Far East Journal of Mathematical Sciences (FJMS)
Volume 59, Number 1, 2011, Pages 103-112
Published Online: December 2011
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

ON DIVISOR PARTITIONS

Neville Robbins

Mathematics Department
San Francisco State University
San Francisco, CA 94132
U. S. A.
e-mail: nrobbins@sfsu.edu

Abstract

If n is a natural number, let $p^*(n)$ denote the number of partitions of n all of whose parts are divisors of n. We obtain some formulas for $p^*(n)$ when n has few divisors.

1. Introduction

If n is a natural number, let $p^*(n)$ denote the number of partitions of n all of whose parts are divisors of n. This function appears to have been previously only scarcely investigated, in spite of a paper by Gupta with a somewhat misleading title. (See [1].) It is known that $p^*(n)$ is the coefficient of x^n in the series expansion of $\prod_{d|n}(1-x^d)^{-1}$. (See [2].) Using this identity, T. D. Noe wrote a Mathematica program to generate a table of $p^*(n)$ for $1 \le n \le 1000$. (See [2].) Unlike other partition functions, the value

2010 Mathematics Subject Classification: 11P83.
Keywords and phrases: divisor partitions.
Received July 29, 2011

of $p^*(n)$ depends on the prime factorization of *n*. Using counting techniques and specialized summation identities, we obtain formulas for $p^*(n)$ when *n* has 5 or fewer divisors, or satisfies other restrictions.

2. Preliminaries

The notation $a^m b^n c^r$ will represent the partition consisting of $m\ a's$, $n\ b's$, and $r\ c's$, where $a > b > c$. However, $(q^m)^n$ will represent *n* copies of q^m (ordinary exponential notation):

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \tag{1}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2}$$

Remarks. (1) and (2) are well-known summation formulas that are easily proven by induction on *n*.

3. The Main Results

Theorem 1. *If q is prime, then* $p^*(q) = 2$.

Proof. The only divisor partitions of *q* are *q* and 1^q. □

Theorem 2. *If q is prime, then* $p^*(q^2) = q + 2$.

Proof. Aside from the trivial partition q^2, each divisor partition of q^2 has the form $q^k 1^{q^2 - kq}$, where $0 \le k \le q$. Thus we have

$$p^*(q^2) = 1 + \sum_{k=0}^{q} 1 = 1 + (q+1) = q + 2. \qquad \square$$

Remarks. Theorem 1 and a slightly weaker version of Theorem 2 are stated in [2].

Theorem 3. *If q is prime, then* $p^*(q^3) = 1 + \dfrac{(q+1)(q^2+2)}{2}$.

Proof. Aside from the trivial partition q^3, each divisor partition of q^3 has the form $(q^2)^j q^k 1^m$, where $0 \le j \le q$ and $jq^2 + kq + m = q^3$, so that $0 \le k \le q^2 - qj$. Therefore, we have

$$p^*(q^3) = 1 + \sum_{j=0}^{q} \sum_{k=0}^{q^2-qj} 1 = 1 + \sum_{j=0}^{q} (q^2 - qj + 1) = 1 + (q+1)(q^2+1) - q\sum_{j=0}^{q} j$$

$$= 1 + (q+1)(q^2+1) - \frac{q^2(q+1)}{2} = 1 + \frac{(q+1)(q^2+2)}{2}. \qquad \square$$

Theorem 4. *If q is prime, then* $p^*(q^4) = 1 + \left(\dfrac{q+1}{12}\right)(2q^5 + q^4 + 3q^3 + 6q^2 + 12)$.

Proof. Aside from the trivial partition q^4, each divisor partition of q^4 has the form $(q^3)^j (q^2)^k q^m 1^r$, where $0 \le j \le q$ and $jq^3 + kq^2 + mq + r = q^4$, so that $0 \le k \le q^2 - qj$ and $0 \le m \le q^3 - jq^2 - kq$. Therefore, we have

$$p^*(q^4) = 1 + \sum_{j=0}^{q} \sum_{k=0}^{q^2-qj} \sum_{m=0}^{q^3-q^2j-qk} 1 = 1 + \sum_{j=0}^{q} \sum_{k=0}^{q^2-qj} (q^3 + 1 - q^2 j - qk)$$

$$= 1 + \sum_{j=0}^{q} \left\{ (q^3 + 1 - q^2 j)(q^2 + 1 - qj) - q \sum_{k=0}^{q^2-qj} k \right\}$$

$$= 1 + \sum_{j=0}^{q} \left\{ (q^3 + 1 - q^2 j)(q^2 + 1 - qj) - \frac{1}{2} q(q^2 - qj)(q^2 + 1 - qj) \right\}.$$

