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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract

This paper is concerned with the nonlinear neutral delay difference equation with positive and negative coefficients

$$\Delta[x(n) + R(n)x(n-m)] + p(n)f(x(n-k)) - q(n)f(x(n-\ell)) = 0, \quad n \geq n_0, \qquad (*)$$

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where Δ is the forward difference operator defined by

$$\Delta x(n) = x(n+1) - x(n).$$

Sufficient conditions are obtained under which for every solution of equation (*) is bounded and tends to a constant as $n \to \infty$.

1. Introduction

Consider the following nonlinear neutral delay difference equation with positive and negative coefficients:

$$\Delta[x(n) + R(n)x(n-m)]$$

$$+ p(n) f(x(n-k)) - q(n) f(x(n-\ell)) = 0, \quad n \geq n_0, \tag{1}$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of nonnegative real numbers, $f \in C(R, R)$, $\{R(n)\}$ is a sequence of real numbers, k, ℓ, m are positive integers, n_0 is a nonnegative integer and $N(n_0) = \{n_0, n_0 + 1, \ldots\}$. We note that when $f(x) = x$, equation (1) reduces to the linear difference equation

$$\Delta[x(n) + R(n)x(n-m)] + p(n)x(n-k) - q(n)x(n-\ell) = 0, \; n \in N(n_0). \tag{2}$$

The asymptotic behavior of solutions of (2) or its special case of $q(n) \equiv 0$ has been studied by many authors see, for e.g., [6, 13]. In [10], it is proved that if $R(n) = 0$, $q_n = 0$, $\{p(n)\}$ is a positive sequence and k is a positive integer such that $\lim_{n\to\infty} \sup \sum_{i=n-k}^{n} p(i) < 1$ and $\sum_{n=n_0}^{\infty} p(n) = \infty$, then every solution of equation (2) tends to zero as $n \to \infty$. While in [9], the authors studied equation (2) with $R(n) \equiv r$, $|r| < 1$, $\{p(n)\}$ is a positive sequence and m, k are positive integers such that

$$\lim_{n\to\infty} \sup \left\{ |\mu| \left(1 + \frac{p(n+m+k)}{p(n+k)}\right) + \sum_{i=n-k}^{n+k} p(i) \right\} < 2 \quad \text{and} \quad \sum_{n=n_0}^{\infty} p(n) = \infty,$$

then every solution of equation (2) tends to zero as $n \to \infty$.

The purpose of this paper is to derive sufficient conditions under which every solution of equation (1) is bounded and tends to a constant as $n \to \infty$. Let $\rho = \max\{m, k, \ell\}$. By a solution of equation (1), we mean a sequence $\{x(n)\}$ of real numbers which is defined for $n \geq n_0 - \rho$ and satisfies equation (1) for $n \geq n_0$. It is easy to see that for any given n_0 and initial conditions of the form $x(n_0 + j) = b_j$, $j = -\rho, -\rho + 1, -\rho + 2, \ldots, 0$. Equation (1) has a unique solution $\{x(n)\}$ which is defined for $n \geq n_0 - \rho$ and satisfies the above initial conditions.

As it is customary, a solution of (1) is said to be *nonoscillatory*, if it is eventually positive or eventually negative. Otherwise, it will be called *oscillatory*.

In what follows, for the sake of convenience, when we write a sequence inequality without specifying its domain of validity, we mean that it holds for all sufficiently large n.

