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EXISTENCE AND MULTIPLICITY OF SYMMETRIC POSITIVE SOLUTIONS FOR THREE-POINT BOUNDARY VALUE PROBLEM

Zheng Wu and Lianglong Wang*

School of Mathematical Science
Anhui University
Hefei, 230039, P. R. China
e-mail: wangll@ahu.edu.cn

Abstract

This paper is concerned with the existence and multiplicity of symmetric positive solutions for the following second-order three-point boundary value problem:

$$u''(t) + a(t) f(t, u(t)) = 0, \quad 0 < t < 1,$$

$$u(t) = u(1-t), \quad u'(0) - u'(1) = u\left(\frac{1}{2}\right),$$

where $a : (0, 1) \to [0, \infty)$ is symmetric on $(0, 1)$ and may be singular at $t = 0$ and $t = 1$, $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous and

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*Corresponding author

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$f(\cdot, u)$ is symmetric on $[0, 1]$ for all $u \in [0, \infty)$. By using Leggett-Williams' fixed point theorem, sufficient conditions are obtained that guarantee the existence of at least three symmetric positive solutions to the above boundary value problem. As applications, three examples are given to illustrate the main results and their differences.

1. Introduction

The existence and multiplicity of positive solutions for second-order nonlinear boundary value problem have been studied by many authors using the fixed point theorems, see [1-3] and the references therein.

In this paper, the existence of symmetric positive solutions for the following second-order three-point boundary value problems (BVP):

$$u''(t) + a(t) f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(t) = u(1-t), \quad u'(0) - u'(1) = u\left(\frac{1}{2}\right), \tag{1.2}$$

is studied, where $a : (0, 1) \to [0, \infty)$ is symmetric on $(0, 1)$ and may be singular at $t = 0$ and $t = 1$, $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous and $f(\cdot, u)$ is symmetric on $[0, 1]$ for all $u \in [0, \infty)$.

The three-point boundary value problems for ordinary differential equations arise in a variety of applied mathematics and physics. For instance, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multi-point BVP; also, many problems in the theory of elastic stability can be handled by multi-point problems (see [4]).

The existence and multiplicity of positive solutions for nonlinear second-order three-point boundary value problem has been studied by many authors by applying the Leray-Schauder continuations theorem, nonlinear alternative of Leray-Schauder, coincidence degree theory, or Krasnoselskii's fixed point theorem and so on. For example, see [5-14] and the references therein. Recently, Henderson and Thompson [15] and Li and Zhang [16] studied the multiple symmetric positive and nonnegative solutions of second-order ordinary differential equations. Yao [17] considered the existence and iterations of n symmetric positive solutions for a singular two-point boundary value problem. Kosmatov [18] obtained sufficient conditions for the existence of positive solutions for an m-point boundary value problem.

Very recently, Sun [19] studied the existence of symmetric positive solutions to the BVP (1.1)-(1.2) by using Krasonselskii's fixed-point theorem. Motivated by the papers mentioned above, in this paper, we investigate the BVP (1.1)-(1.2) and provide sufficient conditions for the existence of at least three symmetric positive solutions of BVP (1.1)-(1.2).

This paper is organized as follows: In the next section, we present some necessary definitions and preliminary lemmas that will be used to prove our main results. In Section 3, we discuss the existence of at least three symmetric positive solutions for the BVP (1.1)-(1.2). Finally, some examples are given to illustrate our main results in Section 4.

2. Preliminaries

In this section, we provide some background material from the theory of cones in Banach spaces, in order that this paper be self-contained. We also state a fixed point theorem due to Leggett and Williams [20] for multiple fixed points of a cone preserving operator.

Definition 2.1. Let E be a real Banach space. Then a nonempty convex closed set $P \subset E$ is said to be a *cone* provided that

(1) $au \in P$ for all $u \in P$ and all $a \geq 0$ and

(2) $u, -u \in P$ implies $u = 0$.

