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POINTWISE MULTIPLIERS FROM $A_{\alpha}^{p}(\mathbb{B}_n)$ INTO $A_{\beta}^{q}(\mathbb{B}_n)$

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Abstract

Let p, q, α and β be four real numbers such that $p > 0$, $q > 0$, $\alpha > -1$ and $\beta > -1$. Let g be a holomorphic function in the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$. Then g is called a pointwise multiplier from the weighted Bergman space $A_{\alpha}^{p}(\mathbb{B}_n)$ into the other one $A_{\beta}^{q}(\mathbb{B}_n)$ if $\{fg : f \in A_{\alpha}^{p}(\mathbb{B}_n)\} \subset A_{\beta}^{q}(\mathbb{B}_n)$. In the case $n = 1$, Zhao [3] completely characterized the pointwise multipliers from $A_{\alpha}^{p}(\mathbb{D})$ into $A_{\beta}^{q}(\mathbb{D})$. In this paper, we prove that his result still holds even in the higher dimensional case $n \geq 2$.

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1. Introduction

Let $n \geq 1$ be a fixed integer. Let $\mathbb{B}_n$ denote the unit ball of $\mathbb{C}^n$. Let ν denote the normalized Lebesgue measure on $\mathbb{B}_n$. For each $\alpha \in \mathbb{R}$, we define a weighted Lebesgue measure ν_α on $\mathbb{B}_n$ by $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z)$, $z \in \mathbb{B}_n$. Here $c_\alpha = \dfrac{\Gamma(n + \alpha + 1)}{\Gamma(n+1)\Gamma(\alpha+1)}$ or $c_\alpha = 1$ if $\alpha > -1$ or $\alpha \leq -1$, respectively. $H(\mathbb{B}_n)$ stands for the space of all holomorphic functions in $\mathbb{B}_n$. The set of all positive real numbers is denoted by $\mathbb{R}_+$.

For any $f \in H(\mathbb{B}_n)$, any $\alpha \in \mathbb{R}$ and any $p \in \mathbb{R}_+$, we define

$$\| f \|_{A_\alpha^p(\mathbb{B}_n)} := \left(\int_{\mathbb{B}_n} | f |^p d\nu_\alpha \right)^{\frac{1}{p}} = \| f \|_{L^p(\nu_\alpha)}.$$

The *weighted Bergman space* $A_\alpha^p(\mathbb{B}_n)$ is defined by

$$A_\alpha^p(\mathbb{B}_n) := \{ f \in H(\mathbb{B}_n) : \| f \|_{A_\alpha^p(\mathbb{B}_n)} < \infty \}.$$

As usual, we define

$$\| f \|_{H^\infty(\mathbb{B}_n)} := \sup_{z \in \mathbb{B}_n} | f(z) | \quad (f \in H(\mathbb{B}_n))$$

and

$$H^\infty(\mathbb{B}_n) := \{ f \in H(\mathbb{B}_n) : \| f \|_{H^\infty(\mathbb{B}_n)} < \infty \}.$$

Let $M_+(\mathbb{B}_n)$ denote the set of all positive Borel measures on $\mathbb{B}_n$. For $\mu \in M_+(\mathbb{B}_n)$, $\alpha \in \mathbb{R}$ and $R \in \mathbb{R}_+$, we define the function $\hat{\mu}_{R,\alpha}$ on $\mathbb{B}_n$ by

$$\hat{\mu}_{R,\alpha}(z) := \frac{\mu(D(z, R))}{(1 - |z|^2)^{n+1+\alpha}} \quad (z \in \mathbb{B}_n),$$

where $D(z, R)$ is the Bergman metric ball with center at z and radius R (cf. p. 27 in [5] and p. 71 in [4]).

For $\alpha \in \mathbb{R}$ and $f \in H(\mathbb{B}_n)$, we define

$$\| f \|_{\mathscr{B}_\alpha(\mathbb{B}_n)} := | f(0) | + \sup_{z\in\mathbb{B}_n} \{(1 - |z|^2)^\alpha |(\nabla f)(z)|\}.$$

The *Bloch type space* $\mathscr{B}_\alpha(\mathbb{B}_n)$ is defined by

$$\mathscr{B}_\alpha(\mathbb{B}_n) := \{ f \in H(\mathbb{B}_n) : \| f \|_{\mathscr{B}_\alpha(\mathbb{B}_n)} < \infty\}.$$

