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AN AXIOMATIC MODEL FOR MCDM USING PAIRWISE COMPARISONS

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Abstract

We show, up to accepting axioms on scales, scoring and aggregation, and using the method of pairwise comparisons, that there is only one model for scoring a set of options with respect to a hierarchy of criteria. The model allows for team decision making with respect to each criterion and we show that aggregation across teams is possible only under the strict use of appropriate scales. We show that there is a clear distinction to be made between the options using a category difference scale and the weighting of criteria using a ratio scale. The interaction between policy, as given by the hierarchy of weighted criteria, and the scores is clearly demonstrated.

1. Introduction

The technique of pairwise comparisons has a long history of analysis and application, [1, 2, 9, 11, 15, 19]. In this paper we establish some basic principles which govern its use in multicriteria problems. In this context there have been various controversies surrounding some popular and well used applications, e.g., the AHP of Saaty [6, 7, 8, 10, 20] and it is one of the aims of this paper to establish an axiomatic and coherent treatment which clears up some of these difficulties.

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The major features of the model we present here are:

(1) The use of category scales in pairwise comparisons, [12, 17, 18].

(2) An axiomatic approach to aggregation across decision makers or criteria, [12].

(3) An axiomatic approach to scoring options, weighting criteria and measuring the relative influence of decision makers.

Aggregation is briefly covered in Section 4, details can be found in [12]. The work on scales is covered in Section 3.

In Section 2 we define the decision problem and its basic modelling. The main results on scoring options and weighting criteria are in Section 5 and we finish with a summary of the main points of the model in Section 6.

1.1. Notation and definitions

N is the set of natural numbers, R is the set of real numbers and R_+ is the set of positive non-zero real numbers. The mathematics used in the modelling is standard and relies upon elementary results in linear algebra and convex subsets of linear spaces, see [12, 13, 14, 15] for more details. All linear spaces are over R.

A part from the linear space theory we also need permutation matrices in Section 5.

Permutation matrices

Let $I(n) = \{1, 2, ..., n\}$. If $p : I(n) \to I(n)$ is a permutation, i.e., a bijection, then the *permutation matrix* $P = (p_{ij})$ of p is the $n \times n$ matrix defined by $p_{ij} = 1$ if $p(i) = j$, $p_{ij} = 0$ otherwise. Note that if $x = (x_1, ..., x_n)^{tr} \in R^n$ is a column vector, then $Px^{tr} = (x_{p(1)}, x_{p(2)}, ..., x_{p(n)})$, i.e., P permutes the components of the vector according to p.

We say that a matrix P is a *permutation matrix* if it is the permutation matrix of some permutation.

We note that if P is a permutation matrix associated to the permutation p, then $P^{-1} = P^T$ and for any $n \times n$ matrix $A = (a_{ij})$ we have $P^{-1}AP = (a_{p(i)p(j)})$, i.e., we permute rows and columns in the same way.

2. The Decision Problem

2.1. Options and the criterion hierarchy

There is a set of n options to be scored with respect to a policy structure given by a hierarchy of criteria as defined in [19]. The overall objective sits at the top of the tree and it is decomposed into subcriteria.

2.2. The criteria

The "leaf" subcriteria are those at the bottom of the hierarchy, and are used to directly assess the options. The basic structural unit in such a hierarchy is that of a criterion C split into leaf subcriteria $C_1, \ldots, C_m$.

If a process can be developed such that the option scores at the leaf subcriteria level can be combined or aggregated to give scores at the criterion level for such a structural unit, then it is clear that we iterate this process as we move up the hierarchy in order to get the overall scores of the options. Thus we limit our analysis to this basic structure and describe such an aggregation process using weighted criteria in Section 4.

2.3. Criterion teams

The options are assessed with respect to each leaf criterion c by a team of decision makers called the *criterion team* T_c.

We assume that each decision maker uses the method of pairwise comparisons and that he/she obtains a set of scores for the options. These scores are obtained from decision makers by the use of appropriate scales which are derived from validated stimulus-response theory in psychophysics [21] and from the standard treatment of scales in terms of allowable transformations, [16]. See Section 3 for details.

