Far East Journal of Mathematical Sciences (FJMS)
Volume 59, Number 1, 2011, Pages 65-72
Published Online: December 2011
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

ZERO-DIVISOR GRAPH IN COMMUTATIVE RINGS WITH DECOMPOSABLE ZERO IDEAL

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Abstract

In this paper, we study some graph properties of zero-divisor graph of a commutative ring R where the zero ideal is decomposable. It is obtained that the girth is 3 and the diameter of the zero-divisor is greater than or equal to 2 for $n \geq 2$. It is also obtained the conditions that the zero-divisor graph is not planar.

1. Introduction

Let R be a commutative ring with nonzero identity and let $Z(R)$ be the set of zero-divisors of R. Let $\Gamma(R)$ be the graph of nonzero zero-divisors of R, in the sense that the vertices of $\Gamma(R)$ are all elements of $Z(R)^* = Z(R) - \{0\}$. For all x, $y \in Z(R)^*$, the vertices x and y are said to be *adjacent* if $xy = 0$. Note that $\Gamma(R)$ is empty if and only if R is an integral domain. For all vertices x, y in $\Gamma(R)$, let $d(x, y)$ be the length of shortest path between x and y. Note that $d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path. The *diameter* of $\Gamma(R)$ is denoted by

2010 Mathematics Subject Classification: 35R35, 49M15, 49N50.

Keywords and phrases: primary decompositions, commutative ring, zero-divisor graph.

Received April 8, 2011

$diam(\Gamma(R))$ and defined by $\sup\{d(x, y) \mid x, y \in Z(R)^*\}$. The *girth* of $\Gamma(R)$, denoted by $\mathrm{gr}(\Gamma(R))$, is the length of shortest cycle in $\Gamma(R)$. We note that $gr(\Gamma(R)) = \infty$ if there is no cycle in the zero-divisor graph. The *degree* of the vertex x of $\Gamma(R)$, denoted by $\deg(x)$, is the number of all edges incident with x. Graph $\Gamma(R)$ is called to be *connected* if there is a path between each pair of its vertices x and y. We will say that $\Gamma(R)$ is *bipartite* if the vertex set of $\Gamma(R)$ may be partitioned into two disjoint sets V_1 and V_2 in such a way that each edge of the graph joins a vertex in V_1 to a vertex in V_2. The graph $\Gamma(R)$ is called to be *complete bipartite* if it is bipartite and each vertex in V_1 is adjacent to all vertices in V_2 and vice versa. It is called that $\Gamma(R)$ is a *star graph* if it is complete bipartite with $|V_1| = 1$.

A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's theorem says that a graph is *planar* if and only if it contains no subdivision of K_5 or $K_{3,3}$ (see [9, Section 9.1]).

The concept of zero-divisor graphs in a commutative ring R was introduced by Beck [7], and then further studied in [3] and [5]. There are so many published articles connected to algebraic theoretic properties of R and graph theoretic properties of $\Gamma(R)$ (see for example, [12], [11], [7] and so on). Anderson and Livingston showed in [3, Theorem 2.4] that, for any commutative ring R, $\Gamma(R)$ is connected. Mulay has shown in [12] that, if $\Gamma(R)$ contains a cycle, then it contains a cycle of length less than or equal to four. In this paper, we continue studying the interplay between $\Gamma(R)$ and R. It is known that $diam(\Gamma(R)) = 1, 2$ or 3 and $gr(\Gamma(R)) = 3, 4$ or ∞ (see [3] and [4]).

An ideal I of ring R is called *decomposable* if there is an integer n such that $I = \mathfrak{q}_0 \cap \cdots \cap \mathfrak{q}_n$, where for all $0 \leq i \leq n$, $\mathfrak{q}_i$ is a $\mathfrak{p}_i$-primary and $\mathfrak{p}_i$ is a prime ideal of R. A decomposition $I = \bigcap_{i=0}^{\infty} \mathfrak{q}_i$ is called *minimal* if $\bigcap_{i \neq j} \mathfrak{q}_i \not\subset \mathfrak{q}_j$ for all $0 \leq j \leq n$ and that $\mathfrak{p}_i \neq \mathfrak{p}_j$ for all $i \neq j$.