Let $\{\alpha, \beta\} \subset \mathbb{R}$ and $\{p, q\} \subset \mathbb{R}_+$. Then a function $g \in H(\mathbb{B}_n)$ is called a *pointwise multiplier* from $A_\alpha^p(\mathbb{B}_n)$ into $A_\beta^q(\mathbb{B}_n)$ if $\{fg : f \in A_\alpha^p(\mathbb{B}_n)\} \subset A_\beta^q(\mathbb{B}_n)$. The set of all pointwise multipliers from $A_\alpha^p(\mathbb{B}_n)$ into $A_\beta^q(\mathbb{B}_n)$ is denoted by $(\mathscr{PM})(A_\alpha^p(\mathbb{B}_n), A_\beta^q(\mathbb{B}_n))$. In [3], Zhao proved the following theorem. Note that $\mathbb{D} := \mathbb{B}_1$ denotes the unit disc in the complex plane $\mathbb{C}$.

Theorem Z ([3, p. 141, Theorem 1])**.** *Let* $\{\alpha, \beta\} \subset (-1, \infty)$ *and* $\{p, q\} \subset \mathbb{R}_+$. *Put* $\gamma = \frac{\beta + 2}{q} - \frac{\alpha + 2}{p}$.

(i) *If* $p \le q$ *and* $\gamma > 0$, *then* $(\mathscr{PM})(A_\alpha^p(\mathbb{D}), A_\beta^q(\mathbb{D})) = \mathscr{B}_{1+\gamma}(\mathbb{D})$.

(ii) *If* $p \le q$ *and* $\gamma = 0$, *then* $(\mathscr{PM})(A_\alpha^p(\mathbb{D}), A_\beta^q(\mathbb{D})) = H^\infty(\mathbb{D})$.

(iii) *If* $p \le q$ *and* $\gamma < 0$, *then* $(\mathscr{PM})(A_\alpha^p(\mathbb{D}), A_\beta^q(\mathbb{D})) = \{0\}$.

(iv) *If* $p > q$, *then* $(\mathscr{PM})(A_\alpha^p(\mathbb{D}), A_\beta^q(\mathbb{D})) = A_\delta^s(\mathbb{D})$, *where* $s = \frac{pq}{p - q}$ *and* $\delta = s\left(\frac{\beta}{q} - \frac{\alpha}{p}\right)$.

The purpose of this paper is to show that the above Theorem Z remains valid even if replacing $\mathbb{D}$ by $\mathbb{B}_n$. Our main result is the following theorem:

Theorem 1. *Let* $\{\alpha, \beta\} \subset (-1, \infty)$ *and* $\{p, q\} \subset \mathbb{R}_+$. *Put* $\gamma = \frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p}$.

(i) *If* $p \leq q$ *and* $\gamma > 0$, *then* $(\mathscr{PM})\,(A_\alpha^p(\mathbb{B}_n),\, A_\beta^q(\mathbb{B}_n)) = \mathscr{B}_{1+\gamma}(\mathbb{B}_n)$.

(ii) *If* $p \leq q$ *and* $\gamma = 0$, *then* $(\mathscr{PM})\,(A_\alpha^p(\mathbb{B}_n),\, A_\beta^q(\mathbb{B}_n)) = H^\infty(\mathbb{B}_n)$.

(iii) *If* $p \leq q$ *and* $\gamma < 0$, *then* $(\mathscr{PM})\,(A_\alpha^p(\mathbb{B}_n),\, A_\beta^q(\mathbb{B}_n)) = \{0\}$.

(iv) *If* $p > q$, *then* $(\mathscr{PM})\,(A_\alpha^p(\mathbb{B}_n),\, A_\beta^q(\mathbb{B}_n)) = A_\delta^s(\mathbb{B}_n)$, *where* $s = \frac{pq}{p-q}$ *and* $\delta = s\left(\frac{\beta}{q} - \frac{\alpha}{p}\right)$.

