma 3.5]. Furthermore, we can assume that the coefficients $a_{i,j}$ are 0 or ± 1 for our purpose. Recall $\mathcal{E}(X_{\mathbb{Q}}) = Aut(\Lambda V, d)$ is represented for the basis v_1, v_2, u_1, u_2, u_3 as

$$\left\{ \begin{pmatrix} a & & & & \\ & b & & & \\ & & a^2 & & \\ & & & b^2 & \\ & & & & c \end{pmatrix}; a, b, c \in \mathbb{Q}^* \right\} \cong \{(a, b, c)\} = \mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}^*.$$

Then the Hasse diagram of inclusions of $\mathcal{E}_{X, S^1}$ is given by the direct calculations as



where $\mathbb{Q}_n^*$ is $\mathbb{Q}^*$ and $\mathbb{Z}_{2,n}$ is $\mathbb{Z}_2 = \{\pm 1\}$ for any index n. All the groups are given by the following relative Sullivan algebras. First, if $Du_1 = v_1^2$, $Du_2 = v_2^2$ and $Du_3 = t^5$, then $\mathcal{E}_\alpha = \{(a, b, 1) \mid a, b \in \mathbb{Q}^*\} \cong \mathbb{Q}^* \times \mathbb{Q}^*$. Next, we have the tables as

$\mathbb{Q}^* \times \mathbb{Z}_2$	Du_1	Du_2	Du_3	τ_i
$(\mathbb{Q}^* \times \mathbb{Z}_2)_1$	v_1^2	$v_2^2 - t^6$	$v_2 t^2$	$\mathcal{E}_{\tau_1} = \{(a, \pm 1, \pm 1)\}$
$(\mathbb{Q}^* \times \mathbb{Z}_2)_2$	$v_1^2 - t^4$	v_2^2	$v_1 t^3$	$\mathcal{E}_{\tau_2} = \{(\pm 1, b, \pm 1)\}$
$(\mathbb{Q}^* \times \mathbb{Z}_2)_3$	v_1^2	$v_2^2 + t^6$	t^5	$\mathcal{E}_{\tau_3} = \{(a, \pm 1, 1)\}$
$(\mathbb{Q}^* \times \mathbb{Z}_2)_4$	$v_1^2 + t^4$	v_2^2	t^5	$\mathcal{E}_{\tau_4} = \{(\pm 1, b, 1)\}$

$\mathbb{Z}_2 \times \mathbb{Z}_2$	Du_1	Du_2	Du_3	β_i
$(\mathbb{Z}_2 \times \mathbb{Z}_2)_1$	$v_1^2 + t^4$	$v_2^2 - t^6$	$v_2 t^2$	$\mathcal{E}_{\beta_1} = \{(\varepsilon, \pm 1, \pm 1)\}$
$(\mathbb{Z}_2 \times \mathbb{Z}_2)_2$	$v_1^2 - t^4$	$v_2^2 + t^6$	$v_1 t^3$	$\mathcal{E}_{\beta_2} = \{(\pm 1, \varepsilon, \pm 1)\}$

Here ε means 1 or –1.

$\mathbb{Q}^*$	Du_1	Du_2	Du_3	μ_i
$\mathbb{Q}_1^*$	v_1^2	$v_2^2 + v_1t^4$	t^5	$\mathcal{E}_{\mu_1} = \{(b^2, b, 1)\}$
$\mathbb{Q}_2^*$	v_1^2	v_2^2	$v_1t^3 + t^5$	$\mathcal{E}_{\mu_2} = \{(1, b, 1)\}$
$\mathbb{Q}_3^*$	v_1^2	$v_2^2 + v_1v_2t$	t^5	$\mathcal{E}_{\mu_3} = \{(a, a, 1)\}$
$\mathbb{Q}_4^*$	v_1^2	v_2^2	$v_2t^2 + t^5$	$\mathcal{E}_{\mu_4} = \{(a, 1, 1)\}$
$\mathbb{Q}_5^*$	$v_1^2 + v_2t$	v_2^2	t^5	$\mathcal{E}_{\mu_5} = \{(a, a^2, 1)\}$