2. Main Results

Theorem 2.1. *Assume that the following conditions hold*:

(A1) $k > \ell$; *there is a constant* $M > 0$ *such that*

$$|x| \leq |f(x)| \leq M|x| \text{ for } x \in R, \text{ and } xf(x) > 0, \text{ for } x \in R,\ x \neq 0, \tag{3}$$

(A2)

$$\limsup_{n\to\infty} |R(n)| = \mu < 1, \tag{4}$$

(A3)

$$H(n) = p(n) - q(n + \ell - k) > 0 \text{ for } n \geq n_1 = n_0 + k - \ell, \tag{5}$$

(A4)

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-\ell-1} q(i + \ell) < \frac{1}{M}, \tag{6}$$

(A5)

$$\lim_{n\to\infty}\sup\left[\sum_{i=n-k}^{n+k} H(i+k) + \frac{q(n+\ell)}{H(n+k)}\sum_{i=n-k+1}^{n} H(i+2k)\right.$$

$$\left. + \mu\left(1+\frac{H(n+m+k)}{H(n+k)}\right) + \sum_{i=n-k}^{n-\ell-1} q(i+\ell)\right] < \frac{2}{M}, \tag{7}$$

then every solution of (1) *is bounded.*

Proof. Let $\{x(n)\}$ be any solution of (1). We shall prove that $\{x(n)\}$ is bounded. For this purpose, we can rewrite (1) in the form

$$\Delta\left[x(n) + R(n)x(n-m) - \sum_{i=n-k}^{n-1} H(i+k)\, f(x(i)) - \sum_{i=n-k}^{n-\ell-1} q(i+\ell)\, f(x(i))\right]$$

$$+ H(n+k)\, f(x(n)) = 0, \quad n \geq n_0. \tag{8}$$

From (4) and (7), we can choose $\varepsilon > 0$ sufficiently small such that $\mu + \varepsilon < 1$ and

$$\lim_{n\to\infty}\sup\left[\sum_{i=n-k}^{n+k} H(i+k) + \frac{q(n+\ell)}{H(n+k)}\sum_{i=n-k+1}^{n} H(i+2k)\right.$$

$$\left. + (\mu+\varepsilon)\left[1+\frac{H(n+m+k)}{H(n+k)}\right]\right] < \frac{2}{M}, \tag{9}$$

also we select $n_1 > n_0$ sufficiently large such that

$$|R(n)| \leq \mu + \varepsilon \text{ for } n > n_1; \tag{10}$$

and noting (A1), we have

$$|R(n)|x^2(n-m) \leq (\mu+\varepsilon) f^2(x(n-m)), \quad n \geq n_1. \tag{11}$$

Now we introduce the three sequences as

$$W_1(n) = \left[x(n) + R(n)x(n-m) - \sum_{i=n-k}^{n-1} H(i+k) f(x(i)) - \sum_{i=n-k}^{n-\ell-1} q(i+\ell) f(x(i)) \right]^2,$$

$$W_2(n) = \sum_{i=n-k}^{n-1} H(i+2k) \sum_{j=i}^{n-1} q(j+\ell) f^2(x(j)),$$

$$W_3(n) = \sum_{i=n-k}^{n-1} H(i+2k) \sum_{j=i}^{n-1} H(j+k) f^2(x(j)) + (\mu+\varepsilon) \sum_{i=n-m}^{n-1} H(i+m+k) f^2(x(i)), \quad n \geq n_2.$$

Calculating $\Delta W_1(n)$, $\Delta W_2(n)$ and $\Delta W_3(n)$, we have

$$\Delta W_1(n) = \Delta \left[\left\{ x(n) + R(n)x(n-m) - \sum_{i=n-k}^{n-1} H(i+k) f(x(i)) - \sum_{i=n-k}^{n-\ell-1} q(i+\ell) f(x(i)) \right\}^2 \right]$$

$$= \Delta \left[x(n) + R(n)x(n-m) - \sum_{i=n-k}^{n-1} H(i+k) f(x(i)) - \sum_{i=n-k}^{n-\ell-1} q(i+\ell) f(x(i)) \right]$$

$$\left[x(n+1) + R(n+1)x(n+1-m) - \sum_{i=n+1-k}^{n} H(i+k)\, f(x(i))\right.$$

$$- \sum_{i=n-k+1}^{n-\ell} q(i+\ell)\, f(x(i)) + x(n) + R(n)x(n-m)$$

$$\left. - \sum_{i=n-k}^{n-1} H(i+k)\, f(x(i)) - \sum_{i=n-k}^{n-\ell-1} q(i+\ell)\, f(x(i))\right]$$