It is worth noting here that the pairwise comparison response from the decision maker is requested in terms of a category difference scale as categories are the only invariants of the allowable transformations we use. These categories form an absolute scale and can be thought of as levels of "desirability" with respect to the relevant criterion. This allows us to combine decision makers' scores, or decision matrices, at the category difference level as they are using the same scale and the conditions of interpersonal communication are satisfied - see [3, 16].

2.4. Aggregation

Aggregation in criterion teams

The team decision for a leaf criterion is to be obtained by combining or aggregating across the teams' decisions. This is achieved by using decision maker influences, see 2.5 below, and the aggregation process as outlined in Section 4.

Aggregation over criteria

We now use criteria weights for each of the leaf subcriteria $C_1, \ldots, C_m$ with respect to C in order to aggregate the scores obtained for each of the subcriteria into overall scores for the options with respect to C. See 2.5 below and Section 4.

2.5. Decision maker influences and criteria weights

The process of aggregation outlined above for decision makers and for criteria is based on giving weights to these objects. These weights reflect the relative importance of either the decision maker in his/her criterion team or the leaf criterion amongst the other leaf criteria with respect to their common overarching criterion. These weights are assumed, for the purpose of this model, to be obtained by pairwise comparison methods - but the underlying scale is a simple ratio scale as we can think of these weights as trade off coefficients between the absolute scales given by the category levels obtained for the options. See 3.3 below. Thus the pairwise comparisons are multiplicative in character - this is in contrast with the comparison of options as there we use a category difference scale.

The influences of the decision makers could also be determined by a team, as could the weights of the criteria. The formal work on aggregation from Section 4 still applies and we can obtain the team weights or influences.

We simplify our discussion below by supposing that decision maker influences are given and we discuss the derivation of criteria weights using consistency ideas in Section 6.

We shall call a weighted set of subcriteria, with respect to a criterion C, a C policy or just a *policy* if C is understood.

The team of decision makers deciding on the policy is called the *policy team*.

3. Scales for Pairwise Comparisons: Scores and Weights

3.1. Response scales

We discuss here the responses of decision makers in a criterion team to the options with respect to a fixed criterion. It is assumed that each decision maker in the team will respond to the stimulus of each option with values which follow a power law.

A scale is defined by the group of allowable transformations of the scale, see [16] for a well presented discussion. As we need the criterion team to have a common view of the options we assume, *a priori*, that the allowable transformations of the scale are given by maps of the form $\phi(\nu) = \alpha\nu^{\beta};\quad \alpha, \beta > 0$ and that we can transform between individual decision makers scales by these maps. Note that these are increasing maps and are homeomorphisms of intervals of strictly positive real numbers. It follows from this that all statements concerning scales are subject to the proviso that they are invariant under allowable transformations.

In more detail, we suppose that there are k decision makers $DM_1, \ldots, DM_k$ in the criterion team. When the ith decision maker makes comparisons using a scale it is assumed that he/she is using an underlying *representative interval* $[m_c^{(i)}, M_c^{(i)}]$.

This establishes for DM_i the context range for the underlying *response scale* for the options with respect to the given criterion. We also assume, without loss of generality, that there is an orientation for any representative interval in that the greater the response value for an option the more desirable that option becomes w.r.t. the criterion. Allowable transformations preserve this orientation.

Between any two decision makers DM_i, DM_j in the criterion team we assume there is an allowable transformation of their response scales:

$$\phi_{i,j} : [m_c^{(i)}, M_c^{(i)}] \to [m_c^{(j)}, M_c^{(j)}], \quad \phi_{ij}(\nu) = \alpha_{ij}\nu^{\beta_{ij}}, \quad \alpha_{ij} > 0, \quad \beta_{ij} > 0,$$

where $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$, $\phi_{ii}(\nu) = \nu$, $\phi_{ij}^{-1} = \phi_{ji}$. It follows that $\beta_{ij}\beta_{jk} = \beta_{ik}$, $\alpha_{ij}^{\beta_{jk}}\alpha_{jk} = \alpha_{ik}$.