2. Preliminaries

Since all the weighted Bergman spaces are *F*-spaces, by using the closed graph theorem ([1, Theorem 2.15]), we can easily prove the following proposition:

Proposition 2. *Suppose* $\{\alpha, \beta\} \subset (-1, \infty)$, $\{p, q\} \subset \mathbb{R}_+$ *and* $g \in H(\mathbb{B}_n)$. *Then the following two conditions are equivalent*:

(i)

$$g \in (\mathscr{PM})\,(A_\alpha^p(\mathbb{B}_n),\, A_\beta^q(\mathbb{B}_n)).$$

(ii)

$$\sup\left\{\frac{\| fg \|_{A_\beta^q(\mathbb{B}_n)}}{\| f \|_{A_\alpha^p(\mathbb{B}_n)}} :\, f \in A_\alpha^p(\mathbb{B}_n)\setminus\{0\}\right\} < \infty.$$

Proposition 3. *Let* $s \in (n, \infty)$, $\alpha = s - (n+1)$ *and* $p \in \mathbb{R}_+$. *Then* $\alpha \in (-1, \infty)$ *and*

$$\int_{\mathbb{B}_n} \frac{|f(w)|^p\{1-|\varphi_z(w)|^2\}^s}{(1-|w|^2)^{n+1}} d\nu(w) = \frac{1}{c_\alpha}\| f\circ\varphi_z\|_{L^p(\nu_\alpha)}^p$$

for any $f \in C(\mathbb{B}_n)$ *and any* $z \in \mathbb{B}_n$, *where* φ_z *is the involutative biholomorphic map of* $\mathbb{B}_n$ *that exchanges* 0 *and z.*

Proof. It is clear that $\alpha \in (-1, \infty)$. By Lemma 1.2 of [5] and Proposition 1.13 of [5], for any $f \in C(\mathbb{B}_n)$ and any $z \in \mathbb{B}_n$,

$$\int_{\mathbb{B}_n} \frac{|f(w)|^p\{1-|\varphi_z(w)|^2\}^s}{(1-|w|^2)^{n+1}} d\nu(w)$$

$$= \int_{\mathbb{B}_n} \frac{|f(w)|^p}{(1-|w|^2)^{n+1}}\left\{\frac{(1-|z|^2)(1-|w|^2)}{|1-\langle w, z\rangle|^2}\right\}^s d\nu(w)$$

$$= \int_{\mathbb{B}_n} |f(w)|^p \frac{(1-|z|^2)^s(1-|w|^2)^{s-n-1}}{|1-\langle w, z\rangle|^{2s}} d\nu(w)$$

$$= \int_{\mathbb{B}_n} |f(w)|^p \frac{(1-|z|^2)^{n+1+\alpha}(1-|w|^2)^{\alpha}}{|1-\langle w, z\rangle|^{2(n+1+\alpha)}} d\nu(w)$$

$$= \frac{1}{c_\alpha}\int_{\mathbb{B}_n} |f(w)|^p \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle w, z\rangle|^{2(n+1+\alpha)}} d\nu_\alpha(w)$$

$$= \frac{1}{c_\alpha}\int_{\mathbb{B}_n} |(f\circ\varphi_z)(w)|^p d\nu_\alpha(w)$$

$$= \frac{1}{c_\alpha}\| f\circ\varphi_z\|_{L^p(\nu_\alpha)}^p. \qquad \square$$

The next lemma is in p. 260 of [5] as Exercise 7.7. For the completeness, we prove it here.

Lemma 4. *For any* $\alpha \in (1, \infty)$, *it holds that*

$$\mathscr{B}_\alpha(\mathbb{B}_n) = \left\{ f \in H(\mathbb{B}_n) : \sup_{z\in\mathbb{B}_n}\{(1-|z|)^{\alpha-1}|f(z)|\} < \infty\right\}.$$

Proof. For any $f \in H(\mathbb{B}_n)$ and any $z \in \mathbb{B}_n$,

$$| f(z) - f(0) | = \left| \int_0^1 \langle (\nabla f)(tz), \bar{z} \rangle dt \right| \le \int_0^1 | (\nabla f)(tz) | dt$$

$$= \int_0^1 (1 - | tz |^2)^{\alpha} | (\nabla f)(tz) | (1 - | tz |^2)^{-\alpha} dt$$

$$\le \sup_{w \in \mathbb{B}_n} \{(1 - | w |^2)^{\alpha} | (\nabla f)(w) |\} \int_0^1 (1 - | tz |^2)^{-\alpha} dt$$

$$\le \frac{1}{\alpha - 1} (1 - | z |)^{1-\alpha} \sup_{w \in \mathbb{B}_n} \{(1 - | w |^2)^{\alpha} | (\nabla f)(w) |\}$$

$$\le \frac{2^{\alpha-1}}{\alpha - 1} (1 - | z |^2)^{1-\alpha} \sup_{w \in \mathbb{B}_n} \{(1 - | w |^2)^{\alpha} | (\nabla f)(w) |\}. \qquad (1)$$