$$\leq -H(n+k)\left[2x(n)\, f(x(n)) - |R(n)|x^2(n-m) - |R(n)|\, f^2(x(n))\right.$$

$$- \sum_{i=n-k}^{n-1} H(i+k)\, f^2(x(i)) - \sum_{i=n-k}^{n} H(i+k)\, f^2(x(n))$$

$$\left. - \sum_{i=n-k}^{n-\ell-1} q(i+\ell)\, f^2(x(i)) - \sum_{i=n-k}^{n-\ell-1} q(i+\ell)\, f^2(x(n))\right]$$

$$= -H(n+k)\left[2x(n)\, f(x(n)) - |R(n)|x^2(n-m) - |R(n)|\, f^2(x(n))\right.$$

$$- \sum_{i=n-k}^{n-1} H(i+k)\, f^2(x(i)) - \sum_{i=n-k}^{n} H(i+k)\, f^2(x(n))$$

$$\left. - \sum_{i=n-k}^{n-\ell-1} q(i+\ell)\, f^2(x(i)) - \sum_{i=n-k}^{n-\ell-1} q(i+\ell)\, f^2(x(n))\right], \quad (12)$$

$$\Delta W_2(n) = q(n+\ell)\, f^2(x(n)) \sum_{i=n-k+1}^{n} H(i+2k)$$

$$- H(n+k) \sum_{i=n-k}^{n-1} q(i+\ell)\, f^2(x(i))$$

$$\le q(n+\ell)\, f^2(x(n)) \sum_{i=n-k+1}^{n} H(i+2k)$$

$$- H(n+k) \sum_{i=n-k}^{n-l-1} q(i+\ell)\, f^2(x(i)), \tag{13}$$

$$\Delta W_3(n) = H(n+k)\, f^2(x(n)) \sum_{i=n-k+1}^{n} H(i+2k)$$

$$- H(n+k) \sum_{i=n-k}^{n-1} H(i+k)\, f^2(x(i))$$

$$+ (\mu+\varepsilon) H(n+m+k)\, f^2(x(n)) - (\mu+\varepsilon) H(n+k)\, f^2(x(n-m)). \tag{14}$$

Set $W(n) = W_1(n) + W_2(n) + W_3(n)$, $n \ge n_1$. By (12)-(14) and (A1), we get

$$\Delta W(n) = \Delta W_1 + \Delta W_2 + \Delta W_3$$

$$\le -H(n+k) \Bigg[2x(n)\, f(x(n)) - \sum_{i=n-k}^{n+k} H(i+k)\, f^2(x(n))$$

$$- \sum_{i=n-k}^{n-\ell-1} q(i+\ell)\, f^2(x(n)) - \frac{q(n+\ell)}{H(n+k)}\, f^2(x(n)) \sum_{i=n-k+1}^{n} H(i+2k)$$

$$- (\mu+\varepsilon)\, f^2(x(n)) \frac{H(n+m+k)}{H(n+k)} - (\mu+\varepsilon)\, f^2(x(n)) \Bigg]$$

$$\le -H(n+k)\, f^2(x(n)) \Bigg[\frac{2}{M} - \Bigg[\sum_{i=n-k}^{n+k} H(i+k)$$

$$+\frac{q(n+\ell)}{H(n+k)}\sum_{i=n-k+1}^{n} H(i+2k)$$

$$+\sum_{i=n-k}^{n-\ell-1} q(i+\ell)+(\mu+\varepsilon)\left(1+\frac{H(n+m+k)}{H(n+k)}\right)\Bigg]\Bigg], \tag{15}$$

