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BELTON AND GEAR REVISED

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Abstract

This paper focuses on issues arising from using pairwise comparisons in order to rank items. In particular, the use of aggregation in multiattribute/team decision making has led to rank reversal if the incorrect model is used. Here we show that the linear space model derived from Barzilai axioms [Oper. Res. Lett. 6 (1987), 131-134] and extended by Foster [Linear models for pairwise comparisons and measures of agreement, Ph.D. Thesis, Brunel University, 1994] gives inbuilt normalization procedure and aggregation procedure that no rank reversal can occur. We also give examples of the failures of various suggested normalization procedures. We investigate the suggested method for normalization given in Belton [Omega 11 (1983), 227-230] and show that it is not an accurate method because it might cause rank reversal.

0. Introduction

(a) The *Analytic Hierarchy Process* is considered as a useful tool for the solution of diverse problems and also to develop a theory for modelling unstructured problem in different fields. This method is presented by Thomas Saaty. In general, using the AHP method [15] in solving a decision problem involves four main steps [20]:

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Step 1. Setting up the decision hierarchy by breaking down the decision problem into a hierarchy of interrelated decision elements.

Step 2. Collecting input data by pairwise comparisons of decision elements, i.e., comparing the options under each criterion.

Step 3. Using the *eigenvalue* method [15] to estimate the relative weights of decision elements on each level of the hierarchy.

Step 4. Aggregating the relative weights of decision elements by using the *weighted arithmetic mean* to get the final scores for the decision options. This method uses a semantic scale and associated 1-9 ratio scale. A complete set of pairwise comparisons is elicited for the relative importance of elements at each level of the hierarchy with respect to the level above. If each of n options is compared with others, then a comparison $n \times n$ matrix $C = [c_{ij}]$ is created which has $c_{ij} > 0, c_{ij} = c_{ij}^{-1}$, for every i, j. Using the Perron-Frobenius theorem for this positive matrix we have the existence of a maximal eigenvalue $\lambda \geq n$ and the associated one dimensional eigenspace has a unique normalized positive eigenvector $(w_1, w_2, \ldots, w_n)$, where $\sum_{i=1}^{n} w_i = 1, w_i > 0, i = 1, \ldots, n$. Saaty then used the index $C = \frac{\lambda - n}{n - 1}$ as a measure of consistency and the normalized eigenvector as the normalized weights of the options [9].

(b) *Linear model*

We used two equivalent mathematical models which are the multiplicative model and the additive model. Both models are based upon obtaining a comparison matrix from the comparisons but differ in

(1) mathematical treatment of that matrix,

(2) the scales that are used.

The main difference between the work of Barzilai et al. [2] and Foster and Algie [10] is that Barzilai worked on a multiplicative model whereas Foster worked on the additive model. So, the bijection map Ln allows linear space techniques to be used to analyze reciprocal matrices.

In our previous work we explained the required properties and axioms for both models and we represent axiom three, i.e., interlevel consistency as in the following diagram:

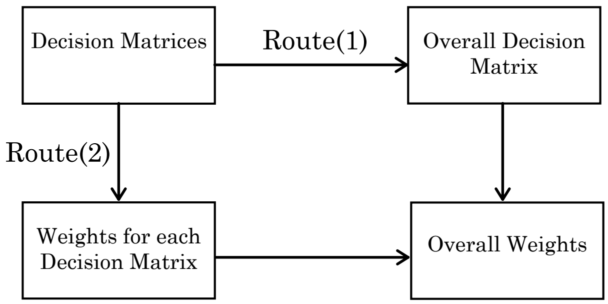

Figure 1

Interlevel consistency. We mean that the two routes given in Figure 1 result in the same set of weights, and in any reasonable model the two routes should coincide. This is called *interlevel consistency* [2].

Interlevel consistency is guaranteed by adopting the axiom that scoring is a linear map on the vector space of decision matrices. Thus this axiom can be considered as asserting the equality of the two routes. Note that aggregation could be over criteria or decision makers.