3.2. Category scales

Given the decision makers' response scales and the allowable transformations between these scales we have to identify an invariant scale common to the team. This we achieve by using the construction in [12]. There, given a response scale $[m_c, M_c]$ for a decision maker for a criterion c, we showed, using a minimization of error principle, that there are natural scale points of the form $m_c r_c^i$, $i = 0, \ldots, t$, where $r_c = (M_c/m_c)^{1/2^t}$, i.e., they form a geometric progression with common ratio r_c which we will call the *progression factor* for the criterion c. The dividing points are called the *category boundaries*. The index i of a category boundary $m_c r_c^i$ is called the *category*. We note the important fact that these categories are invariant under allowable transformations as geometric progression is sent to a geometric progression by allowable maps, i.e., if $m_c r^i$ is a category boundary, then $\phi(m_c r_c^i) = \alpha m_c^{\beta}(r_c^{\beta})^i$, $\phi(\nu) = \alpha\nu^{\beta}$; $\alpha, \beta > 0$ an allowable transformation.

Hence we see immediately that categories are independent of the representative interval of a scale and hence are invariants of the scales we are dealing with if we allow the scale group to be the group of all allowable transformations. We shall call these scales *category scales*. We also note that the scale given by the categories is an absolute scale, i.e., the categories have only the identity transformation as an allowable transformation.

We note that the number of category scale points N in such a category scale is of the form $2^t + 1$ and once a representative interval is given and $N = 2^t + 1$ is given then the value of r_c is easily determined for that representative interval by $m_c r_c^{N-1} = M_c$.

We note from the method of building scales in [12] that $N - 1$ is a power of two and so we adopt the well used category scale with 9 points.

Category difference scale and decision matrices

When comparisons are made between options i and j, say, the decision maker is asked to make the comparison in the following terms:

"Which option is more desirable with respect to c and indicate the difference in desirability on this 0-8 scale."

The decision maker reports back the perceived category differences of the options, i.e., if option i is perceived to be associated to category a and option j to category b, then the decision maker reports back $a - b$ for the comparison of option i to option j. Note that $b - a$ should then be reported back as the comparison of j to i. Thus the response scale lies in the range of integers from -8 to 8 if we adopt the fixed value of $N = 9$.

Decision matrices using pairwise comparisons in a category scale

Comparing each pair of options gives a decision matrix with entries in the range $[-8, 8]$ and we observe that this matrix is skew symmetric.

3.3. Comparison scales for weights

By weights we mean either the values associated to criteria which reflect their relative importance in terms of the overall objective or the weights we assign to decision makers in a criterion or policy team to reflect their relative influences. For simplicity we restrict discussion to weights for criteria; all we say here also applies to decision maker influences. We model these weights as giving trades off between the criterion decisions - the scores of the options with respect to a criterion give levels of desirability, on an absolute scale, and we can translate between the scores between the criteria using the relative weights.

Thus if criterion c_i has weight w_i, $i = 1, 2$, then a score of s with respect to criterion c_1 is worth a score of $s\frac{w_2}{w_1}$ with respect to c_2.

We are assuming that these relative weights are found by using the method of pairwise comparisons. An important distinction between finding scores for options and finding weights for criteria is that we are assuming a ratio scale for weights which is directly estimated by decision makers. Thus when a decision maker is asked to compare two criteria the response is assumed to be an estimate of the relative weights. It may be argued that there is no point including criteria which are hopelessly mismatched, i.e., if some of the relative weights are very large (or very small) and that if we consider the scale as expressing domination, then the scale should be, for example, a 1-9 scale as in [19], where the scale points refer to linguistic comparisons and are somehow transformed into relative weights. If we followed the method for building of scales based upon a minimization of error principle as in Section 2 we can construct the scale with $2^k + 1$ scale points, $t_i = N^{i/2^k}$, $i = 0, \ldots, 2^k$.

We call the scale which these comparisons lie on a *multiplicative scale*.