which together with (7) implies

$$\sum_{n=n_1}^{\infty} H(n+k)\, f^2(x(n))<\infty \tag{16}$$

and hence for any positive integer s we have

$$\lim_{n\to\infty}\sum_{i=n-s}^{n-1} H(i+k)\, f^2(x(i))=0. \tag{17}$$

Noting (7), there is a sufficiently large positive integer $n_3 \geq n_2$ such that

$$q(n+\ell)\sum_{i=n-k+1}^{n} H(i+2k)<\frac{2}{M}H(n+k) \text{ for } n\geq n_3 \tag{18}$$

and thus for $n \geq n_3 + k$ we have

$$q(j+1)\sum_{i=j-k+1}^{j} H(i+2k)<\frac{2}{M}H(j+k), \tag{19}$$

where $j = n-k,\ n-k+1,\ \ldots,\ n-1.$ Therefore, we have

$$H(n+k)+H(n-k+1)+\cdots+H(n+2k-1)q(n+\ell-1)$$

$$<\frac{2}{M}H(n+k-1),$$

hence for $n \geq n_3 + k$ we have

$$W_2(n)=H(n+k)q(n-k+1)\, f^2(x(n-k))+[H(n+k)+H(n+k+1)]$$

$$q(n-k+\ell+1)\, f^2(x(n-k+1)) + \cdots$$

$$\leq \frac{2}{M} \sum_{i=n-k}^{n-1} H(i+k)\, f^2(x(i)), \tag{20}$$

$$W_3(n) = \sum_{i=n-k}^{n-1} H(i+2k) \sum_{j=i}^{n-1} H(j+k)\, f^2(x(j))$$

$$+ (\mu + \varepsilon) \sum_{i=n-m}^{n-1} H(i+m+k)\, f^2(x(i))$$

$$\leq \frac{2}{M} \sum_{j=n-k}^{n-1} H(j+k)\, f^2(x(j)) + 2 \sum_{i=n-k}^{n-1} H(i+k)\, f^2(x(i)), \tag{21}$$

(20) and (21) together with (17) imply $\lim_{n\to\infty} W_2(n) = 0$ and $\lim_{n\to\infty} W_3(n) = 0.$ On the other hand, by (7) and (15), we see that $W(n)$ is eventually decreasing. In view of $W(n) \geq 0$, $\lim_{n\to\infty} W(n) = \gamma$ exists and is finite, thus, $\lim_{n\to\infty} W(n) = \lim_{n\to\infty} W_1(n) = \gamma$, that is

$$\lim_{n\to\infty} \left[x(n) + R(n)x(n-m) - \sum_{i=n-k}^{n-1} H(i+k)\, f(x(i)) - \sum_{i=n-k}^{n-\ell-1} q(i+\ell)\, f(x(i)) \right]^2 = \gamma. \tag{22}$$

Let

$$z(n) = x(n) + R(n)x(n-m) - \sum_{i=n-k}^{n-1} H(i+k)\, f(x(i)) - \sum_{i=n-k}^{n-\ell-1} q(i+\ell)\, f(x(i)).$$

Then

$$\Delta z(n) + H(n+k) f(x(n)) = 0, \quad n \geq n_2 \tag{23}$$

and

$$\lim_{n\to\infty} z^2(n) = \gamma,$$

that is $\lim_{n\to\infty} |z(n)| = \sqrt{\gamma}$.

We claim that $\{z(n)\}$ converges. In fact, this is clear if $\gamma = 0$. If $\gamma > 0$, then it suffices to show that $\{z(n)\}$ is eventually positive or eventually negative. Otherwise, choose a number $0 < \varepsilon_1 < \sqrt{\gamma}$ and let N be a positive integer such that

$$\sqrt{\gamma} - \varepsilon_1 < |z(n)| < \sqrt{\gamma} + \varepsilon_1, \quad n \geq N, \tag{24}$$

and let $A = \{n \geq N : z(n) < 0\}$, $B = \{n \geq N, z(n) > 0\}$. Since $\{z(n)\}$ is neither eventually positive nor eventually negative, it follows that A and B are unbounded; then there exists a divergent sequence of integers $\{n_j\}$ such that $N \leq n_1 < n_2 < \cdots < n_j < \cdots$, $n_j \in k$, $n_j + 1 \in A$. Then $z(n_j + 1) < 0$ and $z(n_j) > 0$. Furthermore, by (24),

$$2(-\sqrt{\gamma} - \varepsilon_1) < z(n_j + 1) - z(n_j) < 2(-\sqrt{\gamma} + \varepsilon_1), \quad j \geq 1.$$