By (1), we obtain for any $f \in H(\mathbb{B}_n)$,

$$\sup_{z \in \mathbb{B}_n} \{(1 - | z |^2)^{\alpha-1} | f(z) |\} \le \left(1 + \frac{2^{\alpha-1}}{\alpha - 1}\right) \| f \|_{\mathscr{B}_\alpha(\mathbb{B}_n)}. \qquad (2)$$

Conversely, suppose

$$f \in H(\mathbb{B}_n) \text{ and } \sup_{z \in \mathbb{B}_n} \{(1 - | z |^2)^{\alpha-1} | f(z) |\} < \infty.$$

Then $f \in A^1_{\alpha-1}(\mathbb{B}_n)$. The Bergman integral formula (Theorem 2.2 of [5]) thus gives

$$f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+\alpha}} d\nu_{\alpha-1}(w) \quad (z \in \mathbb{B}_n). \qquad (3)$$

Differentiating inside the integral sign, we have for $j \in \{1, ..., n\}$,

$$(D_j f)(z) = \int_{\mathbb{B}_n} \frac{(n + \alpha)\overline{w}_j f(w)}{(1 - \langle z, w \rangle)^{n+\alpha+1}} d\nu_{\alpha-1}(w) \quad (z \in \mathbb{B}_n). \qquad (4)$$

By (4), for $z \in \mathbb{B}_n$,

$$|(\nabla f)(z)|$$

$$\le n(n+\alpha)c_{\alpha-1} \sup_{w\in\mathbb{B}_n}\{|f(w)|(1-|w|^2)^{\alpha-1}\}\int_{\mathbb{B}_n} \frac{d\nu(w)}{|1-\langle z, w\rangle|^{n+\alpha+1}}. \tag{5}$$

By Proposition 1.4.10 of [2],

$$\int_{\mathbb{B}_n} \frac{d\nu(w)}{|1-\langle z, w\rangle|^{n+\alpha+1}} \le \frac{C}{(1-|z|^2)^\alpha} \quad (z \in \mathbb{B}_n), \tag{6}$$

where C is a positive constant depending only on α and n. By (5) and (6),

$$\sup_{z\in\mathbb{B}_n}\{(1-|z|^2)^\alpha|(\nabla f)(z)|\}$$

$$\le n(n+\alpha)c_{\alpha-1}C \sup_{w\in\mathbb{B}_n}\{|f(w)|(1-|w|^2)^{\alpha-1}\}. \tag{7}$$

By (7), we have

$$\| f \|_{\mathscr{B}_\alpha(\mathbb{B}_n)} \le \{1+n(n+\alpha)c_{\alpha-1}C\} \sup_{w\in\mathbb{B}_n}\{|f(w)|(1-|w|^2)^{\alpha-1}\}. \tag{8}$$

(2) and (8) together show that

$$\mathscr{B}_\alpha(\mathbb{B}_n) = \{f \in H(\mathbb{B}_n) : \sup_{z\in\mathbb{B}_n}\{(1-|z|)^{\alpha-1}|f(z)|\} < \infty\}. \qquad \square$$

3. Proof of Theorem 1 in the Case $p \le q$

Lemma 5. *Let* $\alpha \in (-1, \infty)$, $\{p, q\} \subset \mathbb{R}_+$ *and* $\mu \in M_+(\mathbb{B}_n)$. *Suppose* $p \le q$. *Then the following two conditions are equivalent*:

(i)

$$\sup\left\{\frac{\| f \|_{L^q(\mu)}}{\| f \|_{A_\alpha^p(\mathbb{B}_n)}} : f \in A_\alpha^p(\mathbb{B}_n)\backslash\{0\}\right\} < \infty.$$

(ii)

$$\sup_{z\in\mathbb{B}_n}\left\{\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{|1-\langle z, w\rangle|^{\frac{2q}{p}(n+1+\alpha)}}d\mu(w)\right\}<\infty.$$