4. Aggregation

4.1. Aggregating scores for options

In order to define aggregation across team members' decisions in a criterion team each decision maker must have the same objective scale, up to allowable transformations for the options with respect to that criterion, see 3.1. Also the comparisons are reported by the decision maker using a difference category scale, which we take to lie in the range -8 to 8, see 3.2. In order to use aggregation we have to extend the difference category scale to include all possible weighted means of decisions.

This means that we use the interval $S = [-8, 8]$ and call S the *extended scale*.

We assume that there are k decision makers and that they have each been given weights reflecting their relative influence on the decisions. Let w_i, $i = 1, \ldots, k$ be the weights of the decision makers. We show below that it is the relative influence weights that determine the aggregation. For this reason we assume that $w_i \in Z_+$ as we can approximate arbitrarily closely any set of relative weights by positive integers.

Consider then two decision makers i, j who have compared options A and B and we obtain the comparisons $s_i, s_j \in [-8, 8]$ respectively. We consider these as weighted decisions, i.e., as pairs $(s_i, w_i), (s_j, w_j), w_i, w_j \in N$.

The mathematical model of aggregation

In order to introduce a model for the process of combining such weighted decisions we write:

$$\text{agg}((s_1, w_1), (s_2, w_2)) = (s, w)$$

to mean that given the weighted decisions $(s_1, w_1), (s_2, w_2)$ and that there is agreement to combine then in this case the aggregated weighted decision is (s, w).

It is not yet clear that we can always do this, but we now introduce a set of axioms to show the existence of such an aggregation function under minimal assumptions.

In the following scale points are in S and all weights in N.

Axiom 1. $\text{agg}((s_1, w), (s_2, w)) = ((s_1 + s_2)/2, 2w)$.

Axiom 2. $\text{agg}((s, w_1), (s, w_2)) = (s, w_1 + w_2)$.

Axiom 3. $\text{agg}((s, 0), (s_1, w_1)) = (s_1, w_1), \forall s, s_1 \in S, w_1 \in N$.

In order to extend the aggregation function to more than two weighted decisions we introduce the following assumptions:

Axiom 4. The aggregation function, when defined, is both commutative and associative and is continuous for fixed weights.

This last axiom allows us to extend the aggregate function to more than two weighted decision makers. The continuity condition establishes that if this aggregation function holds for all teams of weighted decision makers who allow all possible aggregations within their teams, then there is a unique aggregation function, i.e., the following theorem then follows from these axioms.

Theorem on aggregation

It is proved in [12] that if the k decision makers in a criterion team have influence weights λ_i, $i = 1, \ldots, k$ and if an option has scores s_i, $i = 1, \ldots, k$ for each of the decision makers, then the aggregated score for that option is the normalized weighted average

$$\sum_{i=1}^{k} \lambda_i^* s_i, \qquad \lambda_i^* = \frac{\lambda_i}{\sum_{j=1}^{k} \lambda_j}, \quad i = 1, \ldots, k.$$

The same aggregation technique applies to aggregating criterion decisions over weighted criteria.

4.2. Aggregating weights

A similar aggregation process takes place if the decision makers in a policy team are giving relative weights to criteria and the decision makers are weighted with respect to their influences on the policy. The only difference is that we use a multiplicative scale. A similar analysis follows for determining decision maker influences.

5. Scoring and Weighting using Pairwise Comparisons

In this section, we indicate how the work on scales and aggregation in Sections 3 and 4 together with three axioms concerning the interaction between scoring and aggregation imply that there is only one method of scoring a decision matrix.

5.1. Characterization of scores for options

We assume that there are n options and a decision maker is using a category scale derived from a fixed context range for a fixed criterion.

This category scale allows the decision maker to compare the options two at a time and each comparison is a category difference estimation lying in the range $W = \{-8, -7, \ldots, 0, \ldots, 7, 8\}$. Let $c(i, j)$ be the comparison of option i to option j, and let $C = (c(i, j))$ be the $n \times n$ skew-symmetric matrix with entries in W. The set of skew symmetric matrices with entries in the scale W we denote by $\sum_n(W)$. As we are aggregating these decision matrices over the decision makers in the team we extend the scale to $S = [-8, 8]$ to include all possible weighted averages of scale points. We call the set of all such decision matrices *generalized decision matrices* and denote this set by $\sum_n = \sum_n(S)$.