The conclusion follows if we simplify, and make use of identities (1) and (2). $\square$

Before deriving the formula for $p^*(pq)$, where p and q are distinct primes, we will need the following lemma:

Lemma 1. *Let p, q be positive integers such that* $(p, q) = 1.$ *Then*

$$\sum_{j=1}^{q-1}\left[\frac{jp}{q}\right] = \frac{1}{2}(p-1)(q-1).$$

Proof. Consider the set of lattice points (a, b), where $1 \le a \le q-1$ and $1 \le b \le p-1.$ The total number of these lattice points is $(p-1)(q-1).$ Let line $\mathcal{L}$ have the equation: $qy = px.$ Therefore, $\mathcal{L}$ does not pass through any lattice point. Now, if (a, b) is a lattice point below $\mathcal{L},$ then (b, a) is a corresponding point above $\mathcal{L}.$ Thus there are equally many lattice points above and below $\mathcal{L}.$ The conclusion now follows. □

Theorem 5. *If p and q are distinct primes, then*

$$p^*(pq) = 2 + \frac{(p+1)(q+1)}{2}.$$

Proof. The divisor partitions of pq are $pq,$ q^p and $p^i q^j 1^h,$ where $ip + jq + h = pq,$ so that $0 \le i \le q-1$ and $jq \le p(q-i).$ Therefore, we have

$$p^*(pq) - 2 = \sum_{i=0}^{q-1}\sum_{j=0}^{\left[\left(\frac{p}{q}\right)(q-i)\right]} 1 = \sum_{i=0}^{q-1}\left(1 + \sum_{j=1}^{\left[\left(\frac{p}{q}\right)(q-i)\right]} 1\right) = q + \sum_{i=0}^{q-1}\sum_{j=1}^{\left[\left(\frac{p}{q}\right)(q-i)\right]} 1.$$

If we let $k = q - i$ and then invoke Lemma 1, we get

$$p^*(pq) - 2 = q + \sum_{k=1}^{q}\sum_{j=1}^{\left[\frac{pk}{q}\right]} 1 = q + \sum_{k=1}^{q}\left[\frac{pk}{q}\right] = q + p + \sum_{k=1}^{q-1}\left[\frac{pk}{q}\right]$$

$$= p + q + \frac{(p-1)(q-1)}{2} = \frac{(p+1)(q+1)}{2}.$$

The conclusion now follows. □

We conclude by obtaining formulas for $p^*(n)$ in several cases, where n has 6 divisors.

Theorem 6. *If q is an odd prime, then*

$$p^*(4q) = 4 + 2\left(1 + \left[\frac{q}{4}\right]\right)\left(1 + \left[\frac{q}{2}\right] - \left[\frac{q}{4}\right]\right)$$

$$+ 2\left(1 + \left[\frac{q}{2}\right]\right)\left(1 + q - \left[\frac{q}{2}\right]\right)$$

$$+ \left(1 + \left[\frac{3q}{4}\right]\right)\left(1 + \left[\frac{3q}{2}\right] - \left[\frac{3q}{4}\right]\right) + (1 + q)^2.$$

Proof. $4q$ has 4 divisor partitions that are all multiples of q, namely: $4q$, $(2q)^2$, $(2q)q^2$, q^4. There are also 6 types of divisor partitions, where some of the parts may be 1, 2 or 4, namely:

Type (i) $q^3 4^a 2^b 1^c$ Type (ii) $(2q)q4^a 2^b 1^c$ Type (iii) $q^2 4^a 2^b 1^c$

Type (iv) $(2q)4^a 2^b 1^c$ Type (v) $q4^a 2^b 1^c$ Type (vi) $4^a 2^b 1^c$

In each case, we have $4a + 2b + c = jq$, where $1 \le j \le 4$, namely $j = 1$ in Types (i) and (ii), $j = 2$ in Types (iii) and (iv), $j = 3$ in Type (v), and $j = 4$ in Type (vi). For a given value of j, the number of corresponding divisor partitions of $4q$ is

$$\sum_{a=0}^{\left[\frac{jq}{4}\right]} \sum_{b=0}^{\left[\frac{jq}{2}\right]-2a} 1 = \sum_{a=0}^{\left[\frac{jq}{4}\right]}\left(1 + \left[\frac{jq}{2}\right] - 2a\right) = \sum_{a=0}^{\left[\frac{jq}{4}\right]}\left(1 + \left[\frac{jq}{2}\right]\right) - 2\sum_{a=0}^{\left[\frac{jq}{4}\right]} a$$

$$= \left(1 + \left[\frac{jq}{4}\right]\right)\left(1 + \left[\frac{jq}{2}\right]\right) - \left[\frac{jq}{4}\right]\left(1 + \left[\frac{jq}{4}\right]\right)$$

$$= \left(1 + \left[\frac{jq}{4}\right]\right)\left(1 + \left[\frac{jq}{2}\right] - \left[\frac{jq}{4}\right]\right).$$