Throughout, R is a commutative ring with $1 \neq 0$, $Z(R)^*$ is the set of nonzero zero-divisors, $Nil(R)$ is the ideal of nilpotent elements of R and the zero ideal of R is decomposable.

2. Some Properties of Zero-divisor Graph

Let the zero ideal of R has a minimal primary decomposition as $\bigcap_{i=0}^{n} \mathfrak{q}_i$, where $\mathfrak{q}_i$ is a $\mathfrak{p}_i$-primary for all $0 \leq i \leq n$. Then $Nil(R) = \bigcap_{i=0}^{n} \mathfrak{p}_i$ and $Z(R) = \bigcup_{i=0}^{n} \mathfrak{p}_i$. There is $x_i \in \bigcap_{j \neq i} \mathfrak{q}_j$ and $x_i \notin \mathfrak{q}_i$ for all $0 \leq i \leq n$ such that $\mathfrak{p}_i = \sqrt{0 :_R x_i}$. Let $\mathfrak{p}_0$ be a minimal ideal of R.

Proposition 2.1. *With the above notation, the subgraph H of $\Gamma(R)$ with the vertex set $\{x_i \mid 0 \leq i \leq n\}$ is a complete graph K_{n+1}.*

Proof. If 0 is a primary ideal, then H trivially is K_1. So, let $n \geq 1$ and let $0 \leq i, \; j \leq n$ such that $i \neq j$. Since $x_i \in \bigcap_{j \neq i} \mathfrak{q}_j$, we have

$$x_i x_j \in \left(\bigcap_{j \neq i} \mathfrak{q}_j\right) \cap \mathfrak{q}_i = 0.$$

This completes the proof. □

Corollary 2.2. *With the above notation, if $n \geq 4$, then $\Gamma(R)$ is not planar.*

Proof. Since K_5 is not planar, the result follows from [9, Proposition 9.1.10]. □

We are going to study the graph properties of $\Gamma(R)$ in the case of $0 = \bigcap_{i=0}^{n} \mathfrak{q}_i$. If $n = 0$, then 0 is a primary ideal.

Proposition 2.3. *Suppose that the zero ideal of R is $\mathfrak{p}$-primary. Then*

(i) *If $|\mathfrak{p}| = 1$, then $\Gamma(R)$ is empty.*

(ii) *If $|\mathfrak{p}| = 2$, then $diam(\Gamma(R)) = 0$ and $gr(\Gamma(R)) = \infty$.*

(iii) *If $|\mathfrak{p}| = 3$, then $diam(\Gamma(R)) = 1$ and $gr(\Gamma(R)) = \infty$.*

(iv) *If $|\mathfrak{p}| \geq 4$, then $1 \leq diam(\Gamma(R)) \leq 2$ and $gr(\Gamma(R)) = 3$ or ∞.*

Proof. We note that $Z(R) = Nil(R) = \mathfrak{p}$.

(i) In this case, $\mathfrak{p} = 0$ and R is an integral domain.

(ii) This is trivial.

(iii) Let $|\mathfrak{p}| = 3$. Then we may assume that $\mathfrak{p} = \{0, x, y\}$. This implies that $xy = 0$.

(iv) This immediately follows from [1, Theorem 2.2] and [4, Theorem 2.3]. Because in the last statement $gr(\Gamma(R)) \neq 4$. ☐

If the zero ideal of R is $\mathfrak{p}$-primary with $|\mathfrak{p}| \leq 5$, then $\Gamma(R)$ is planar. Let $|\mathfrak{p}| = 6$. If $\Gamma(R) = K_5$, then it is not planar and if $\Gamma(R) \neq K_5$, then it is planar.

Theorem 2.4. *Let zero ideal of R be* $\mathfrak{p}$*-primary with* $\mathfrak{p} \notin Max(R)$ *and* $|\mathfrak{p}| \geq 7$. *Then* $\Gamma(R)$ *is not planar*.