Proof. See [4, p. 69], (a) $\Leftrightarrow$ (b) of Theorem 50. □

Proposition 6. *Let* $\{\alpha, \beta\}\subset(-1, \infty)$ *and* $\{p, q\}\subset\mathbb{R}_+$. *Put* $\gamma=\frac{n+1+\beta}{q}-\frac{n+1+\alpha}{p}$. *Suppose* $p\le q$. *Then for any* $g\in H(\mathbb{B}_n)$, *the following inequalities hold*:

$$C_1\sup_{z\in\mathbb{B}_n}\{(1-|z|^2)^\gamma|g(z)|\}^q$$

$$\le\sup_{z\in\mathbb{B}_n}\left\{\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{|1-\langle z, w\rangle|^{\frac{2q}{p}(n+1+\alpha)}}|g(w)|^q d\nu_\beta(w)\right\}$$

$$\le C_2\sup_{z\in\mathbb{B}_n}\{(1-|z|^2)^\gamma|g(z)|\}^q,$$

where C_1 *and* C_2 *are both positive constants depending only on* α, β, p, q *and* n.

Proof. Put

$$s=\frac{q}{p}(n+1+\alpha),\quad \alpha_0=\beta-\gamma q. \tag{1}$$

Then by the assumptions and (1),

$$s\in(n, \infty),\quad \alpha_0=s-(n+1). \tag{2}$$

Fix $g\in H(\mathbb{B}_n)$. Define

$$G(z)=(1-|z|^2)^\gamma|g(z)|\quad(z\in\mathbb{B}_n). \tag{3}$$

Then $G \in C(\mathbb{B}_n)$, and so, by (1) ~ (3) and Proposition 3,

$$\frac{1}{c_{\alpha_0}} \sup_{z\in\mathbb{B}_n} \| G \circ \varphi_z \|_{L^q(\nu_{\alpha_0})}^q$$

$$= \sup_{z\in\mathbb{B}_n} \left\{ (1-|z|^2)^{\frac{q}{p}(n+1+\alpha)} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\beta}{|1-\langle z, w\rangle|^{\frac{2q}{p}(n+1+\alpha)}} |g(w)|^q d\nu(w) \right\}$$

$$= \frac{1}{c_\beta} \sup_{z\in\mathbb{B}_n} \left\{ \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{|1-\langle z, w\rangle|^{\frac{2q}{p}(n+1+\alpha)}} |g(w)|^q d\nu_\beta(w) \right\}. \tag{4}$$

Since $\nu_{\alpha_0}(\mathbb{B}_n) = 1$, by (3),

$$\sup_{z\in\mathbb{B}_n} \| G \circ \varphi_z \|_{L^q(\nu_{\alpha_0})}^q \le \sup_{w\in\mathbb{B}_n} |G(w)|^q = \sup_{w\in\mathbb{B}_n} \{(1-|w|^2)^\gamma |g(w)|\}^q. \tag{5}$$

By (4) and (5),

$$\sup_{z\in\mathbb{B}_n} \left\{ \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{|1-\langle z, w\rangle|^{\frac{2q}{p}(n+1+\alpha)}} |g(w)|^q d\nu_\beta(w) \right\}$$

$$\le \frac{c_\beta}{c_{\alpha_0}} \sup_{z\in\mathbb{B}_n} \{(1-|z|^2)^\gamma |g(z)|\}^q. \tag{6}$$

Conversely, choose any $R \in \mathbb{R}_+$. By using Lemma 2.24 of [5], we have

$$|g(z)|^q \le \frac{C_1}{(1-|z|^2)^{n+1+\beta}} \int_{D(z,R)} |g|^q d\nu_\beta \quad (z \in \mathbb{B}_n), \tag{7}$$

where C_1 is a positive constant depending only on β, R and n. By (7),

$$\sup_{z\in\mathbb{B}_n} \{(1-|z|^2)^\gamma |g(z)|\}^q$$

$$\le \sup_{z\in\mathbb{B}_n} \left\{ \frac{C_1}{(1-|z|^2)^{n+1+\beta-\gamma q}} \int_{D(z,R)} |g|^q d\nu_\beta \right\}$$

$$= \sup_{z\in\mathbb{B}_n}\left\{\frac{C_1}{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}\int_{D(z,R)}|g|^q dv_\beta\right\}$$

$$= C_1 \sup_{z\in\mathbb{B}_n}\left\{\int_{D(z,R)}\frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{(1-|z|^2)^{\frac{2q}{p}(n+1+\alpha)}}|g(w)|^q dv_\beta(w)\right\}. \tag{8}$$