Note that every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if $y - x \in P$.

Definition 2.2. A map α is said to be a *nonnegative continuous concave functional* on E if $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y),$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Definition 2.3. An operator is called *completely continuous* if it is continuous and maps bounded sets into pre-compact sets.

Definition 2.4. The function u is said to be *symmetric* on $[0, 1]$ if

$$u(t) = u(1-t), \quad t \in [0, 1].$$

Definition 2.5. The function u is called a *symmetric positive solution* of the BVP (1.1)-(1.2) if u is symmetric and positive on $[0, 1]$ and satisfies the differential equation (1.1) with the boundary value condition (1.2).

Definition 2.6. For numbers $0 < a < b$ and α nonnegative continuous concave functional on E, define convex sets P_r and $P(\alpha, a, b)$, respectively, by

$$P_r = \{y \in P : \| y \| < r\}$$

and

$$P(\alpha, a, b) = \{y \in P : a \leq \alpha(y), \| y \| < b\}.$$

To obtain multiple symmetric positive solutions of BVP (1.1)-(1.2), the following fixed point theorem of Leggett and Williams will be fundamental.

Theorem 2.1 [20]**.** *Let* $A : \overline{P}_c \to \overline{P}_c$ *be a completely continuous and* α *be a nonnegative continuous concave functional on P such that* $\alpha(x) \leq \| x \|$ *for all* $x \in \overline{P}_c$. *Suppose there exist* $0 < d < a < b \leq c$ *such that*

(C1) $\{x \in P(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$, *and* $\alpha(Ax) > a$, *for* $x \in P(\alpha, a, b)$,

(C2) $\| Ax \| \leq d$, *for* $x \in \overline{P}_d$, *and*

(C3) $\alpha(Ax) > a$, *for* $x \in P(\alpha, a, c)$ *with* $\| Ax \| \geq b$.

Then A has at least three fixed points x_1, x_2 *and* x_3 *satisfying*

$$\| x_1 \| < d, \ a < \alpha(x_2) \text{ and } \| x_3 \| > d \text{ with } \alpha(x_3) < a.$$

We shall consider the Banach space $C[0, 1]$ endowed with the norm $\| u \| = \max_{0 \le t \le 1} | u(t) |.$ Denote

$$C^+[0, 1] = \{u \in C[0, 1] : u(t) \ge 0,\ t \in [0, 1]\}.$$

Lemma 2.1 [19]. *Let* $y \in C[0, 1]$ *be symmetric on* $[0, 1]$. *Then the three-point BVP*

$$u''(t) + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(t) = u(1 - t), \quad u'(0) - u'(1) = u\left(\frac{1}{2}\right), \tag{2.2}$$

has a unique symmetric solution

$$u(t) = \int_0^1 G(t, s) y(s) ds, \tag{2.3}$$

where $G(t, s) = G_1(t, s) + G_2(s)$, *here*

$$G_1(t, s) = \begin{cases} t(1 - s), & 0 \le t \le s \le 1, \\ s(1 - t), & 0 \le s \le t \le 1, \end{cases} \qquad G_2(s) = \begin{cases} 1 - \dfrac{s}{2}, & 0 \le s \le \dfrac{1}{2}, \\ \dfrac{1 + s}{2}, & \dfrac{1}{2} \le s \le 1. \end{cases}$$

Lemma 2.2 [19]. *Let* $t, s \in [0, 1]$. *Then* $G(t, s)$ *satisfies* $\frac{3}{4} G(s, s) \le G(t, s) \le G(s, s)$.

Lemma 2.3 [19]. *Let* $y \in C^+[0, 1]$. *Then the unique symmetric solution* $u(t)$ *of BVP* (2.1)-(2.2) *is nonnegative on* $[0, 1]$, *and if* $y(t) \not\equiv 0$; *then* $u(t) > 0,\ t \in [0, 1]$.

