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# ON INTEGRAL REPRESENTATION OF CAUSAL OPERATORS

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## Abstract

In this work we advance a contribution toward tackling the solvability of the problem of representing causal operators with classical Volterra integral operators. We have combined notions from Nemytskii operator theory and frequency domain methods to achieve our results.



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## 1. Introduction and Preliminaries

**Definition 1.1.** Let $S(\Omega, \mathbb{R})$ be the linear space of real-valued measurable functions defined on $\Omega \subset \mathbb{R}^N$, an operator $T : S \to S$ given by

$$(Tx)(t) = \Phi(t) + \int_{t_0}^{t} k(t - s) g(x(s)) ds \tag{1}$$

is called a *classical nonlinear Volterra operator* of convolution type, where $\Phi, k \in S$ and $g$ is some continuous function.

**Definition 1.2.** Let $E$ be a function space with elements defined from the interval $[0, \tau)$ into $\mathbb{R}^N$ and $V : E \to E$ an operator. Then $V$ is called a *causal operator* if for each $x, y \in E$ such that $x(s) = y(s)$ for $0 \leq s \leq t$, then $(Vx)(s) = (Vy)(s)$ for $0 \leq s \leq t$ with $t < \tau$ arbitrary.

Causal operators are often called *nonanticipative operators*. They are also labeled *abstract Volterra operators* in the language of Tonelli and Corduneanu [1]. Apart from the classical Volterra integral operators, a good and more general example of abstract Volterra operators is the Nemytskii operators defined below:

**Definition 1.3.** Let $F = F(t, x(t))$ satisfy the Carathéodory conditions:

(C1) For a fixed $x$, the function $F$ is continuous in $t$.

(C2) For a fixed $t$, the function $F$ is measurable in $x$.

Then the operator $T$ given by $(Tx)(t) = F(t, x(t))$ is called *Nemytskii operator*.

Nemytskii operators are, generally, real-valued composition operators $(t, x_1, ..., x_n) \mapsto h(t, x_1, ..., x_n)$ but for our purpose, Definition 1.3 suffices. Abstract Volterra operators have been studied by various authors some of whom are Tonelli [8], Tychonoff [9] and Corduneanu [1] in the investigations of certain evolutionary processes dependent on heredity like feedback control equations in automatic control theory. Many properties of abstract Volterra operators including properties of Nemytskii operators have been treated by Corduneanu in [1, 2].

An important theorem due to Krasnoselski (treated in [3, 4] and [1]) on integrability of Nemytskii operators is given below:

**Theorem 1.1.** *Let* $p_i$, $i = 1, \ldots, k$, *and* $q$ *be finite,* $\mu(t) > 0$, $\mu \in L_q[0, \infty)$, $c > 0$ *and the Nemytskii operator F satisfies the Carathéodory conditions* (C1) *and* (C2) *in Definition* 1.3 *together with the following growth condition*:

$$| F(t, x_1, \ldots, x_k) | \leq \mu(t) + c(| x_1 |^{\frac{p_1}{q}} + \cdots + | x_k |^{\frac{p_k}{q}}). \tag{2}$$

*Then F maps* $L_{p_1}[0, \infty) \times \cdots \times L_{p_k}[0, \infty)$ *into* $L_q[0, \infty)$.

We must observe that if $p_i = 2$ for all $i = 1, \ldots, k$ and $\mu \in L_2[0, \infty)$ (i.e., $q = 2$), then (2) becomes

$$| F(t, x_1, \ldots, x_k) | \leq \mu(t) + c(| x_1 | + \cdots + | x_k |). \tag{3}$$

In this case, $F$ maps $L_2[0, \infty) \times \cdots \times L_2[0, \infty)$ into $L_2[0, \infty)$.