Properties of generalized decision matrices

An important property of $\sum_n$ is that it is convex. There are two obvious properties of $\sum_n$ that we need:

Property 1. Let P be an $n \times n$ permutation matrix, see 1.1 above, and let $C \in \sum_n$. Then $P^{-1}CP \in \sum_n$.

Property 2. Let $\lambda_i \in Z_+$, $i = 1, \ldots, k$ be the influence weights given to the k decision makers in a criterion team and let C_i, $i = 1, \ldots, k$ be the corresponding decision matrices for the decision makers. Then $C \in \sum_n$, where

$$C = \sum_{i=1}^{k} \lambda_i^* C_i, \quad \lambda_i^* = \frac{\lambda_i}{\sum_{j=1}^{k} \lambda_j}, \quad i = 1, \ldots, k,$$

i.e., the team decision matrix formed by aggregation is also a generalized decision matrix. We let $Z_n = (z_{ij}),\ z_{ij} = 0,\ \forall i, j$ be the zero decision matrix, clearly $Z_n \in \sum_n$.

Characterizing the score function

We assume that there is a function giving the relative scores of the options:

$$\text{Score} : \sum_n \to R^n.$$

Using variants of the axioms developed in [4, 5] we show that there is only one possible scoring method given by:

$$\text{Score}(C) = \alpha C.L^{tr},$$

where $L = (1/n, 1/n, \ldots, 1/n)$ and $\alpha \in R_+$, i.e., the score associated to option i is a multiple of the mean of the ith row of C.

The axioms are:

Axiom Sc1. The scores are independent of the order of the options, i.e.,

$$\text{Score}(P^{-1}CP) = P.\text{Score}(C), \quad \forall C \in \Sigma_n,$$

where P is a permutation matrix.

Axiom Sc2. Scoring an aggregation of decision matrices is the same as aggregating the scores of each decision matrix, i.e., scoring and aggregation "commute", i.e.,

$$\text{Score}\left(\sum_{i=1}^{k} \lambda_i C_i\right) = \sum_{i=1}^{k} \lambda_i \text{Score}(C_i), \quad \lambda_i \geq 0, \quad i = 1, \ldots, k, \quad \sum_{i=1}^{k} \lambda_i = 1.$$

Axiom Sc3. $\text{Score}(Z_n) = (0, 0, \ldots, 0)$.

Note that in these axioms we are careful to make sure that all matrices lie in Σ_n, by Properties 1, 2 above, and hence that Score is defined only on decision matrices.

We now consider the influence weights for decision makers and weights for criteria.

5.2. Characterization of weights

We have seen that the scores of options can be characterised by insisting on the commutativity of aggregation and scoring (and two other axioms). However in the case of influence weights for members of a criterion team or the weights of the criteria there is a marked difference as there is, in general, no underlying objective scale as the weights are purely relative and give trade off information between levels of relative desirability of the options.

In order to simplify the following discussion we concentrate on the weights of the criteria. The same comments apply to finding influence weights of the decision makers in a criterion team.

We note that all criterion scores for the options are in terms of a common scale of relative desirability. So in finding the relative weights of the criteria we can make pairwise comparisons which ask for the relative worth of scores with respect to one criterion as against another. There is no a priori limit on the size of these relative weights as we have seen that these weights are normalized before we aggregate, [12, 14]. Thus when we make pairwise comparisons for obtaining weights we produce decision matrices $D = (d_{ij})$ which are reciprocal matrices with entries in $R_{++} = \{x : x > 0\}$, i.e., $d_{ij} = 1/d_{ji}, \forall i, j$. Let this set of reciprocal matrices be denoted by Γ_k which is an abelian group under multiplication, see [5].