Therefore, in view of (23),

$$0 < 2(\sqrt{\gamma} - \varepsilon_1) < H(n_j + k) f(x(n_j)) < 2(\sqrt{\gamma} + \varepsilon_1), \quad j \geq 1. \tag{25}$$

On the other hand, by (16) and (25), we see that $\{f(x(n_j))\}$ converges to zero. Noting that $\{H(n)\}$ is bounded, we get $H(n_j + k) f(x(n_j)) \to 0$ as $j \to \infty$ which contradicts (25). Thus $\{z(n)\}$ must converge. So,

$$\lim_{n\to\infty} z(n) = \lim_{n\to\infty}\left[x(n) + R(n)x(n-m) - \sum_{i=n-k}^{n-1} H(i+k) f(x(i))\right.$$

$$- \sum_{i=n-k}^{n-\ell-1} q(i+\ell) f(x(i)) \Bigg] = \beta, \tag{26}$$

where $\beta = \sqrt{\gamma}$ or $\beta = -\sqrt{\gamma}$ and is finite. In view of (23), we have

$$\sum_{i=n-k}^{n-1} H(i+k) f(x(i)) = z(n-k) - z(n),$$

so,

$$\lim_{n\to\infty} \sum_{i=n-k}^{n-1} H(i+k) f(x(i)) = 0. \tag{27}$$

By (26) and (27), we have

$$\lim_{n\to\infty} \left[x(n) + R(n)x(n-m) - \sum_{i=n-k}^{n-\ell-1} q(i+\ell) f(x(i)) \right] = \beta. \tag{28}$$

Next, we shall show that $\{|x(n)|\}$ is bounded. If $\{|x(n)|\}$ is unbounded, then there exists a divergent sequence of integers $\{n_j\}$ such that $|x(n_j)| \to \infty$ as $j \to \infty$, and

$$|x(n_j)| = \sup_{n_0-\rho \le n \le n_j} |x(n)|. \tag{29}$$

Noting (6) and (29), we have

$$\left| x(n_j) + R(n_j)x(n_j - m) - \sum_{i=n_j-k}^{n_j-\ell-1} q(i+\ell) f(x(i)) \right|$$

$$\ge |x(n_j)| \left[1 - (\mu + \varepsilon) - M \sum_{i=n_j-k}^{n_j-\ell-1} q(i+\ell) \right] \to \infty \text{ as } j \to \infty$$

which contradicts (28). So $\{|x(n)|\}$ is bounded. The proof of Theorem 2.1 is complete. □

Theorem 2.2. *Let* (A1), (4) *and* (5) *hold. Assume that* $R(n) \geq 0$ *or* $R(n) \leq 0$ *for sufficiently large n and*

$$\lim_{n\to\infty} |R(n)| = \mu < 1 \tag{30}$$

and

$$\lim_{n\to\infty} \sum_{i=n-k}^{n-\ell-1} q(i+\ell) = 0, \tag{31}$$

$$\lim_{n\to\infty} \sup\Bigg[\sum_{i=n-k}^{n+k} H(i+k) + \frac{q(n+\ell)}{H(n+k)} \sum_{i=n-k+1}^{n} H(i+2k)$$

$$+ \mu\left(1 + \frac{H(n+m+k)}{H(n+k)}\right)\Bigg] < \frac{2}{M}. \tag{32}$$

Then every solution of (1) *tends to a constant as* $n \to \infty$.