Lemma 2.4 [19]. *Let* $y \in C^+[0, 1]$. *Then the unique symmetric solution* $u(t)$ *of BVP* (2.1)-(2.2) *satisfies*

$$\min_{t \in [0,1]} u(t) \ge \frac{3}{4} \| u \|. \tag{2.4}$$

We assume the following conditions throughout the paper:

(H_1) $a:(0,1)\to[0,+\infty)$ is continuous, symmetric on $(0,1)$ and

$$0<\int_0^1 G(s,s)a(s)ds<+\infty.$$

(H_2) $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous and $f(\cdot,u)$ is symmetric on $[0,1]$ for all $u\in[0,\infty)$.

Denote

$$P=\{u\in C^{+}[0,1]: u(t) \text{ is symmetric, concave on } [0,1]$$
$$\text{and } \min_{t\in[0,1]} u(t)\geq\frac{3}{4}\|u\|\}.$$

Obviously, P is a positive cone in $C[0,1]$. Define an operator $A:P\to C[0,1]$ by

$$(Au)(t)=\int_0^1 G(t,s)a(s)f(s,u(s))ds,\quad t\in[0,1]. \tag{2.5}$$

It is easy to see that the BVP (1.1)-(1.2) has a solution $u=u(t)$ if and only if u is a fixed point of the operator A defined by (2.5).

Lemma 2.5 [19]**.** *Suppose that* (H_1) *and* (H_2) *hold. Then* A *is completely continuous and* $A(P)\subset P$.

3. Main Results

We shall use the following notation:

$$f^0=\lim_{x\to+0}\sup\max_{t\in[0,1]}\frac{f(t,x)}{x},\qquad f^\infty=\lim_{x\to+\infty}\sup\max_{t\in[0,1]}\frac{f(t,x)}{x}$$

and

$$\Lambda=\left(\int_0^1 G(s,s)a(s)ds\right)^{-1}.$$

Let the nonnegative continuous concave functional $\alpha : P \to [0, \infty)$ be defined by

$$\alpha(u) = \min_{0 \le t \le 1} u(t), \quad \forall u \in P.$$

Obviously, $\frac{3}{4}\| u \| \le \alpha(u) \le \| u \|$.

Theorem 3.1. *Suppose* $(H_1), (H_2)$ *hold. If the following conditions are satisfied*:

(H_3) $f^0 < \Lambda$ *and* $f^\infty < \eta\Lambda, \eta \in [0, 1)$ (*particularly*, $f^0 = f^\infty = 0$),

(H_4) *there exists a constant* $a > 0, b > \frac{4}{3}a$ *such that*

$$f(t, x) > \frac{4}{3}\Lambda a, \quad (s, x) \in [0, 1] \times [a, b],$$

then the BVP (1.1)-(1.2) *has at least three symmetric positive solutions*.

Proof. First, from $f^\infty < \eta\Lambda$, $\eta \in [0, 1)$, there exists a real number $N > b$, such that $f(s, x) \le \eta\Lambda x$, for $(s, x) \in [0, 1] \times [N, \infty)$. Take

$$c \ge \max\left\{N, \frac{M}{\Lambda(1 - \eta)}\right\},$$

where $M = \max\{f(s, x) : (s, x) \in [0, 1] \times [0, N]\}$. Now choose $u \in \overline{P_c}$. Thus,

$$\begin{aligned} 0 \le (Au)(t) &= \int_0^1 G(t, s)a(s) f(s, u(s))ds \\ &= \int_{|u(s)| \le N} G(t, s)a(s) f(s, u(s))ds \\ &\quad + \int_{|u(s)| > N} G(t, s)a(s) f(s, u(s))ds \end{aligned}$$