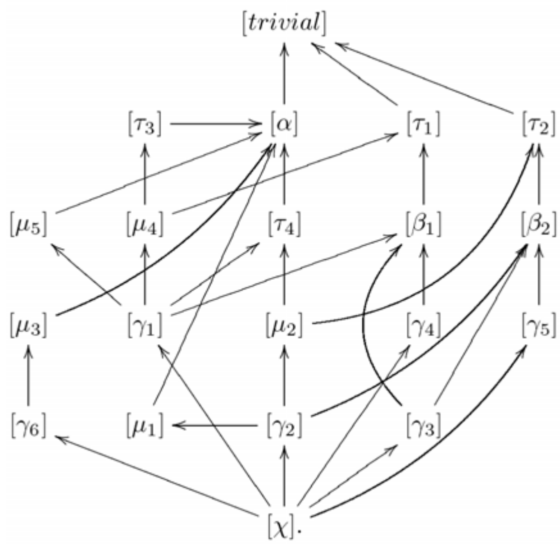
$\mathbb{Z}_2$	Du_1	Du_2	Du_3	γ_i
$\mathbb{Z}_{2,1}$	$v_1^2 + v_2t$	v_2^2	$v_2t^2 + t^5$	$\mathcal{E}_{\gamma_1} = \{(\pm1, 1, 1)\}$
$\mathbb{Z}_{2,2}$	v_1^2	$v_2^2 + v_1t^4$	$v_1t^3 + t^5$	$\mathcal{E}_{\gamma_2} = \{(1, \pm1, 1)\}$
$\mathbb{Z}_{2,3}$	v_1^2	$v_2^2 + t^6$	$v_1t^3 + v_2t^2$	$\mathcal{E}_{\gamma_3} = \{(\pm1, \pm1, \pm1)\}$
$\mathbb{Z}_{2,4}$	$v_1^2 + t^4$	$v_2^2 + v_1t^4$	v_2t^2	$\mathcal{E}_{\gamma_4} = \{(1, \pm1, \pm1)\}$
$\mathbb{Z}_{2,5}$	$v_1^2 - t^4$	$v_2^2 + v_2t^3$	v_1t^3	$\mathcal{E}_{\gamma_5} = \{(\pm1, 1, \pm1)\}$
$\mathbb{Z}_{2,6}$	$v_1^2 + t^4$	$v_2^2 + v_1v_2t$	t^5	$\mathcal{E}_{\gamma_6} = \{(\pm1, \pm1, 1)\}$

Finally, if $Du_1 = v_1^2$, $Du_2 = v_2^2$ and $Du_3 = v_1t^3 + v_2t^2 + t^5$, then $\mathcal{E}_\chi = \{(1, 1, 1)\} = e$.

Thus there are 19-type free S^1-actions with

$$\mathcal{P}_{X,S^1} = \{[trivial], [\alpha], [\tau_1], ..., [\tau_4], [\beta_1], [\beta_2], [\mu_1], ..., [\mu_5], [\gamma_1], ..., [\gamma_6], [\chi]\}$$

and the Hasse diagram is



4. S^1-depth

Define the S^1-depth of X as the height of Hasse diagram of $\mathcal{P}_{X,S^1}$, i.e.,

$$S^1\text{-depth}(X) := \max\{n \mid [trivial] > [\mu_{i_1}] > [\mu_{i_2}] > \cdots > [\mu_{i_n}]$$

$$\text{for } \mathcal{P}_{X,S^1} = \{[\mu_i]\}\}.$$

Of course, $S^1\text{-depth}(X) = 0$ if the rational toral rank [5] of X is zero and $S^1\text{-depth}(X) = \infty$ if there does not exist such an integer. Since $\mathcal{E}((X \times Y)_{\mathbb{Q}}) \supset \mathcal{E}(X_{\mathbb{Q}}) \times \mathcal{E}(Y_{\mathbb{Q}})$ as groups, we have

$$S^1\text{-depth}(X \times Y) \geq S^1\text{-depth}(X) + S^1\text{-depth}(Y).$$

We easily see $S^1\text{-depth}(S^4 \times S^6 \times S^3) = 1$. On the other hand, in Section 3(3), we have $S^1\text{-depth}(S^4 \times S^6 \times S^9) = 5$ from

$$[trivial] > [\alpha] > [\tau_4] > [\mu_2] > [\gamma_2] > [\chi]$$

$$(\text{or } [trivial] > [\alpha] > [\tau_3] > [\mu_4] > [\gamma_1] > [\chi]).$$

Example 4.1. S^1-depth$(S^3 \times S^5 \times S^9 \times S^{13} \times S^{19}) \geq 5$.