Obviously, the classical Volterra integral operators

$$(Tx)(t) = \Phi(t) + \int_0^t k(t, s) g(x(s)) ds \tag{4}$$

are Nemytskii operators whenever $\Phi, x \in L_2[0, \infty)$ and $g$ satisfies the sector condition

$$0 < \frac{g(x) - g(y)}{x - y} \leq k_1, \quad \forall x, y \in L_2[0, \infty) \text{ and } k_1 > 0, \ g(0) = 0. \tag{5}$$

Also, if $\Phi, x \in L_2[0, \infty)$ and $g$ satisfies the sector condition (5), then the Nemytskii operator $F(t, \Phi(t), x(t)) = \Phi(t) + \int_0^t k(t, s) g(x(s)) ds$ satisfies the growth condition (3) trivially.

A mathematical problem (see, for example, Corduneanu [1, p. 116]) is posed as: *under what conditions can we assert that an abstract Volterra operator* $G(t, x(t))$ *is representable in the form*

$$(Gx)(t) = x^0 + \int_0^t (Vx)(s) ds \tag{6}$$

*with V another abstract Volterra operator*? In Corduneanu [2], the open problem was solved for linear functional differential equations using Riesz representation theorem in the Hilbert space $L_2([0, \tau), \mathbb{R}^N)$ and in the Banach space $L_1([0, \tau), \mathbb{R}^N)$ using semigroup properties. The need arises to extend and generalizes these results to more general nonlinear causal operator equations, i.e., not necessarily linear functional differential equations) in $L_2[0, \infty)$ space thereafter extend it to arbitrary Banach space. Results in $L_2[0, \infty)$ are easily applicable in the space $L^2_{loc}([0, T), \mathbb{R}^N)$. It remains open to solve the problem for nonlinear functional equations (not necessarily linear functional differential equations).

The purpose of this work is to contribute to the solvability of the problem above by giving conditions under which an abstract Volterra operator $G(t, x(t))$ can be represented by a convolution type nonlinear Volterra integral operator (1). In the sequel, we shall make use of the following frequency domain result (see, for example, Leonov [5, p. 48], Nohel and Shea [7, p. 283]).

**Theorem 1.2.** *Let* $\Phi, k \in L_2[0, \infty)$ *and g satisfies the sector condition* (5). *If k satisfies the frequency domain condition*

$$\operatorname{Re}\left\{(1 + i\omega q)\int_0^\infty e^{-i\omega t}\Phi(t)\,dt\right\} \geq 0,\ q > 0,\ \omega \in [0, \infty), \tag{7}$$

*then the classical operator T given by* (1) *has a fixed point* $\sigma(t) \in L_2[0, \infty)$.

## 2. Main Results

**Theorem 2.1.** *Let* $G(t, x(t))$ *be a Nemytskii operator in the sense of Definition* 1.3 *such that* $G(t, x(t)) = 0, \forall t < 0, T$ *is a classical nonlinear Volterra operator of convolution type* (1). *Suppose* $G(t, x(t))$ *satisfies the growth condition* (3) *together with the following conditions*:

(a) $G(0, x(0)) = \Phi(0)$ *for all* $x \in L_2[0, \infty)$.

(b) $$\frac{\int_0^\infty e^{-i\omega t}[G(t, x(t)) - \Phi(t)]\,dt}{\int_0^\infty e^{i\omega t} g(x(t))\,dt} = constant \neq 0 \text{ for all } x \in L_2[0, \infty).$$

*Further, assume that g satisfies the sector condition* (5) *and* $\Phi$ *satisfies the frequency domain condition* (7). *Then there exists* $k_1 \in L_2[0, \infty)$ *such that*

$$G(t,\ x(t)) = \Phi(t) + \int_{t_0}^{t} k_1(t-s)\, g(x(s))\, ds. \tag{8}$$

*If, in addition, we have that the set of common fixed points of T and G is not empty, then* $(G)(t) = (T)(t)$.