$$\leq \int_{|u(s)|\leq N} G(s, s)a(s)f(s, u(s))ds$$

$$+ \int_{|u(s)|>N} G(s, s)a(s)f(s, u(s))ds$$

$$\leq \int_{|u(s)|\leq N} G(s, s)a(s)Mds$$

$$+ \int_{|u(s)|>N} G(s, s)a(s)\eta\Lambda x ds$$

$$\leq \int_0^1 G(s, s)a(s)(M + \eta\Lambda x)ds$$

$$\leq (M + \eta\Lambda c)\int_0^1 G(s, s)a(s)ds$$

$$= \Lambda c\left(\eta + \frac{M}{\Lambda c}\right)\int_0^1 G(s, s)a(s)ds$$

$$= c\left(\eta + \frac{M}{\Lambda c}\right) \leq c.$$

Then $\| Au \| \leq c,$ i.e., $Au \in \overline{P}_c.$ Hence

$$A : \overline{P}_c \to \overline{P}_c. \tag{3.1}$$

Let

$$u_0(t) = -\frac{2}{3}a\left(t - \frac{1}{2}\right)^2 + \frac{4}{3}a.$$

Then $\| u_0 \| = \frac{4}{3}a < b,$ $\alpha(u) = \frac{7}{6}a > a.$ Thus $u_0 \in \{u \in P(\alpha, a, b) : \alpha(u) > a\},$ we obtain

$$\{u \in P(\alpha, a, b) : \alpha(u) > a\} \neq \varnothing. \tag{3.2}$$

For $u \in P(\alpha, a, b),$ we have

$$\alpha(u) \geq a, \quad \| u \| \geq b \text{ and } a \geq u(t) \geq b, \quad t \in [0, 1],$$

then

$$\begin{aligned}\alpha(Au) &= \min_{0\le t\le 1}(Au)(t)\\ &= \min_{0\le t\le 1}\int_0^1 G(t, s)a(s)f(s, u(s))ds\\ &\ge \frac{3}{4}\min_{0\le t\le 1}\int_0^1 G(s, s)a(s)f(s, u(s))ds\\ &> \frac{3}{4}\times\frac{4}{3}\Lambda a\int_0^1 G(s, s)a(s)ds = a. \end{aligned}\tag{3.3}$$

This shows that condition (C1) of Theorem 2.1 is satisfied.

Second, from $f^0 < \Lambda,$ there exists a real number $d \in (0, a)$ such that $f(s, x) < \Lambda x$ for $(s, x) \in [0, 1]\times[0, d]$. For every $u \in P, \| u \| \le d,$

$$\begin{aligned}0 \le (Au)(t) &= \int_0^1 G(t, s)a(s)f(s, u(s))ds\\ &\le \int_0^1 G(s, s)a(s)f(s, u(s))ds\\ &< \int_0^1 G(s, s)a(s)\Lambda u(s)ds\\ &\le \Lambda\| u \|\int_0^1 G(s, s)a(s)ds = \| u \| \le d, \quad 0 \le t \le 1.\end{aligned}$$

Then

$$\| Au \| < \| u \| \le d, \quad \forall u \in \overline{P}_d. \tag{3.4}$$

This shows that condition (C2) of Theorem 2.1 is satisfied.

We finally show that (C3) of Theorem 2.1 also holds. For $u \in P(\alpha, a, c)$ and $\| Au \| \ge b,$ we have

$$\alpha(Au) \ge \frac{3}{4}\| Au \| \ge \frac{3}{4}b > a. \tag{3.5}$$

So, condition (C3) of Theorem 2.1 is satisfied. Therefore, the BVP (1.1)-(1.2) has at least three symmetric positive solutions u_1, u_2, u_3 satisfying

$$\| u_1 \| < d,\ a < \alpha(u_2),\ \text{and } \| u_3 \| > d \ \text{ with } \alpha(u_3) < a.$$

The proof is completed. □

Theorem 3.2. *Suppose* (H_1), (H_2) *hold. In addition, assume that there exist numbers* a, b, c, d *with* $0 < d < a < \frac{4}{3}a < b \leq c$ *such that the following conditions are satisfied*:

(H_5) $f(s, x) \leq \Lambda c$ *for* $(s, x) \in [0, 1] \times [0, c]$,

(H_6) $f(s, x) < \Lambda d$ *for* $(s, x) \in [0, 1] \times [0, d]$,

(H_7) $f(s, x) > \frac{4}{3}\Lambda a$ *for* $(s, x) \in [0, 1] \times [a, b]$.