By Lemma 2.20 of [5],

$$\left(\frac{|1-\langle z, w\rangle|}{1-|z|^2}\right)^{\frac{2q}{p}(n+1+\alpha)} < C_2 \quad (z\in\mathbb{B}_n,\ w\in D(z, R)), \tag{9}$$

where C_2 is a positive constant depending only on p, q, α, R and n. By (8) and (9),

$$\sup_{z\in\mathbb{B}_n}\{(1-|z|^2)^\gamma|g(z)|\}^q$$

$$\le C_1C_2 \sup_{z\in\mathbb{B}_n}\left\{\int_{D(z,R)}\frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{|1-\langle z, w\rangle|^{\frac{2q}{p}(n+1+\alpha)}}|g(w)|^q dv_\beta(w)\right\}$$

$$\le C_1C_2 \sup_{z\in\mathbb{B}_n}\left\{\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{|1-\langle z, w\rangle|^{\frac{2q}{p}(n+1+\alpha)}}|g(w)|^q dv_\beta(w)\right\}. \tag{10}$$

The assertion of the proposition follows from (6) and (10). □

Proposition 7. *Let* $\{\alpha, \beta\}\subset(-1, \infty)$, $\{p, q\}\subset\mathbb{R}_+$ *and* $g\in H(\mathbb{B}_n)$. *Put* $\gamma = \dfrac{n+1+\beta}{q} - \dfrac{n+1+\alpha}{p}$. *Suppose* $p\le q$. *Then the following three conditions are equivalent*:

(i)

$$g\in(\mathscr{PM})(A_\alpha^p(\mathbb{B}_n),\ A_\beta^q(\mathbb{B}_n)).$$

(ii)

$$\sup_{z\in\mathbb{B}_n}\left\{\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\frac{q}{p}(n+1+\alpha)}}{|1-\langle z, w\rangle|^{\frac{2q}{p}(n+1+\alpha)}}|g(w)|^q d\nu_\beta(w)\right\}<\infty.$$

(iii)

$$\sup_{z\in\mathbb{B}_n}\{(1-|z|^2)^\gamma|g(z)|\}<\infty.$$

Proof. Define $\mu_g\in M_+(\mathbb{B}_n)$ by $d\mu_g=|g|^q d\nu_\beta$. Then

$$\|f\|_{L^q(\mu_g)}=\|fg\|_{A_\beta^q(\mathbb{B}_n)} \text{ for all } f\in H(\mathbb{B}_n).$$

Hence, the present proposition follows from Proposition 2, Lemma 5 and Proposition 6. □

Proof of Theorem 1 in the case $p\le q$

When $\gamma>0$, by Lemma 4,

$$\mathscr{B}_{1+\gamma}(\mathbb{B}_n)=\{g\in H(\mathbb{B}_n): \sup_{z\in\mathbb{B}_n}\{(1-|z|)^\gamma|g(z)|<\infty\}. \tag{1}$$

By Proposition 7 and (1), we obtain

$$(\mathscr{PM})\,(A_\alpha^p(\mathbb{B}_n), A_\beta^q(\mathbb{B}_n))=\mathscr{B}_{1+\gamma}(\mathbb{B}_n).$$

When $\gamma=0$, by Proposition 7, we obtain

$$(\mathscr{PM})(A_\alpha^p(\mathbb{B}_n), A_\beta^q(\mathbb{B}_n))=\{g\in H(\mathbb{B}_n): \sup_{z\in\mathbb{B}_n}|g(z)|<\infty\}=H^\infty(\mathbb{B}_n).$$

When $\gamma<0$, it is easily shown that

$$\{g\in H(\mathbb{B}_n): \sup_{z\in\mathbb{B}_n}\{(1-|z|^2)^\gamma|g(z)|\}<\infty\}$$

$$=\{g\in H(\mathbb{B}_n): \lim_{|z|\to 1-0}|g(z)|=0\}=\{0\}. \tag{2}$$

By Proposition 7 and (2), we obtain

$$(\mathscr{PM})(A_\alpha^p(\mathbb{B}_n),\ A_\beta^q(\mathbb{B}_n)) = \{0\}.$$

The proof of Theorem 1 in the case $p \le q$ is now completed.