Proof. Let $\{x(n)\}$ be any solution of (1). From the proof of Theorem 2.1, we know that $\{|x(n)|\}$ is bounded and (28) holds. Now we shall prove that $\lim_{n\to\infty} x(n)$ exists and is finite. Noting condition (31), we obtain

$$0 \leq \left|\sum_{i=n-k}^{n-\ell-1} q(i+\ell) f(x(i))\right| \leq \sum_{i=n-k}^{n-\ell-1} q(i+\ell) |f(x(i))|$$

$$\leq M \sum_{i=n-k}^{n-\ell-1} q(i+\ell) |x(i)| \to 0 \text{ as } n \to \infty$$

which together with (28) gives

$$\lim_{n\to\infty} [x(n) + R(n)x(n-m)] = \beta. \tag{33}$$

If $\mu = 0$, then $\lim_{n\to\infty} x(n) = \beta$ which is finite.

If $0 < \mu < 1$, then we let

$$\lim_{n\to\infty} \sup x(n) = u_1, \quad \lim_{n\to\infty} \inf x(n) = u_2$$

and let $\{a_i\}$ and $\{b_i\}$ be two sequences such that $a_i \to \infty$, $b_i \to \infty$ as $i \to \infty$ and

$$\lim_{i\to\infty} x(a_i) = u_1, \quad \lim_{i\to\infty} x(b_i) = u_2,$$

for $n > n_3$, we have the following two cases:

Case (i). If $0 < R(n) < 1$ for $n > n_3$, then we have

$$u_1 = \lim_{i\to\infty} x(a_i) = \lim_{i\to\infty} [x(a_i) - R(a_i)x(a_i - m) + R(a_i)x(a_i - m)]$$

$$= \beta + \mu \lim_{i\to\infty} x(a_i - m) \le \beta + \mu u_1$$

and

$$u_2 = \lim_{i\to\infty} x(b_i) = \lim_{i\to\infty} [x(b_i) - R(b_i)x(b_i - m) + R(b_i)x(b_i - m)]$$

$$= \beta + \mu \lim_{i\to\infty} x(b_i - m) \ge \beta + \mu u_2.$$

Thus $u_1 \le \dfrac{\beta}{1-\mu} \le u_2$, which together with $u_1 \ge u_2$ implies $u_1 = u_2 = \dfrac{\beta}{1-\mu}$. This shows that $\lim_{n\to\infty} x(n)$ exists and is finite.

Case (ii). If $-1 < R(n) < 0$ for $n > n_3$, then we have

$$\beta = \lim_{i\to\infty} [x(a_i) - R(a_i)x(a_i - m)] = u_1 + \mu \lim_{i\to\infty} x(a_i - m) \ge u_1 + \mu u_2$$

and

$$\beta = \lim_{i\to\infty} [x(b_i) - R(b_i)x(b_i - m)] = u_2 + \lim_{i\to\infty} x(b_i - m) \le u_2 + \mu u_1.$$

Thus $0 \le u_1 - u_2 \le \mu(u_1 - u_2)$, so that $u_1 = u_2 = \dfrac{\beta}{1+\mu}$. This shows that $\lim_{n\to\infty} x(n)$ exists and is finite. The proof of Theorem 2.2 is complete. □

Theorem 2.3. *Assume that the conditions of Theorem* 2.2 *imply that every oscillatory solution of* (1) *tends to zero as* $n \to \infty$.

In Theorem 2.2, taking $f(x) \equiv x$ we have

Corollary 2.1. *Assume that* $k > \ell$, (5) *and* (31) *hold and*

$$\lim_{n\to\infty} \sup\left[\sum_{i=n-k}^{n+k} H(i+k) + \frac{q(n+\ell)}{H(n+k)} \sum_{i=n-k+1}^{n} H(i+2k)\right.$$

$$\left. + \mu\left(1 + \frac{H(n+m+k)}{H(n+k)}\right)\right] < 2. \tag{34}$$

Then every solution of equation (2) *tends to a constant as* $n \to \infty$.