One important property of Barzilai's axioms is that the solution leads to interlevel consistent decisions, i.e., the requirement that f be a linear homomorphism preserves this structure. Also, more important, however, the geometric mean solution satisfies:

$$f(A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_r^{\alpha_r}) = [f(A_1)]^{\alpha_1} [f(A_2)]^{\alpha_2} \cdots [f(A_r)]^{\alpha_r}$$

which guarantees interlevel consistency. Whereas, in the original AHP the eigenvector solution does not satisfy this property. Barzilai [13] observes that the AHP, since it is based upon ratio information, should be converted into a variant with a multiplicative structure with the geometric row means of the decision matrices to calculate the relative scores, and with a geometric mean aggregation rule to calculate the final scores of the alternatives. Consequently, one could aggregate in two different ways without affecting the final scores: either by combining first the decision matrices into one matrix from which one obtains the final scores, or by combining the relative scores under the respective criteria or for decision makers into a vector of final scores.

Besides, Crawford and Williams [6] derive the geometric mean from statistical considerations and show that it is preferable to the eigenvector solution in several important respects.

Belton and Gear introduced two examples in their paper [4] and found that in certain circumstances the Analytical Hierarchy Process (AHP) [15] can give contradictory results, i.e., rank reversal phenomenon. In addition, Belton pointed out that the root of this inconsistency, is the normalization procedure in the second step of the AHP at which the vectors which denote the relative importance of options with respect to individual criteria are normalized so that their entries sum to 1. We represent these examples as follows:

(1) By using first, the eigenvector method applying the two routes in Figure 1 to each example and secondly, the linear method applying the two routes to each example.

(2) By applying the SMART method to Belton's example (2) using the normalization procedure that Saaty used.

(3) Belton suggested that a reasonable procedure to give a consistent ranking is that one should normalize the eigenvectors at the second step of the AHP in order that the maximum entry is one rather than the entries summing to one. We investigate this normalization procedure and we find that Belton's suggested method does not work perfectly because rank reversal still occurs while using this normalization procedure. We apply the suggested normalization procedure to one of Saaty's examples, i.e., the example of buying a house [16] and still we find that rank reversal occurs.

The phenomenon of rank reversal can be simply stated as follows: The ranking of options determined by any method of scoring may be altered by the addition of another option for consideration. This characteristic of the methodology has been well known for years, and has been discussed in a number of articles by critics and by defenders of the AHP [16].

Belton used two examples to illustrate the uncertainty of the AHP We use the same two examples by applying the two routes to each example by using the eigenvector method and linear method.

1. The Application of the Two Routes to Belton's Examples using the Eigenvector Method and the Linear Method

Example 1. Using the eigenvector method and the linear method.

In this example we demonstrate the use of the eigenvector method and the linear method. Suppose that the decision maker wishes to evaluate three options $\{A, B, C\}$ on three criteria $\{a, b, c\}$. The decision matrices specified for this example are:

Criterion $[a]$ Criterion $[b]$ Criterion $[c]$

$$\begin{bmatrix} 1 & 1/9 & 1 \\ 9 & 1 & 9 \\ 1 & 1/9 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 9 & 9 \\ 1/9 & 1 & 1 \\ 1/9 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 8/9 & 8 \\ 9/8 & 1 & 9 \\ 1/8 & 1/9 & 1 \end{bmatrix}$$

In this example it is assumed that all criteria are considered equally important. Therefore the criteria weights are (1/3, 1/3, 1/3).

Results

Using the AHP method

$$\{0.44, 0.46, 0.10\} \text{ Route (1) } B > A > C$$

$$\{0.45, 0.47, 0.08\} \text{ Route (2) } B > A > C.$$

Using the linear method

$$\{0.44, 0.46, 0.10\} \text{ Routes (1) and (2) } B > A > C.$$

Analysis of results in example 1: When using the AHP method gives similar ranking of the options but different relative weights by applying Routes (1) and (2). Whereas, using the linear method, we find out that both routes give the same weights that are equal to the weights in route (1) while using the AHP [16]. Recall that if the matrix is consistent, then the normalized geometric mean solution is equal to the normalized eigenvector solution. The two solutions are always the same, regardless of consistency, if the dimension is less than or equal to three [6]. This result does not hold for inconsistent matrices with dimension greater than three [6].

Example 2. Using the AHP method and the linear method.