4. Proof of Theorem 1 in the Case $p > q$

Lemma 8. *Let* $\alpha \in (-1, \infty)$, $\{p, q\} \subset \mathbb{R}_+$ *and* $\mu \in M_+(\mathbb{B}_n)$. *Suppose* $p > q$. *Then the following two conditions are equivalent*:

(i)

$$\sup\left\{\frac{\| f \|_{L^q(\mu)}}{\| f \|_{A_\alpha^p(\mathbb{B}_n)}} : f \in A_\alpha^p(\mathbb{B}_n)\setminus\{0\}\right\} < \infty.$$

(ii)

$$\hat{\mu}_{R,\alpha} \in L^{\frac{p}{p-q}}(\nu_\alpha) \text{ for all } R \in \mathbb{R}_+.$$

Proof. See [4, p. 73], (a) $\Leftrightarrow$ (c) of Theorem 54. □

Proposition 9. *Let* $\{\alpha, \beta\} \subset (-1, \infty)$, $\{p, q\} \subset \mathbb{R}_+$ *and* $g \in H(\mathbb{B}_n)$. *Define* $\mu_g \in M_+(\mathbb{B}_n)$ *by* $d\mu_g = |g|^q d\nu_\beta$. *Suppose* $p > q$. *Then the following two conditions are equivalent*:

(i)

$$g \in (\mathscr{PM})(A_\alpha^p(\mathbb{B}_n),\ A_\beta^q(\mathbb{B}_n)).$$

(ii)

$$(\hat{\mu}_g)_{R,\alpha} \in L^{\frac{p}{p-q}}(\nu_\alpha) \text{ for all } R \in \mathbb{R}_+.$$

Proof. By the definition of μ_g,

$$\| f \|_{L^q(\mu_g)} = \| fg \|_{A_\beta^q(\mathbb{B}_n)} \text{ for all } f \in H(\mathbb{B}_n).$$

The present proposition thus follows from Proposition 2 and Lemma 8. □

Proposition 10. *Let* $\{\alpha, \beta\} \subset (-1, \infty)$ *and* $\{p, q\} \subset \mathbb{R}_+$. *Suppose* $p > q$. *Put* $s = \frac{pq}{p-q}$ *and* $\delta = s\left(\frac{\beta}{q} - \frac{\alpha}{p}\right)$. *Then for any pair* $\{f, g\} \subset H(\mathbb{B}_n)$,

$$\| fg \|_{A_\beta^q(\mathbb{B}_n)} \le C \| f \|_{A_\alpha^p(\mathbb{B}_n)} \| g \|_{A_\delta^s(\mathbb{B}_n)},$$

where C is a positive constant depending only on α, β, p, q *and n.*

Proof. Put

$$p_0 = \frac{p}{p-q}, \quad q_0 = \frac{p_0}{p_0 - 1}. \tag{1}$$

Then

$$\{p_0, q_0\} \subset (1, \infty), \quad \frac{1}{p_0} + \frac{1}{q_0} = 1. \tag{2}$$

By the assumptions,

$$\frac{sq}{s-q} = p, \quad \left(\beta - \frac{q\delta}{s}\right)\frac{s}{s-q} = \alpha. \tag{3}$$

By (1) ~ (3),

$$\begin{aligned}
&\| fg \|_{A_\beta^q(\mathbb{B}_n)}^q \\
&= \int_{\mathbb{B}_n} | fg |^q \, d\nu_\beta \\
&= c_\beta \int_{\mathbb{B}_n} | f(z) |^q (1 - | z |^2)^{\beta - \frac{\delta}{p_0}} \cdot | g(z) |^q (1 - | z |^2)^{\frac{\delta}{p_0}} \, d\nu(z) \\
&\le c_\beta \left[\int_{\mathbb{B}_n} \left\{ | f(z) |^q (1 - | z |^2)^{\beta - \frac{\delta}{p_0}} \right\}^{q_0} d\nu(z) \right]^{\frac{1}{q_0}}
\end{aligned}$$

$$\cdot\left[\int_{\mathbb{B}_n}\left\{|g(z)|^q(1-|z|^2)^{\frac{\delta}{p_0}}\right\}^{p_0}d\nu(z)\right]^{\frac{1}{p_0}}$$

$$= c_\beta\left[\int_{\mathbb{B}_n}|f(z)|^{\frac{sq}{s-q}}(1-|z|^2)^{\left(\beta-\frac{q\delta}{s}\right)\frac{s}{s-q}}d\nu(z)\right]^{\frac{s-q}{s}}$$

$$\cdot\left[\int_{\mathbb{B}_n}|g(z)|^s(1-|z|^2)^\delta d\nu(z)\right]^{\frac{q}{s}}$$