Indeed, we put the model $(\Lambda(v_1, v_2, v_3, v_4, v_5), 0)$ with $|v_1| = 3$, $|v_2| = 5$, $|v_3| = 9$, $|v_4| = 13$ and $|v_5| = 19$.

If μ_1 is given by $Dv_1 = Dv_2 = Dv_3 = Dv_4 = 0$ and $Dv_5 = t^{10}$, then $\mathcal{E}_{\mu_1} = \{(a, b, c, d, 1); a, b, c, d \in \mathbb{Q}^*\} = \mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}^*$.

If μ_2 is given by $Dv_1 = Dv_2 = Dv_3 = Dv_4 = 0$ and $Dv_5 = v_1v_4t^2 + t^{10}$, then $\mathcal{E}_{\mu_2} = \{(a, b, c, a^{-1}, 1)\} = \mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}^*$.

If μ_3 is given by $Dv_1 = Dv_2 = Dv_3 = Dv_4 = 0$ and $Dv_5 = v_1v_4t^2 + v_2v_3t^3 + t^{10}$, then $\mathcal{E}_{\mu_3} = \{(a, b, b^{-1}, a^{-1}, 1)\} = \mathbb{Q}^* \times \mathbb{Q}^*$.

If μ_4 is given by $Dv_1 = Dv_2 = Dv_3 = 0$, $Dv_4 = v_1v_3t$ and $Dv_5 = v_1v_4t^2 + v_2v_3t^3 + t^{10}$, then $\mathcal{E}_{\mu_4} = \{(a, a^2, a^{-2}, a^{-1}, 1)\} = \mathbb{Q}^*$.

If μ_5 is given by $Dv_1 = Dv_2 = 0$, $Dv_3 = v_1v_2t$, $Dv_4 = v_1v_3t$ and $Dv_5 = v_1v_4t^2 + v_2v_3t^3 + t^{10}$, then $\mathcal{E}_{\mu_5} = \{(1, 1, 1, 1, 1)\} = e$.

Thus we have

$$[\mathit{trivial}] > [\mu_1] > [\mu_2] > [\mu_3] > [\mu_4] > [\mu_5]$$

from the sequence of $\mathcal{E}_{X, S^1}$,

$$\mathbb{Q}^{*\times 5} \supset \mathbb{Q}^{*\times 4} \supset \mathbb{Q}^{*\times 3} \supset \mathbb{Q}^{*\times 2} \supset \mathbb{Q}^* \supset e.$$

References

[1] Y. Félix, S. Halperin and J.-C. Thomas, Rational Homotopy Theory, Springer GTM, Vol. 205, 2001.

[2] Y. Félix, J. Oprea and D. Tanré, Algebraic Models in Geometry, Oxford GTM, Vol. 17, 2008.

[3] M. Gerstenhaber, On the deformation of rings and algebras, Ann. of Math. 79 (1964), 59-103.

[4] M. Gerstenhaber, On the deformation of rings and algebras IV, Ann. of Math. 99 (1974), 257-276.

[5] S. Halperin, Rational homotopy and torus actions, London Math. Soc. Lecture Note Series 93, Cambridge Univ. Press, 1985, pp. 293-306.

[6] P. Hilton, G. Mislin and J. Roitberg, Localization of nilpotent groups and spaces, North-Holland Math. Studies 15 (1975).

[7] F. Klein, Erlangen Program, 1872.

[8] V. Puppe, Cohomology of fixed sets and deformation of algebras, Manuscripta Math. 23 (1978), 343-354.

[9] J. W. Rutter, Spaces of homotopy self-equivalences, Springer L.N.M., Vol. 1662, 1997.

[10] H. Shiga and M. Tezuka, Rational fibrations, homogeneous spaces with positive Euler characteristic and Jacobians, Ann. Inst. Fourier 37 (1987), 81-106.

[11] D. Sullivan, Infinitesimal computations in topology, Publ. I.H.E.S. 47 (1977), 269-331.