In this example we demonstrate the use of the rank reversal. Now, suppose an additional option, D is introduced to the problem. Besides, option D is considered an exact copy of option B. The resulting decision matrices are:

Criterion $[a]$ Criterion $[b]$ Criterion $[c]$

$$\begin{bmatrix} 1 & 1/9 & 1 & 1/9 \\ 9 & 1 & 9 & 1 \\ 1 & 1/9 & 1 & 1/9 \\ 9 & 1 & 9 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 9 & 9 & 9 \\ 1/9 & 1 & 1 & 1 \\ 1/9 & 1 & 1 & 1 \\ 1/9 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 8/9 & 8 & 8/9 \\ 9/8 & 1 & 9 & 1 \\ 1/8 & 1/9 & 1 & 1/9 \\ 9/8 & 1 & 9 & 1 \end{bmatrix}$$

It is assumed that the decision maker would still consider the criteria to be of equal importance and these matrices are all consistent. Consequently, the normalized geometric mean solution is equal to the normalized eigenvector solution [6].

Results

Using the AHP method

$\{0.30, 0.31, 0.08, 0.31\}$ Route (1) $D \sim B > A > C$

$\{0.36, 0.29, 0.06, 0.29\}$ Route (2) $A > B \sim D > C$.

Using the linear method

$\{0.30, 0.31, 0.08, 0.31\}$ Routes (1) and (2) $D \sim B > A > C$.

Analysis of results in example 2: Using the AHP there is a rank reversal by applying Route (2) between option A and option B as it is found in Belton's example. Whereas, there is no rank reversal by applying Route (1). Obviously, there is a difference in the solutions we obtain using Routes (1) and (2) in the AHP method. On the other hand, using the L.M.

gives the same scoring for both routes that is equal to the weights in Route (1) by the AHP method. Rank reversal occurs while using Route (2) in example 2. We are insisting on interlevel consistency, i.e., Route (1) = Route (2). But we can clearly see that on applying the AHP this does not occur. This is due to the weighted arithmetic aggregation procedure that takes place in Route (2). Barzilai referred to this property as the interlevel inconsistency of Saaty's solution. We refer this to Interlevel Consistency Axiom. On the other hand, using the weighted geometric aggregation rule in the linear method implies Interlevel Consistency to occur.

2. Applying the Normalization Procedure to SMART Method

We use the normalization procedure while generating scores using the SMART method.

	Options			
	A	B	C	D
Criteria				
C1	1	9	1	9
C2	9	1	1	1
C3	8	9	1	9

Option $A[(1+9+8)(1/3)] = 6$.

Option $B[(9+1+9)(1/3)] = 6.33$.

Option $C[(1+1+1)(1/3)] = 1$.

Option $D[(9+1+9)(1/3)] = 6.33$.

Thus, the rank order of options will be:

$$B \sim D > A > C.$$

So, the rank order of the options remains unchanged before normalization.

Now, let us normalize the scores of options on each criterion and then find the rank order of the options.

	Options			
	A	B	C	D
Criteria				
C1	1/20	9/20	1/20	9/20
C2	9/12	1/12	1/12	1/12
C3	8/27	9/27	1/27	9/27

Option $A[(1/20 + 9/12 + 8/27)(1/3)] = 0.365.$

Option $B[(9/20 + 1/12 + 9/27)(1/3)] = 0.288.$

Option $C[(1/20 + 1/12 + 1/27)(1/3)] = 0.056.$

Option $D[(9/20 + 1/12 + 9/27)(1/3)] = 0.288.$

As a result, the rank order of options will be:

$$A > B \sim D > C.$$

So, the rank order of the options is changed after the normalization process.

3. Suggested Normalization Procedure by Belton

Belton and Gear suggested that one should normalize the eigenvectors at the second step of the AHP so that, the maximum entry is one rather than the entries summing to one. We examine this normalization procedure and find that it does not work perfectly. Rank reversal still occurs while using this normalization procedure. We apply this normalization procedure to one of Saaty's examples, i.e., the example of buying a house [16]. The following outcomes are as results of using the L.M. in that example. This solution is the solution that we consider the exact solution because linear space analysis [9] with the axioms discussed before shows the correct normalization and aggregation procedures. In this case for the multiplicative model the normalization should be

$$(y_1, y_2, \dots, y_n) = \prod_{i=1}^{n} y_i = 1$$

for the multiplicative case.