The conclusion now follows. □

A more convenient version of Theorem 6 is given by Theorem 6a below:

Theorem 6a. *If q is an odd prime, then*

$$p^*(4q) = \begin{cases} 35m^2 + 41m + 16 \text{ if } q = 4m+1, \\ 35m^2 + 76m + 45 \text{ if } q = 4m+3. \end{cases}$$

Proof. This follows directly from Theorem 6. □

Our final theorem concerns a formula for $p^*(2q^2)$, where q is an odd prime. We need two preliminary lemmas.

Lemma 2. *If q is an odd prime, and n is the number of divisor partitions of* $2q^2$ *of the form* $(q^2)^1(2q)^a q^b 2^c 1^d$, *then*

$$n = \frac{1}{48}(q+1)(q+3)(2q^2 + q + 9).$$

Proof. By the hypothesis, we have $a(2q) + bq + 2c + d = q^2$, so we have $0 \le a \le \frac{q-1}{2}$, $0 \le b \le q - 2a$, $0 \le c \le \left[\frac{q^2 - (2a+b)q}{2}\right]$. Therefore, we have

$$n = \sum_{a=0}^{\frac{q-1}{2}} \sum_{b=0}^{q-2a} \sum_{c=0}^{\left[\frac{q^2-(2a+b)q}{2}\right]} 1 = \sum_{a=0}^{\frac{q-1}{2}} \sum_{b=0}^{q-2a} \left(1 - qa + \left[\frac{q^2 - bq}{2}\right]\right).$$

Let $\sum_{b=0}^{q-2a}\left(1 - qa + \left[\frac{q^2 - bq}{2}\right]\right) = S_0 + S_1$, where S_0, S_1 are the sums taken over even, odd values of b, respectively. Then

$$S_0 = \sum_{j=0}^{\frac{q-1}{2}-a} \left(1 - qa + \frac{q^2 - 1}{2} - jq\right) = \sum_{j=0}^{\frac{q-1}{2}-a} \left(\frac{q^2 + 1}{2} - qa - qj\right)$$

$$= \sum_{j=0}^{\frac{q-1}{2}-a}\left(\frac{q^2+1}{2}-qa\right)-q\sum_{j=0}^{\frac{q-1}{2}-a} j = \left(\frac{q+1}{2}-a\right)\left(\frac{q^2+1}{2}-qa\right)$$

$$-\frac{q}{2}\left(\frac{q-1}{2}-a\right)\left(\frac{q+1}{2}-a\right)=\left(\frac{q+1}{2}-a\right)\left(\frac{q^2+1}{2}-qa-\frac{q^2}{4}+\frac{q}{2}+\frac{qa}{2}\right)$$

$$= \left(\frac{q+1}{2}-a\right)\left(\frac{q^2}{4}+\frac{q}{4}+\frac{2}{4}-\frac{qa}{2}\right).$$

Similarly, we have

$$S_1 = \sum_{j=0}^{\frac{q-1}{2}-a}\left(1-aq+\frac{q^2-q}{2}-jq\right)$$

$$= \left(\frac{q+1}{2}-a\right)\left(1-aq+\frac{q^2-q}{2}\right)-\frac{q}{2}\left(\frac{q-1}{2}-a\right)\left(\frac{q+1}{2}-a\right)$$

$$= \left(\frac{q+1}{2}-a\right)\left(\frac{q^2-q+4}{4}-\frac{aq}{2}\right).$$

Now

$$\sum_{b=0}^{q-2a}\left(1-qa+\left[\frac{q^2-bq}{2}\right]\right) = S_0+S_1 = \left(\frac{q+1}{2}-a\right)\left(\frac{q^2+3}{2}-aq\right)$$