$$= c_\beta\left[\int_{\mathbb{B}_n}|f(z)|^p(1-|z|^2)^\alpha d\nu(z)\right]^{\frac{q}{p}}\left[\int_{\mathbb{B}_n}|g(z)|^s(1-|z|^2)^\delta d\nu(z)\right]^{\frac{q}{s}}$$

$$= c_\beta c_\alpha^{-\frac{q}{p}}c_\delta^{-\frac{q}{s}}\|f\|^q_{A^p_\alpha(\mathbb{B}_n)}\|g\|^q_{A^s_\delta(\mathbb{B}_n)}.$$

This completes the proof. □

Proposition 11. *Let* $\{\alpha, \beta\} \subset (-1, \infty)$ *and* $\{p, q, R\} \subset \mathbb{R}_+$. *Put* $s = \frac{pq}{p-q}$ *and* $\delta = s\left(\frac{\beta}{q} - \frac{\alpha}{p}\right)$. *Suppose* $p > q$. *Then for any* $g \in H(\mathbb{B}_n)$,

$$\int_{\mathbb{B}_n}|g|^s d\nu_\delta \le C\int_{\mathbb{B}_n}|(\hat{\mu}_g)_{R,\alpha}|^{\frac{p}{p-q}}d\nu_\alpha,$$

where $d\mu_g = |g|^q d\nu_\beta$ *and* C *is a positive constant depending only on the six numbers* $\{\alpha, \beta, p, q, R, n\}$.

Proof. By the definition of $(\hat{\mu}_g)_{R,\alpha}$,

$$\int_{\mathbb{B}_n}|(\hat{\mu}_g)_{R,\alpha}|^{\frac{p}{p-q}}d\nu_\alpha$$

$$= \int_{\mathbb{B}_n}\left\{\frac{1}{(1-|z|^2)^{n+1+\alpha}}\int_{D(z,R)}|g|^q d\nu_\beta\right\}^{\frac{p}{p-q}}d\nu_\alpha(z). \tag{1}$$

By Lemma 2.24 of [5], we have

$$|g(z)|^q \leq \frac{C_1}{(1-|z|^2)^{n+1+\beta}} \int_{D(z,R)} |g|^q dv_\beta \quad (z \in \mathbb{B}_n), \tag{2}$$

where C_1 is a positive constant depending only on β, R and n. By (1) and (2),

$$\int_{\mathbb{B}_n} |(\hat{\mu}_g)_{R,\alpha}|^{\frac{p}{p-q}} dv_\alpha$$

$$\geq \int_{\mathbb{B}_n} \left\{ \frac{1}{(1-|z|^2)^{n+1+\alpha}} \frac{(1-|z|^2)^{n+1+\beta}}{C_1} |g(z)|^q \right\}^{\frac{p}{p-q}} dv_\alpha(z)$$

$$= c_\alpha C_1^{-\frac{p}{p-q}} \int_{\mathbb{B}_n} (1-|z|^2)^{\frac{p(\beta-\alpha)}{p-q}+\alpha} |g(z)|^{\frac{pq}{p-q}} dv_\alpha(z)$$

$$= c_\alpha C_1^{-\frac{p}{p-q}} \int_{\mathbb{B}_n} (1-|z|^2)^{\delta} |g(z)|^s dv(z)$$

$$= c_\alpha C_1^{-\frac{p}{p-q}} c_\delta^{-1} \int_{\mathbb{B}_n} |g|^s dv_\delta.$$

This completes the proof. □

Proof of Theorem 1 in the case $p > q$

By Proposition 10,

$$g \in A_\delta^s(\mathbb{B}_n) \Rightarrow g \in (\mathscr{PM})(A_\alpha^p(\mathbb{B}_n), A_\beta^q(\mathbb{B}_n)). \tag{1}$$

By Proposition 9 and Proposition 11,

$$g \in (\mathscr{PM})(A_\alpha^p(\mathbb{B}_n), A_\beta^q(\mathbb{B}_n)) \Rightarrow g \in A_\delta^s(\mathbb{B}_n). \tag{2}$$

(1) and (2) together show that

$$(\mathscr{PM})(A_\alpha^p(\mathbb{B}_n), A_\beta^q(\mathbb{B}_n)) = A_\delta^s(\mathbb{B}_n).$$

The proof of Theorem 1 is now finished. □

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