Proof. Since $\mathfrak{p} \in Ass(R)$, there is a nonzero $x \in R$ such that $\mathfrak{p} = (0 :_R x)$. It is easy to see that by non-maximality of $\mathfrak{p}$, Rx is infinite. Let y_1, y_2, y_3 be disjoint elements of $\mathfrak{p}$. Then nonzero elements of Rx are adjacent to y_1, y_2, y_3. So, a subdivision of $\Gamma(R)$ is isomorphic to $K_{3,3}$ and the proposition follows from [9, Theorem 9.1.7 and Proposition 9.1.10]. ☐

Proposition 2.5. *Let zero ideal of R be* $\mathfrak{p}$*-primary with* $|\mathfrak{p}| \geq 7$ *and* $|R/\mathfrak{p}| \geq 4$. *Then* $\Gamma(R)$ *is not planar*.

Proof. Suppose that $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in R/\mathfrak{p}$ are nonzero disjoint elements. Then u_1x, u_2x and u_3x are disjoint, where $\mathfrak{p} = (0 :_R x)$. There are distinct elements $v_1, v_2, v_3 \in \mathfrak{p}$ which are adjacent to u_1x, u_2x, u_3x as required. ☐

For the case $n = 1$, we have the following.

Theorem 2.6. *Let R be a non-integral domain commutative ring and* $0 = \mathfrak{q}_0 \cap \mathfrak{q}_1$ *be a primary decomposition of zero ideal*. *Then the following hold*:

(i) *Let* $\mathfrak{p}_1$ *be not minimal*. *If* $|\mathfrak{p}_0| = 2$, *then* $\Gamma(R)$ *is a star graph and if* $|\mathfrak{p}_0| \geq 3$, *then* $gr(\Gamma(R)) = 3$.

(ii) *Let* $\mathfrak{p}_1$ *be minimal. If* $\mathfrak{p}_0 \cap \mathfrak{p}_1 = 0$, *then* $\Gamma(R)$ *is a complete bipartite graph. So, if* $|\mathfrak{p}_i| \geq 3$, *then* $gr(\Gamma(R)) = 4$.

Proof. (i) There is a minimal ideal $\mathfrak{q}$ of R such that $\mathfrak{q} \subset \mathfrak{p}$. This forces $\mathfrak{q} = \mathfrak{p}_0$. Hence, $Nil(R) = \mathfrak{p}_0 \subset \mathfrak{p}_1 = Z(R)$. If $|\mathfrak{p}_0| = 2$ and x is the nonzero element of $\mathfrak{p}_0$, then for all $y, z \in \mathfrak{p}_1 \backslash \mathfrak{p}_0$, $yz \neq 0$. So, for all $y \in \mathfrak{p}_1$, it must be $yx = 0$. This shows that $\Gamma(R)$ is a star graph. If $|\mathfrak{p}_0| \geq 3$, then the statement follows by [1, Theorem 2.12].

(ii) Assume that $V_1 = \mathfrak{p}_0 \backslash \{0\}$ and $V_2 = \mathfrak{p}_1 \backslash \{0\}$. For all $x, y \in Z(R)$ with $xy = 0$, there is no the case $x, y \in \mathfrak{p}_i$ $(i = 0, 1)$. Because $x, y \in \mathfrak{p}_i$ implies that $0 = xy \in \mathfrak{p}_j$ for $j \neq i$. Therefore, $x \in \mathfrak{p}_j$ or $y \in \mathfrak{p}_j$. So, there is a nonzero element in $\mathfrak{p}_0 \cap \mathfrak{p}_1$. This shows that $\Gamma(R)$ is bipartite. Let us now, $x, y \in Z(R)^*$ with $x \in V_1$ and $y \in V_2$. Then $xy \in \mathfrak{p}_0 \cap \mathfrak{p}_1 = 0$. So, $\Gamma(R)$ is a complete bipartite graph. It is clear that in all complete bipartite graphs $K_{r,s}$ with $r, s \geq 2$, the girth is 4. $\square$