**Proof.** By Theorem 1.1 and the sector condition (5), it is clear that $T$ is a mapping from $L_2[0, \infty)$ into itself while Theorem 1.2 and the frequency domain condition (7) ensure that $T$ has a fixed point in $L_2[0, \infty)$. Next, taking condition (a) into account, we define the operator $F_1$ by

$$F_1(t,\ x(t)) = G(t,\ x(t)) - \Phi(t). \tag{9}$$

We shall show that there exists $k_1 \in L[0, \infty)$ such that

$$F_1(t,\ x(t)) = \int_0^t k_1(t-s)\, g(x(s))\, ds. \tag{10}$$

Taking Fourier transform of (9) and dividing the result by the Fourier transform of $g(x(t))$, we obtain for all $x \in L_2[0, \infty)$ (taking conditions (b) of Theorem 2.1 into account)

$$\frac{\int_0^\infty e^{-i\omega t} F_1(t,\ x(t))\, dt}{\int_0^\infty e^{-i\omega t} g(x(t))\, dt} = \frac{\int_0^\infty e^{-i\omega t}[G(t,\ x(t)) - \Phi(t)]\, dt}{\int_0^\infty e^{-i\omega t} g(x(t))\, dt} = \text{constant} \neq 0. \tag{11}$$

Let $\dfrac{\int_0^\infty e^{-i\omega t} F_1(t,\ x(t))\, dt}{\int_0^\infty e^{-i\omega t} g(x(t))\, dt} = z$ for some complex number $z$. Since $F_1$ satisfies the growth condition (2), we observe that $F_1(t,\ x(t)) \in L_2[0, \infty),\ \forall x \in L_2[0, \infty)$ which implies that $\int_0^\infty e^{-i\omega t} F_1(t,\ x(t))\, dt$ is invertible. But $\int_0^\infty e^{-i\omega t} F_1(t,\ x(t))\, dt$ $= z\int_0^\infty e^{-i\omega t} g(x(t))\, dt$ so that on defining $k_1$ by $z = \int_0^\infty e^{-i\omega t} k_1(t)\, dt$, we obtain

$$\int_0^\infty e^{-i\omega t} F_1(t,\ x(t))\, dt = \left(\int_0^\infty e^{-i\omega t} k_1(t)\, dt\right)\left(\int_0^\infty e^{-i\omega t} g(x(t))\, dt\right). \tag{12}$$

By convolution theory and inversion formula, equation (12) yields (10) which is the desired result.

Further, if the common fixed point set of $T$ and $G$ is not empty, i.e., $F(T) \cap F(G) \neq \emptyset$ (where $F(T)$ and $F(G)$ denote the fixed points sets of $T$ and $G$, respectively). Using the fact that $F(T) \cap F(G) \neq \emptyset$, we set $\sigma(t) \in L_2[0, \infty)$ as a common fixed point of $T$ and $G$ and from equation number (11), we have:

$$\frac{\int_0^\infty e^{-i\omega t} F_1(t, \sigma(t))\, dt}{\int_0^\infty e^{-i\omega t} g(\sigma(t))\, dt} = \frac{\int_0^\infty e^{-i\omega t} [\sigma(t) - \Phi(t)]\, dt}{\int_0^\infty e^{-i\omega t} g(\sigma(t))\, dt} = \text{constant}$$

$$= \frac{\int_0^\infty e^{-i\omega t} \int_0^t k(t-s) g(\sigma(t))\, dt}{\int_0^\infty e^{-i\omega t} g(\sigma(s))\, dt} = \text{constant}$$

$$= \frac{\left(\int_0^\infty e^{-i\omega t} k(t)\, dt\right)\left(\int_0^\infty e^{-i\omega t} g(\sigma(t))\, dt\right)}{\int_0^\infty e^{-i\omega t} g(\sigma(s))\, dt} = \text{constant.}$$

This yields $\int_0^\infty e^{-i\omega t} F_1(t, x(t))\, dt = \left(\int_0^\infty e^{-i\omega t} k(t)\, dt\right)\left(\int_0^\infty e^{-i\omega t} g(\sigma(t))\, dt\right)$ which by inversion formula and convolution theory, yields (1). Therefore, $G(t, x(t))$ is given by the RHS of (1) which means that $(G)(t) = (T)(t)$ provided $F(T) \cap F(G) \neq \emptyset$. End of the proof.

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