Example: Buying a house [Saaty, p. 36]					using G.M.M.	
Decision matrix under criterion (1)			G.M.M.	Norm. G.M.M.		
1	6	8	3.634	**0.754**		
0.167	1	4	0.874	**0.181**		
0.125	0.25	1	0.315	**0.065**		
			4.823	1		
			1			
Decision matrix under criterion (2)			G.M.M.	Norm. G.M.M.		
1	7	0.2	1.119	**0.233**		
0.143	1	0.125	0.261	**0.054**		
5	8	1	3.42	**0.712**		
			4.8	1		
			1			
Decision matrix under criterion (3)			G.M.M.	Norm. G.M.M.		
1	8	6	3.634	**0.754**		
0.125	1	0.25	0.315	**0.065**		
0.167	4	1	0.874	**0.181**		
			4.823	1		
			1			

Decision matrix under criterion (4)			G.M.M.	Norm. G.M.M.		
1	1	1	1	**0.333**		
1	1	1	1	**0.333**		
1	1	1	1	**0.333**		
			3	1		
			1			
Decision matrix under criterion (5)			G.M.M.	Norm. G.M.M.		
1	5	4	2.714	**0.674**		
0.2	1	0.333	0.405	**0.101**		
0.25	3	1	0.909	**0.226**		
			4.028	1		
			1			
Decision matrix under criterion (6)			G.M.M.	Norm. G.M.M.		
1	8	6	3.634	**0.747**		
0.125	1	0.2	0.292	**0.06**		
0.167	5	1	0.941	**0.193**		
			4.868	1		
			1			
Decision matrix under criterion (7)			G.M.M.	Norm. G.M.M.		
1	0.5	0.5	0.63	**0.2**		
2	1	1	1.26	**0.4**		
2	1	1	1.26	**0.4**		
			3.15	1		
			1			

Decision matrix under criterion (8) G.M.M. Norm. G.M.M.						
1	0.143	0.2	0.306	**0.072**		
7	1	3	2.759	**0.649**		
5	0.333	1	1.186	**0.279**		
			4.25	1		
			1			
SOLUTION ROUTE (2)			Using G.M.M.			
			G.M.M.	Norm. G.M.M.		
			1.081	**0.36**		
			0.991	**0.33**		
			0.934	**0.311**		
			3.005	1		

So, the ranking of the three houses while using the L.M. is:

House A then House B and then House C.

The following is the ranking of the three houses by using the AHP method [16] and the next solution by applying the suggested normalization procedure by Belton.

EVEC1	EVEC2	EVEC3	EVEC4	EVEC5	EVEC6	EVEC7	EVEC8	Criteria Weights	Final Scores
0.754	0.233	0.754	0.333	0.674	0.747	0.2	0.072	**0.173**	**0.3959**
0.181	0.055	0.065	0.333	0.101	0.06	0.4	0.65	**0.054**	**0.341**
0.065	0.713	0.181	0.333	0.226	0.193	0.4	0.278	**0.188**	**0.2631**
								0.018	1.0001
1	1.001	1		1.001	1	1	1	**0.031**	
0.754	**0.713**	**0.754**	**0.333**	**0.674**	**0.747**	**0.4**	**0.65**	**0.036**	123
								0.167	
								0.333	
								1	

Normalization step so that the maximum entry is one rather the entries summing to one.

EVEC1	EVEC2	EVEC3	EVEC4	EVEC5	EVEC6	EVEC7	EVEC8		
1	0.3268	1	1	1	1	0.5	0.1108		
0.2401	0.0771	0.0862	1	0.1499	0.0803	1	1		
0.0862	1	0.2401	1	0.3353	0.2584	1	0.4277		
	Criteria Weights	**Final Scores**	**Norm. Final Scores**						
	0.173	**0.584**	**0.358**						
	0.054	**0.5874**	**0.36**						
	0.188	**0.4612**	**0.282**						
	0.018	1.6326	1						
	0.031								
	0.036	ORDER	213						
	0.167								
	0.333								
	1								

So, the ranking of the three houses while using the AHP is:

House A then House B and then House C.

On the other hand, the ranking of the three houses while using Belton's suggested normalization procedure are:

House B then House A and then House C.

So, rank reversal occurs while using Belton's normalization procedure.

4. Summary

We see that the use of the aggregation and normalization leads to a crucial difference between the AHP method and the L.M. The AHP method used first the eigenvector technique to give a set of n relative scores and then to normalize these scores so that the sum is one. Also, the AHP used arithmetic aggregation rule to give final scores. Whereas, the linear method first used the geometric mean to give a set of n relative scores, these scores are already normalized because of the structure of the linear multiplicative model. Also, the linear method used the geometric mean aggregation rule to give the final scores. Moreover, Belton's suggested normalization procedure also leads to rank reversal.

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