Example 2.7. (1) Consider the idealization $R = \mathbb{Z}(+)\mathbb{Z}_2$. It is easy to see that $Nil(R) = \{(0, 0), (0, 1)\}$ is a prime ideal, say, $\mathfrak{p}_0$. Moreover,

$$\mathbb{Z}(R) = \{(n, \bar{s}) | n \text{ is even and } \bar{s} \in \mathbb{Z}_2\}$$

is a prime ideal, say $\mathfrak{p}_1$. On the other hand,

$$0 = \mathfrak{p}_0 \cap \mathfrak{q},$$

where $\mathfrak{q} = \{(m, 0) | 4 | m\}$ is $\mathfrak{p}_1$-primary. By Theorem 2.6, $\Gamma(R)$ is a star graph.

(2) In $R = \mathbb{Z}_3 \times \mathbb{Z}_3$, we can compute that $Nil(R) = \{(0, 0)\}$,

$$Z(R) = \{(0, 0), (0, 1), (1, 0), (0, 2), (2, 0)\},$$

$$\mathfrak{p}_0 = \{(0, 0), (0, 1), (0, 2)\} \text{ and } \mathfrak{p}_1 = \{(0, 0), (1, 0), (2, 0)\}.$$

Note that $\mathfrak{p}_0 \cap \mathfrak{p}_1 = 0$ and $\mathfrak{p}_0 \cup \mathfrak{p}_1 = Z(R)$. It is easy to see that $\Gamma(R)$ is complete bipartite.

Theorem 2.8. *Let* $0 = \mathfrak{q}_0 \bigcap \mathfrak{q}_1$ *be a minimal primary decomposition with* $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, $\mathfrak{p}_0 \subseteq \mathfrak{p}_1$ *and* $|\mathfrak{p}_0| \geq 7$. *Then* $\Gamma(R)$ *is not planar.*

Proof. There are nonzero $x, t \in R$, where $\mathfrak{p}_0 = 0 :_R x$ and $\mathfrak{p}_1 = 0 :_R t$. The elements x, t and $x + t$ are disjoint and nonzero. There exist nonzero elements v_1, v_2, v_3 in $\mathfrak{p}_0$ adjacent to $x, t, x + t$. So, a subdivision graph of $\Gamma(R)$ is isomorphic to $K_{3,3}$. □

Remark 2.9. Considering the notation of Theorem 2.8, let x, t be not nilpotent elements. Then the result holds for $|\mathfrak{p}_0| \geq 4$. Because in that case, $x + t \notin \mathfrak{p}_0$.

Corollary 2.10. *Suppose that* $\mathfrak{p}_0$ *and* $\mathfrak{p}_1$ *are minimal prime ideals of R with* $\mathfrak{p}_0 \bigcap \mathfrak{p}_1 = 0$. *If* $|\mathfrak{p}_i^*| \geq 3$, *then* $\Gamma(R)$ *is not planar.*

Proof. It immediately follows from Theorem 2.6(ii). □

Theorem 2.11. *Let* $n \geq 2$ *and* $0 = \mathfrak{q}_0 \bigcap \mathfrak{q}_1 \bigcap \cdots \bigcap \mathfrak{q}_n$ *be a minimal primary decomposition of zero ideal such that* $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i = \sqrt{0 :_R x_i}$ *for* $0 \leq i \leq n$. *Then* $gr(\Gamma(R)) = 3$.