In Theorem 2.2, taking $q(n) \equiv 0$ and $f(x) \equiv x$, we have

Corollary 2.2. *Assume that k is a non-negative integer and* $\{p(n)\}$ *is a positive sequence and*

$$\lim_{n\to\infty} \sup\left[\sum_{i=n-k}^{n+k} p(i+k) + \mu\left(1 + \frac{p(n+m+k)}{p(n+k)}\right)\right] < 2.$$

Then every solution of the equation

$$\Delta[x(n) + R(n)x(n-m)] + p(n)x(n-k) = 0, \quad n \ge n_0,$$

tends to a constant as $n \to \infty$.

Theorem 2.4. *The conditions in Theorem* 2.2 *together with*

(i) *for any* $\alpha > 0$ *there exists* $\delta > 0$ *such that*

$$|f(x)| \ge \delta \text{ for } |x| \ge \alpha \tag{35}$$

and

(ii)

$$\sum_{n=n_0}^{\infty} H(n) = \infty \tag{36}$$

imply that every solution of (1) *tends to zero as* $n \to \infty$.

Proof. By Theorem 2.3, we only have to prove that every nonoscillatory solution of (1) tends to zero as $n \to \infty$. Let $\{x(n)\}$ be an eventually positive solution of (1). We shall prove that $\lim_{n\to\infty} x(n) = 0$. By Theorem 2.1, we rewrite (1) in the form (23). Summing from n_0 to n on both sides of (23), we get

$$\sum_{i=n_0}^{n} H(i+k)\, f(x(i)) = z(n_0) - z(n+1)$$

by using (26) we have $\sum_{i=n_0}^{\infty} H(i+k)\, f(x(i)) < \infty$, which together with (36), yields $\liminf_{n\to\infty} f(x(n)) = 0$. We claim that

$$\liminf_{n\to\infty} x(n) = 0. \tag{37}$$

Let $\{s_m\}$ be such that $s_m \to \infty$ as $m \to \infty$ and $\lim_{n\to\infty} f(x(s_m)) = 0$. Then we must have $\liminf_{n\to\infty} (x(s_m)) = M = 0$. In fact, if $M > 0$, then there is a subsequence $\{s_{m_k}\}$ such that $x(s_{m_k}) \geq M/2$ for sufficiently large k. By (35), we have $f(x(s_{m_k})) \geq \zeta$ for some $\zeta > 0$ and sufficiently large k, which yields a contradiction because $\liminf_{k\to\infty} f(x(s_{m_k})) = 0$. Therefore, by Theorem 2.2, $\lim_{n\to\infty} x(n)$ exists and hence $\lim_{n\to\infty} x(n) = 0$. Thus, the proof is complete. □

3. Example

$$\Delta\left[x(n) + \frac{n-1}{6n}x(n-1)\right] + \left[\frac{2}{(n-1)^2}\right][1+\sin^2 x(n-2)]x(n-2)$$
$$- \frac{1}{n^2}[1+\sin^2 x(n-1)]x(n-1) = 0, \quad n \geq 2, \tag{38}$$

here $p(n) = \frac{2}{(n-1)^2}$, $q(n) = \frac{1}{n^2}$, $R(n) = \frac{n-1}{6n}$, $m = 1$, $k = 2$, $\ell = 1$ by simple calculation, $\mu = \lim_{n\to\infty} | R(n) | = \frac{1}{6} < 1,$

$$| x | \le | (2 + \sin^2 x) x | \le 2| x |, \quad x^2(1 + \sin^2 x) > 0 \ (x \ne 0).$$

The above equation satisfies all the conditions of Theorems 2.1 and 2.2. Therefore, every solution of (38) is bounded and tends to a constant as $n \to \infty$.

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