Then the BVP (1.1)-(1.2) *has at least three symmetric positive solutions* u_1, u_2, u_3 *satisfying*

$$\| u_1 \| < d,\ a < \alpha(u_2),\ \textit{and } \| u_3 \| > d \ \textit{ with } \alpha(u_3) < a.$$

Proof. First, $\forall u \in \overline{P_c}$, we have $0 \leq u \leq c$ and then by (H_5),

$$\begin{aligned} 0 \leq (Au)(t) &= \int_0^1 G(t, s)a(s)f(s, u(s))ds \\ &\leq \int_0^1 G(s, s)a(s)f(s, u(s))ds \\ &\leq \Lambda c \int_0^1 G(s, s)a(s)ds = c. \end{aligned}$$

Therefore, $\| Au \| \leq c$, i.e., $Au \in \overline{P_c}$. Hence,

$$A : \overline{P_c} \to \overline{P_c}. \tag{3.6}$$

Let

$$u_0(t) = -\frac{2}{3}a\left(t - \frac{1}{2}\right)^2 + \frac{4}{3}a.$$

Then $\| u_0 \| = \frac{4}{3}a < b,\quad \alpha(u) = \frac{7}{6}a > a.$ Thus $u_0 \in \{u \in P(\alpha, a, b) : \alpha(u) > a\},$ we obtain

$$\{u \in P(\alpha, a, b) : \alpha(u) > a\} \neq \varnothing. \tag{3.7}$$

For $u \in P(\alpha, a, b),$ we have

$$\alpha(u) \geq a, \| u \| \leq b, \quad \text{and} \quad a \leq u(t) \leq b,\ t \in [0, 1].$$

Then

$$\begin{aligned}\alpha(Au) &= \min_{0 \leq t \leq 1}(Au)(t)\\ &= \min_{0 \leq t \leq 1}\int_0^1 G(t, s)a(s)f(s, u(s))ds\\ &\geq \frac{3}{4}\min_{0 \leq t \leq 1}\int_0^1 G(s, s)a(s)f(s, u(s))ds\\ &> \frac{3}{4} \times \frac{4}{3}\Lambda a\int_0^1 G(s, s)a(s)ds = a. \end{aligned}\tag{3.8}$$

This shows that condition (C1) of Theorem 2.1 is satisfied.

Second, $\forall u \in \overline{P_d},$ we have $0 \leq u \leq d$ and then by $(H_6),$

$$\begin{aligned}0 \leq (Au)(t) &= \int_0^1 G(t, s)a(s)f(s, u(s))ds\\ &\leq \int_0^1 G(s, s)a(s)f(s, u(s))ds\\ &< \Lambda d\int_0^1 G(s, s)a(s)ds = d.\end{aligned}$$

Then

$$\| Au \| < \| u \| \leq d, \quad \forall u \in \overline{P}_d. \tag{3.9}$$

This shows that condition (C2) of Theorem 2.1 is satisfied.

We finally show that (C3) of Theorem 2.1 also holds. For $u \in P(\alpha, a, c)$ and $\| Au \| \geq b,$ we have

$$\alpha(Au) \geq \frac{3}{4} \| Au \| \geq \frac{3}{4} b > a. \tag{3.10}$$

So, condition (C3) of Theorem 2.1 is satisfied. Therefore, the BVP (1.1)-(1.2) has at least three positive solutions u_1, u_2, u_3 satisfying

$$\| u_1 \| < d, \; a < \alpha(u_2), \;\text{ and }\; \| u_3 \| > d \;\text{ with }\; \alpha(u_3) < a.$$

The proof is completed. □

In a similar way, we can get the following result.