Proof. By Proposition 2.1, the subgraph with vertex set $\{x_i \mid 0 \leq i \leq n\}$ is a complete graph. Now, for the case of $n \geq 2$, the cycle $x_0 - - - x_1 - - - x_2 - - - x_0$ exists. □

Theorem 2.12. *Let* $n \geq 2$ *and* $0 = \mathfrak{q}_0 \bigcap \mathfrak{q}_1 \bigcap \cdots \bigcap \mathfrak{q}_n$ *be a minimal primary decomposition. Let for* $0 \leq k \leq n$, $\mathfrak{p}_0, \ldots, \mathfrak{p}_k$ *be the only minimal ideals of R such that* $\left|\bigcap_{i=0}^{k} \mathfrak{p}_k\right| \geq 7$. *Then* $\Gamma(R)$ *is not planar.*

Proof. There are nonzero elements $s_i \in R$ provided $\mathfrak{p}_i = (0 :_R s_i)$. For every $k + 1 \leq j \leq n$ (if $k < n$), there is $0 \leq i \leq k$ that $\mathfrak{p}_i \subseteq \mathfrak{p}_j$. Since $\left|\bigcap_{i=0}^{k} \mathfrak{p}_k\right| \geq 7$, one can see that there are disjoint elements $t_1, t_2, t_3 \in \bigcap_{i=0}^{k} \mathfrak{p}_k$ distinct to s_{i_1}, s_{i_2}, s_{i_3} for $\{i_1, i_2, i_3\} \subseteq \{0, \ldots, n\}$. It is clear that $t_1, t_2, t_3 \in Nil(R)$ adjacent to $s_{i_1}, s_{i_2}, s_{i_3}$. So, there exists a subdivision graph of $\Gamma(R)$ isomorphic to $K_{3,3}$. □

Proposition 2.13. *Let* $Nil(R) = 0$. *Then* $d(x_i) = |\mathfrak{q}_i| - 1$ *for all* $0 \leq i \leq n$.

Proof. By the hypothesis, for all $0 \leq i \leq n$, one has $x_i \notin \mathfrak{p}_i$. So, $(0 :_R x_i) = \mathfrak{q}_i$ by [6, Lemma 4.4]. Hence, $d(x_i)$ is the number of nonzero elements of $\mathfrak{q}_i$. □

Theorem 2.14. *Suppose that* $n \geq 1$. *In the minimal decomposition*

$$0 = \mathfrak{q}_0 \cap \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n,$$

with $Nil(R) \neq 0$, *we have* $diam(\Gamma(R)) \geq 2$.

Proof. Let $diam(\Gamma(R)) \leq 1$. Since $n \geq 1$, it follows from [3, Theorem 2.3] that $diam(\Gamma(R)) = 1$ and $\Gamma(R)$ is connected. This implies that $Z(R)^2 = 0$. So, $\mathfrak{p}_n \subset Nil(R) = \bigcap_{i=0}^{n} \mathfrak{p}_i$. So, $\mathfrak{p}_n \subset \mathfrak{p}_0$, a contradiction. □

Proposition 2.15. *For every* $0 \leq i \leq n$ *and all nonzero elements* $x, y \in \mathfrak{p}_i$, $d(x, y) \leq 2$.

Proof. If $xy = 0$, then there is nothing to prove. Let $xy \neq 0$. Considering $\mathfrak{p}_i = \sqrt{0 : x_i}$, there are $m, n \in \mathbb{N}$ such that $x^n x_i = y^m x_i = 0$. Let m, n be the least integers with these properties. Then $x^n y^{m-1} x_i = x^{n-1} y^m x_i = 0$. Hence, $x --- x^{n-1} y^{m-1} x_i --- y$ is a path with length 2. □

Theorem 2.16. *Let* $Z(R)$ *be an ideal of* R. *Then* $diam(\Gamma(R)) \leq 2$.

Proof. There is $0 \leq k \leq n$ where $Z(R) = \mathfrak{p}_k$. By Proposition 2.15, the proof is completed. □

Corollary 2.17. *Suppose that* $Z(R)$ *is an ideal*. *Then for all pairs* (x, y) *of zero-divisors of* R, $(0 : (x, y)) \neq 0$.

Corollary 2.18. *Let* $n \geq 1$, $Nil(R) \neq 0$ *and* $Z(R)$ *be an ideal of* R. *Then* $diam(\Gamma(R)) = 2$.

Proof. The result follows from Theorems 2.14 and 2.16. □

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