$$= \frac{(q+1)(q^2+3)}{4}-\left(\frac{2q^2+q+3}{2}\right)a+qa^2 \to n$$

$$= \sum_{a=0}^{\frac{q-1}{2}}\left\{\frac{(q+1)(q^2+3)}{4}-\left(\frac{2q^2+q+3}{2}\right)a+qa^2\right\}$$

$$=\left(\frac{q+1}{2}\right)\left(\frac{(q+1)(q+3)}{4}-\frac{(2q^2+q+3)}{2}\right)\left(\frac{(q+1)(q-1)}{8}+\left(\frac{q^2}{6}\right)\left(\frac{q-1}{2}\right)\left(\frac{q+1}{2}\right)\right)$$

$$= \frac{(q+1)(q+3)(2q^2+q+9)}{48}. \qquad \square$$

Lemma 3. *If q is an odd prime, and m is the number of divisor partitions of $2q^2$ of the form $(2q)^a q^b 2^c 1^d$, then*

$$m = \frac{q+1}{12}(4q^3 + 5q^2 + 9q + 12).$$

Proof. By the hypothesis, we have $a(2q) + bq + 2c + d = 2q^2$, so that $0 \le a \le q$, $0 \le b \le 2(q-a)$, $0 \le c \le \left[q^2 - aq - \frac{bq}{2}\right]$. Therefore, we have

$$m = \sum_{a=0}^{q} \sum_{b=0}^{2(q-a)} \sum_{c=0}^{\left[q^2 - aq - \frac{bq}{2}\right]} = \sum_{a=0}^{q} \sum_{b=0}^{2(q-a)} \left(1 + \left[q^2 - aq - \frac{bq}{2}\right]\right).$$

Let $\sum_{b=0}^{2(q-a)}\left(1 + \left[q^2 - aq - \frac{bq}{2}\right]\right) = S_0 + S_1$, where S_0, S_1 are the sums taken over even, odd values of b, respectively. Then

$$S_0 = \sum_{j=0}^{q-a}(1 + q^2 - aq - jq) = (q+1-a)(q^2+1-aq)$$

$$- q\sum_{j=1}^{q-a} j = (q+1-a)(q^2+1-aq) - \frac{q}{2}(q-a)(q+1-a)$$

$$= (q+1-a)\left(\frac{q^2+2-aq}{2}\right).$$

Also,

$$S_1 = \sum_{j=0}^{q-a-1}\left(1 + \left[q^2 - \frac{q}{2}\right] - aq - jq\right)$$

$$= \sum_{j=0}^{q-a-1}\left(1 + \frac{2q^2-q-1}{2} - aq - jq\right)$$

$$= \sum_{j=0}^{q-a-1}\left(\frac{2q^2 - q + 1}{2} - aq - jq\right) = (q-a)\left(\frac{2q^2 - q + 1}{2} - aq\right)$$

$$- q\sum_{j=1}^{q-a-1} j = (q-a)\left(\frac{2q^2 - q + 1}{2} - aq\right) - \frac{q}{2}(q-a-1)(q-a)$$

$$= \frac{q-a}{2}(q^2 + 1 - aq).$$

Therefore, after simplifying, we have

$$S_0 + S_1 = \frac{2q^3 + q^2 + 3q + 2}{2} - \left(\frac{4q^2 + q + 3}{2}\right)a + qa^2.$$

This implies

$$m = \sum_{a=0}^{q}\left\{\frac{2q^3 + q^2 + 3q + 2}{2} - \left(\frac{4q^2 + q + 3}{2}\right)a + qa^2\right\}$$

$$= (q+1)\left(\frac{2q^3 + q^2 + 3q + 2}{2}\right) - \left(\frac{4q^2 + q + 3}{2}\right)\frac{q(q+1)}{2}$$

$$+ \frac{q^2(q+1)(2q+1)}{6}.$$

After simplifying, we obtain $m = \frac{q+1}{12}(4q^3 + 5q^2 + 9q + 12).$ □

Theorem 7. *If q is an odd prime, then*

$$p^*(2q^2) = 2 + \frac{q+1}{16}(6q^3 + 9q^2 + 16q + 25).$$

Proof. Note that $2q^2$ has 2 divisor partitions with largest part greater than or equal to q^2, namely, $2q^2$ and $(q^2)^2$. Therefore, $p^*(2q^2) = 2 + m + n.$ The conclusion now follows, after simplification, from Lemmas 2 and 3. □

References

[1] H. Gupta, Partitions of *n* into divisors of *m*, Indian J. Pure Applied Math. 6 (1975), 1276-1286.

[2] N. Sloane, Online Encyclopedia of Integer Sequences, A018818.