Theorem 3.3. *Suppose* $(H_1), (H_2)$ *hold. In addition, assume that there exist numbers a, b, c, d with* $0 < d < a < b \leq c$ *such that* $(H_5), (H_6)$ *and the following condition is satisfied*:

(H_8) $\quad f(s, x) > \frac{4}{3} \Lambda a$ *for* $(s, x) \in [0, 1] \times [a, c]$,

then the BVP (1.1)-(1.2) *has at least three symmetric positive solutions* u_1, u_2, u_3 *satisfying*

$$\| u_1 \| < d, \; a < \alpha(u_2), \;\textit{ and }\; \| u_3 \| > d \;\textit{ with }\; \alpha(u_3) < a.$$

4. Examples

In this section, we present some examples to illustrate our main results in Section 3.

Example 4.1 [19]**.** Consider the three-point BVP

$$u'' + \frac{48}{225}(1 + \min(t, 1-t))u^2 e^{8-u} = 0, \quad 1 < t < 1, \tag{4.1}$$

$$u(t) = u(1-t), \quad u'(0) - u'(1) = u\left(\frac{1}{2}\right). \tag{4.2}$$

Set $a(t) = \frac{48}{225}e^8$, $f(t, u) = (1 + \min(t, 1-t))u^2 e^{-u}$, then $f^0 = f^\infty = 0$, thus condition (H_3) holds. Furthermore, a direct computation shows that $\Lambda = \frac{9}{2}e^{-8}$. Let $a = 5$, $b = 8 > \frac{4}{3}a$. Then, for $(t, s) \in [0, 1] \times [a, b]$, we have $f(t, x) \geq f(0, x) > f(0, 8) = \frac{16}{9}\Lambda \times 8 > \frac{4}{3}\Lambda a$, which implies that condition (H_4) holds. Hence, by Theorem 3.1, the BVP (4.1)-(4.2) has at least three symmetric positive solutions.

Remark 4.1. Sun [19] obtained that the BVP (4.1)-(4.2) has at least two symmetric positive solutions by Example 4.1. In this paper, we investigate that the BVP (4.1)-(4.2) has at least three symmetric positive solutions by the same example.

Example 4.2. Consider the three-point BVP

$$u'' + \frac{768}{2025}(1 + \min(t, 1-t))u^3 e^{-\frac{1}{8}u} = 0, \quad 1 < t < 1, \tag{4.3}$$

$$u(t) = u(1-t), \quad u'(0) - u'(1) = u\left(\frac{1}{2}\right). \tag{4.4}$$

Set $a(t) = \frac{768}{2025}e^8$, $f(t, u) = (1 + \min(t, 1-t))u^3 e^{-\frac{1}{8}u}$, then $\Lambda = 8$. Set $a = 8$, $b = \frac{4}{3} \times 16 \times 2$, $c = 24^2 \times 2$, $d = \frac{1}{2}$. Furthermore, a direct computation shows that (H_5), (H_6), (H_7) hold. Hence, by Theorem 3.2, the BVP (4.3)-(4.4) has at least three symmetric positive solutions u_1, u_2, u_3 satisfying

$$\| u_1 \| < d,\ a < \alpha(u_2), \text{ and } \| u_3 \| > d \text{ with } \alpha(u_3) < a.$$