Our assumption at this point is that there is a policy team with members having weighted influences. The policy team can decide on the criteria weights by aggregating their weighted decisions, obtained through pairwise comparisons using multiplicativity. The process of aggregation is formally the same except that we obtain weighted geometric means instead of weighted additive means. Thus if we require that this commutes with scoring we obtain that the weights are given by the geometric means of the rows of the multiplicative decision matrices - this is essentially the argument in [5].

In detail, if the policy team has m members with influence weights w_i, $i = 1, ..., m$ and each member produces a multiplicative decision matrix D_i, $i = 1, ..., m$ and if we let $D_i^{w_i} = (d_{pq,i}^{w_i})$, $D_i = (d_{pq,i})$, then the aggregated decision matrix is

$$\prod_{i=1}^{m} D_i^{w_i} = (d_{ij}),$$

where we assume that the influence weights are normalized, i.e., $\sum_{i=1}^{m} w_i = 1$.

Given a multiplicative decision matrix $D = (d_{ij})$, then the weights obtained from D are

$$W_i = \sqrt[k]{\prod_{j=1}^{k} d_{ij}}.$$

We note that these multiplicative weights are normalized in that $\prod_{i=1}^{k} W_i = 1$. In order to use these weights to aggregate across the criteria decisions we have to re-normalise so that the sum of the weights is 1, see Section 4.

5.3. Combining over criteria

If we now aggregate the criterion scores over these weights using the same technique of aggregation as developed in Section 4 we have to normalise the weights to sum 1, i.e., if $s^{(i)}$ is the vector of scores for criterion i and w_i is the weights for criterion i as obtained above, then the overall scores are:

$$s = \frac{1}{W} \sum_{i=1}^{m} w_i s^{(i)}, \quad W = \Sigma_{w_i}.$$

6. Summary of the Model

6.1. Criterion teams

Given a criterion team scoring options with respect to a given criterion we have:

(1) Each decision maker uses the same underlying objective scale when making pairwise comparisons and the comparisons are reported as category differences from the category scale. Hence each decision maker produces a decision matrix in Σ_n.

(2) Each decision maker has an influence weight that reflects his/her relative influence on the decisions. These weights can be considered as a means of trading off stated levels of relative desirability of the options between decision makers with respect to that criterion.

(3) The option scores derived from the decision matrices of each decision maker are characterised by the above axioms.

(4) Axiom Sc2 tells us that we can aggregate across the normalised weighted decision matrices to get a criterion team decision matrix and then score that to get the criterion decision or we can score each of the decision matrices of the team members and then aggregate the scores using the same normalized weights to get the same criterion decision.

(5) The above methodology gives an automatic normalization of the scores, whether team member scores or criterion scores.

(6) We make the point here that all of the difficulties found in the use of other methodologies which use pairwise comparisons, see e.g., [7, 8], are due to the fact that aggregation and the somewhat arbitrary normalization methods used do not commute. But this is in fact due to the failure to use well founded scales for comparisons, as once this is done the model is driven forward in the direction given by this paper. See [14].

6.2. Aggregating over criteria

(7) If all the criterion teams have done their work and produced criterion decisions, then we can aggregate these across the criteria using the relative weights of the criteria. We can regard these weights as once again giving trades off between the criteria in terms of relative desirability in meeting the objective. Hence we obtain overall scores for the options.

(8) The above complete methodology from the use of objective scales for options, through the axiomatic scoring and aggregation processes to the overall normalized score satisfies all the basic requirements of such models.

6.3. Policy and the effect of policy changes

(9) There is now a clear model for the effect of policy changes on the scoring of options - as long as these policy changes are derived from changes in the weights of the criteria which in turn are given by changes in the decision matrices. Policy is seen, at the basic structural unit of the hierarchy, as a set of weights which are normalised to product 1. Thus the natural setting for policy is the Lie group

$$G = \left\{ (x_1, \ldots, x_n) : \prod_{i=1}^{n} x_i = 1, x_i > 0, i = 1, \ldots, n \right\}.$$

We examine this model of policy and policy change in a subsequent paper.

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