Example 4.3. Consider the three-point BVP

$$u'' + \frac{48}{2025}(1 + \min(t, 1 - t))u^2 e^{-\frac{1}{8}u} = 0, \quad 1 < t < 1, \tag{4.5}$$

$$u(t) = u(1 - t), \quad u'(0) - u'(1) = u\left(\frac{1}{2}\right). \tag{4.6}$$

Set $a(t) = \frac{48}{2025} e^8$, $f(t, u) = (1 + \min(t, 1 - t))u^2 e^{-\frac{1}{8}u}$, then $\Lambda = \frac{1}{2}$. Set $a = 8$, $b = 12$, $c = 16$, $d = \frac{1}{2}$. Furthermore, a direct computation shows that (H_5), (H_6), (H_8) hold. Hence, by Theorem 3.3, the BVP (4.5)-(4.6) has at least three symmetric positive solutions u_1, u_2, u_3 satisfying

$$\| u_1 \| < d, \; a < \alpha(u_2), \text{ and } \| u_3 \| > d \text{ with } \alpha(u_3) < a.$$

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References

[1] Ling Hu and Lianglong Wang, Multiple positive solutions of boundary value problems for systems of nonlinear second order differential equations, J. Math. Anal. Appl. 335 (2007), 1052-1060.

[2] Wu Zheng and Wang Lianglong, Multiple positive solutions for singular boundary value problems with *p*-Laplacian, Ann. Diff. Eqs. 23 (2007), 519-524.

[3] Hu Ling and Wang Lianglong, Multiple positive solutions of boundary value problems for a system of nonlinear third order differential equations, Ann. Diff. Eqs. 22 (2006), 266-269.

[4] S. Timoshenko, Theory of Elastic Stability, McGraw-Hill, New York, 1961.

[5] W. Feng and J. R. L. Webb, Solvability of a three-point boundary value problems at resonance, Nonlinear Anal. 30 (1997), 3227-3238.

[6] C. P. Gupta, A shaper condition for the solvability of a three-point second order boundary value problem, J. Math. Anal. Appl. 205 (1997), 586-597.

[7] X. He and W. Ge, Triple solutions for second-order three-point boundary value problems, J. Math. Anal. Appl. 268 (2002), 256-265.

[8] G. Infante, Eigenvalues of some non-local boundary value problems, Proc. Edinb. Math. Soc. 46 (2003), 75-86.

[9] B. Liu, Positive solutions of a nonlinear three-point boundary value problem, Appl. Math. Comput. 132 (2002), 11-28.

[10] R. Ma, Positive solutions for second-order three-point boundary value problems, Appl. Math. Lett. 14 (2001), 1-5.

[11] Y. Sun and L. Liu, Solvability for a nonlinear second-order three-point boundary value problem, J. Math. Anal. Appl. 269 (2004), 265-275.

[12] J. R. L. Webb, Positive solutions of some three-point boundary value problems via fixed point index theory, Nonlinear Anal. 47 (2001), 4319-4332.

[13] Yongping Sun, Optimal existence criteria for symmetric positive solutions to a singular three-point boundary value problem, Nonlinear Anal. 66 (2007), 1051-1063.

[14] Shejun Wang and Hongrui Sun, Optimal existence criteria for symmetric positive solutions to a singular three-point boundary value problem, Nonlinear Anal. 69 (2008), 4266-4276.

[15] J. Henderson and H. B. Thompson, Multiple symmetric positive solutions for a second order boundary value problem, Proc. Amer. Math. Soc. 128 (2000), 2373-2379.

[16] F. Li and Y. Zhang, Multiple symmetric nonnegative solutions of second-order ordinary differential equations, Appl. Math. Lett. 17 (2004), 261-267.

[17] Q. Yao, Existence and iteration of n symmetric positive solutions for a singular two-point boundary value problem, Comput. Math. Appl. 47 (2004), 1195-1200.

[18] N. Kosmatov, Symmetric solutions of a multi-point boundary value problem at resonance, J. Math. Anal. Appl. 309 (2005), 25-36.

[19] Yongping Sun, Existence and multiplicity of symmetric positive solutions for three-point boundary value problem, J. Math. Anal. Appl. 329 (2007), 998-1009.

[20] R. Leggett and L. Williams, Multiple positive fixed solutions of nonlinear operator on ordered Banach spaces, Indiana Univ. Math. J. 28 (1979), 673-688.